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Behaviour of phase-space holes in 2D simulations

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This work shows creation of phase-space holes when an initially perturbed plasma evolves in time, in the absence of external magnetic fields. When such a field is applied to the plasma ab initio, these structures are not created. If such hole structures are created with the magnetic field turned off, the holes are destroyed under the influence of a magnetic field when one is applied to the plasma.

1. Introduction

In this paper we investigate the behaviour of a spatially periodic two-dimensional (2D) electron plasma with a fixed homogeneous ion background.

In one-dimensional (1D) simulations, the creation of phase-space holes as the response of the plasma to an initial perturbation has been investigated by many authors: Berk and Roberts (1967) and Roberts and Berk (1967) used a water-bag model while Bertrand et al. (1988) used Eulerian and PIC codes. The tendency of zero-mean velocity structures (holes) to coalesce was pointed out by Berk et al. (1970) and Bertrand et al. (1988). Holes with a non-null mean velocity were found by Boutros-Ghali and Dupree (1981) and Berman et al. (1986), and were related by Dupree (1982) to weak-turbulence theory.

Two kinds of holes are found in simulations:

(i) time-independent holes without mean velocity; These holes are BGK modes (Bernstein 1958);

(ii) time-dependent holes, moving in phase space with a constant mean velocity.

The behaviour of the first kind of holes in 1D simulations is presented in Ghizzo et al. (1987), and that of the second kind is given in Mineau (1989) – again in 1D simulation with an Eulerian Vlasov code.

Numerical simulations in 2D show that a plasma subject to the same type of initial conditions as in 1D simulations, in the absence of a magnetic field, has the same kind of response as in the 1D case, as presented in Fijalkow et al. (1994). A preliminary publication by Fijalkow (1995) suggests that a constant magnetic field inhibits hole creation.

In the present work we shall present 2D simulations that show the creation and evolution of phase-space coherent structures (holes), in the absence of an external, constant in space and time, magnetic field. We show the non-creation of such coherent structures if a magnetic field is present, and the deflation of these co-

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herent structures, created without a magnetic field, after a field is applied to the plasma.

2. Numerical code, initial conditions

The normalized (to the inverse of the plasma frequency for time t, to the electron Debye length for space variables x and y, and to the electron thermal velocity v_{th} for velocity variables v_x and v_y), Vlasov–Poisson system is given in two dimensions by

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + (E_x + v_y B_z)\frac{\partial f}{\partial x} + (E_y - v_x B_z)\frac{\partial f}{\partial v_y} = 0, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E}(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, x, y, v_x, v_y)\, dv_x\, dv_y - N_0. \tag{2}$$

The Vlasov equation (1) is solved by a time-splitting method connected with a flux-balance technique (Fijalkow 1998a,b) to perform the advection in time of the splitted equations. The Eulerian method used is very stable and fast (less than $0.299\,\mu$s per time step and grid point in phase space on a Cray C90 computer).

Since in our work the plasma is periodic in space, the Poisson equation (2) is solved directly by fast Fourier transform as

$$E_x(k_x, k_y) = -i\frac{k_x}{k_x^2 + k_y^2}\rho(k_x, k_y), \tag{3}$$

$$E_y(k_x, k_y) = -i\frac{k_y}{k_x^2 + k_y^2}\rho(k_x, k_y), \tag{4}$$

so that the solution of (2) needs only three Fourier transforms at each time step.

The initial conditions used in the present study are of two kinds:

(a) a spatially perturbed, Maxwellian in velocity, normalized function

$$f(x, y, v_x, v_y, t = 0) = (2\pi)^{-1} \exp\left(-\frac{v_x^2 + v_y^2}{2}\right)(1 + Q_x \cos k_{0x} x + Q_y \cos k_{0y} y); \tag{5}$$

the relaxation in time of such an initial problem leads to the creation of moving phase-space holes;

(b) a spatially perturbed, double-stream in velocity, function

$$f(x, y, v_x, v_y, t = 0) = \mathcal{N} \exp\left(-\frac{|v_x - v_{x0}|^2 + |v_y - v_{y0}|^2}{2}\right)[1 + 5(v_x^2 + v_y^2)] \times \left[1 + Q_x\left(\frac{\cos \alpha x}{10} + \cos 2\alpha x\right) + Q_y\left(\frac{\cos \beta y}{30} + \cos 3\beta y\right)\right], \tag{6}$$

with

$$\alpha = 2\pi/L_x, \qquad \beta = 2\pi/L_y.$$

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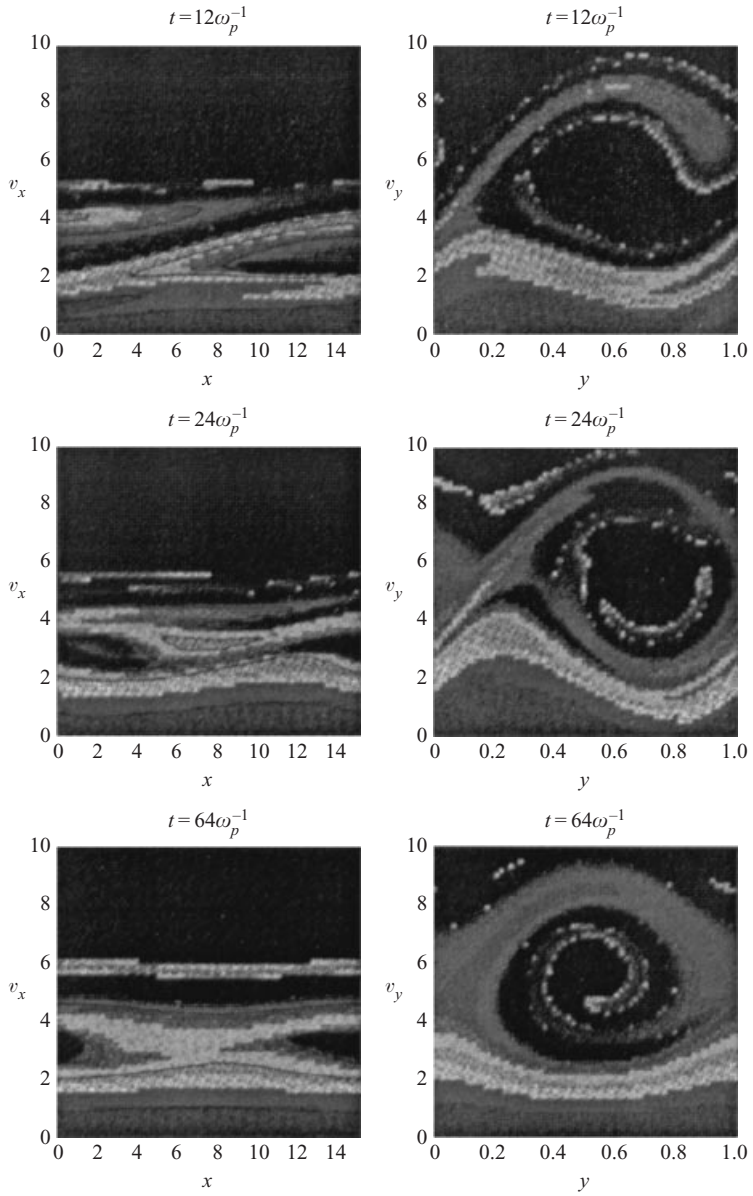


Figure 1. Projection of the 2D phase space onto (x, v_x) and (y, v_y) subspaces. Since the problem is symmetric in the velocity, only the upper halves of the subspaces are shown. The evolution of mobile holes in time is shown for $k_{0x} = 0.4$, $k_{0y} = 0.3$, $Q_x = Q_y = 0.3$ and $B_z = 0$.

L_x and L_y are the (normalized to the Debye length) computational box lengths needed for the creation of a single hole in phase space, in x and y respectively. $\mathcal{N}$ is a normalization factor chosen in such a way the integral of the distribution function $f(x, y, v_x, v_y, t)$ over velocity and space is equal to $L_x^* L_y$.

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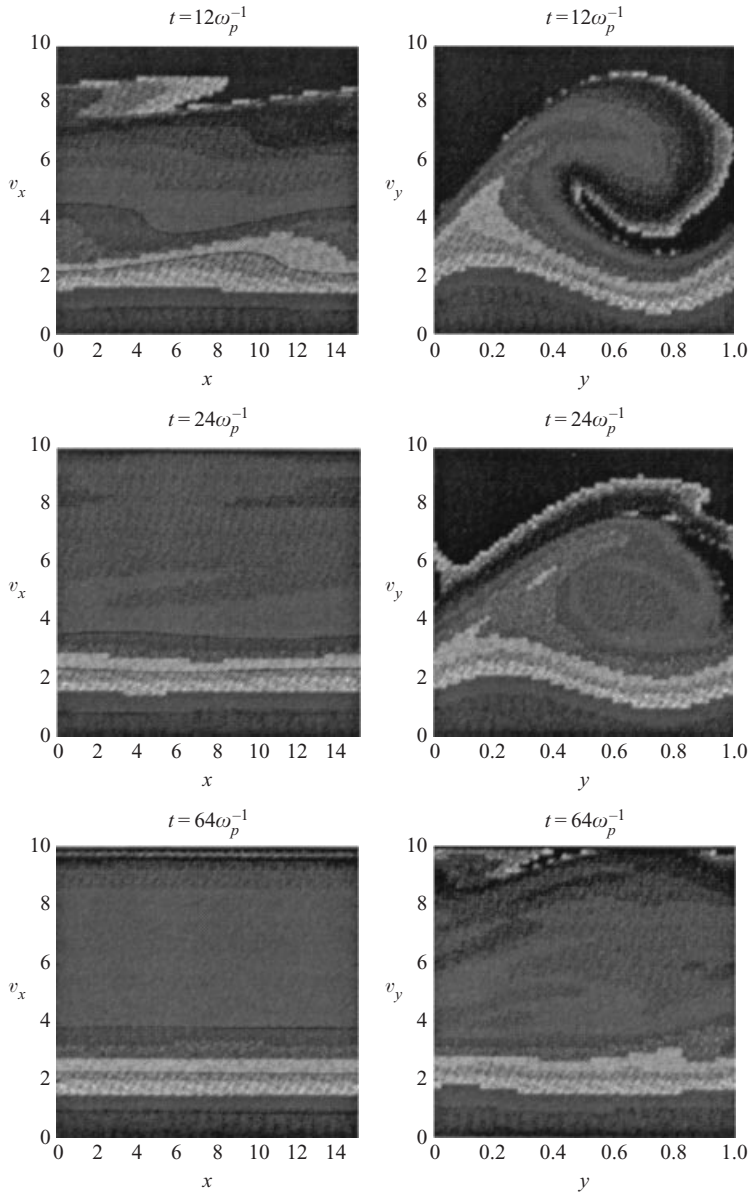


Figure 2. Upper half-subspaces (x, v_x) and (y, v_y). Annihilation of mobile holes in time is shown for $k_{0x} = 0.4$, $k_{0y} = 0.3$, $Q_x = Q_y = 0.3$ and $\omega_c = 0.1\omega_p$.

3. Behaviour of time-dependent phase-space holes

We start the numerical integration of the Vlasov–Poisson system (1), (2), with the initial condition represented by (5).

The chosen values of the wave numbers k_{0x} and k_{0y} are

$$k_{0x} = 0.4, \qquad k_{0y} = 0.3,$$

and the perturbation coefficients Q_x and Q_y are both equal to 0.3.

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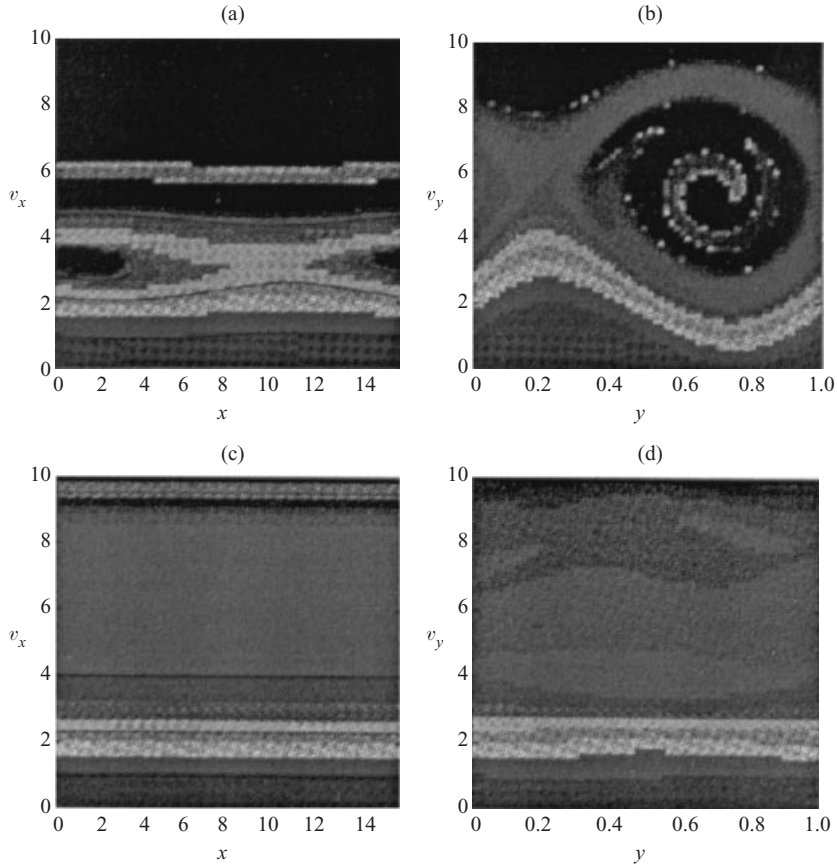


Figure 3. Phase space after a long time $t = 100\omega_p^{-1}$ without (a, b), and with (c, d) a magnetic field ($\omega_c = 0.1\omega_p$). The wavenumbers and perturbation coefficients are the same as in the previous figures.

In all the simulations described in this paper the magnetic field is given in plasma-frequency units.The velocity varies from -8 to $+8$ (in thermal-velocity units) in both x and y. If the gyrofrequency is small compared with the plasma frequency, this implies a small magnetic field. This type of initial conditions is of the same family as those used in linear theory to study Landau damping and in the theory of nonlinear Landau damping in earlier works. The formation of phase-space holes is shown in Fig. 1. Since the phase space is symmetric in velocity, only the upper halves of the figures are shown. After a short time, we can see the appearance of a hole structure at $t = 12\omega_p^{-1}$, and that structure remains for a very long time, as was the case in 1D simulations. A structure of this type moves in phase with a constant mean velocity.

The introduction of a magnetic field alters totally the phase-space behaviour. This can be seen in Fig. 2. Starting the time evolution with the same initial condition as in the previous case, (5), we apply a magnetic field B_{0z} that is constant in time and space. The equivalent gyrofrequency is $\omega_c = 0.1\omega_p$, implying a small value for the field. At $t = 12\omega_p^{-1}$, an attempt to create a hole can be observed, with a sharper effect in the (y, v_y) subspace. At $T = 24\omega_p^{-1}$, the global structure persists, but the

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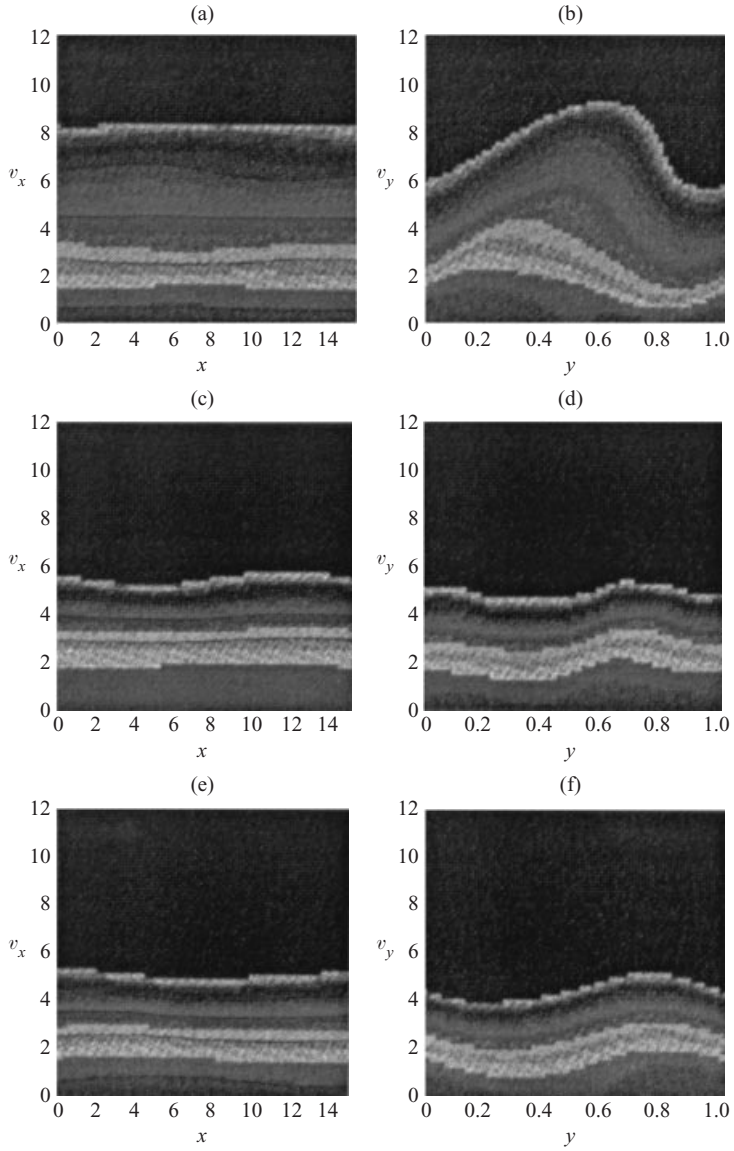


Figure 4. View of the phase-space projections at $t = 12\omega_p^{-1}$ for three values of the gyrofrequency: $\omega_c = 0.3\omega_p$ (a, b), $0.9\omega_p$ (c, d) and $1.7\omega_p$ (e, f).

centre of the structure is not free of matter, as it was at the same time in Fig. 1. At $t = 64\omega_p^{-1}$, the hole structure disappears. To summarize the global behaviour, we present Fig. 3. Here we can compare after a long evolution time ($t = 100\omega_p^{-1}$) the phase-space projections with and without the magnetic field applied previously. Without the field B_{0z}, the holes are perfectly formed in both the (x, v_x) and (y, v_y) subspaces. When the field is present no structure exists at all.

To study the influence of the magnetic field strength on the plasma evolution, the projections of the phase space to the (x, v_x) and (y, v_y) subspaces are presented for three gyrofrequency values, $\omega_{c_1} = 0.3\omega_p$, $\omega_{c_2} = 0.9\omega_p$ and $\omega_{c_3} = 1.7\omega_p$, in Fig. 4

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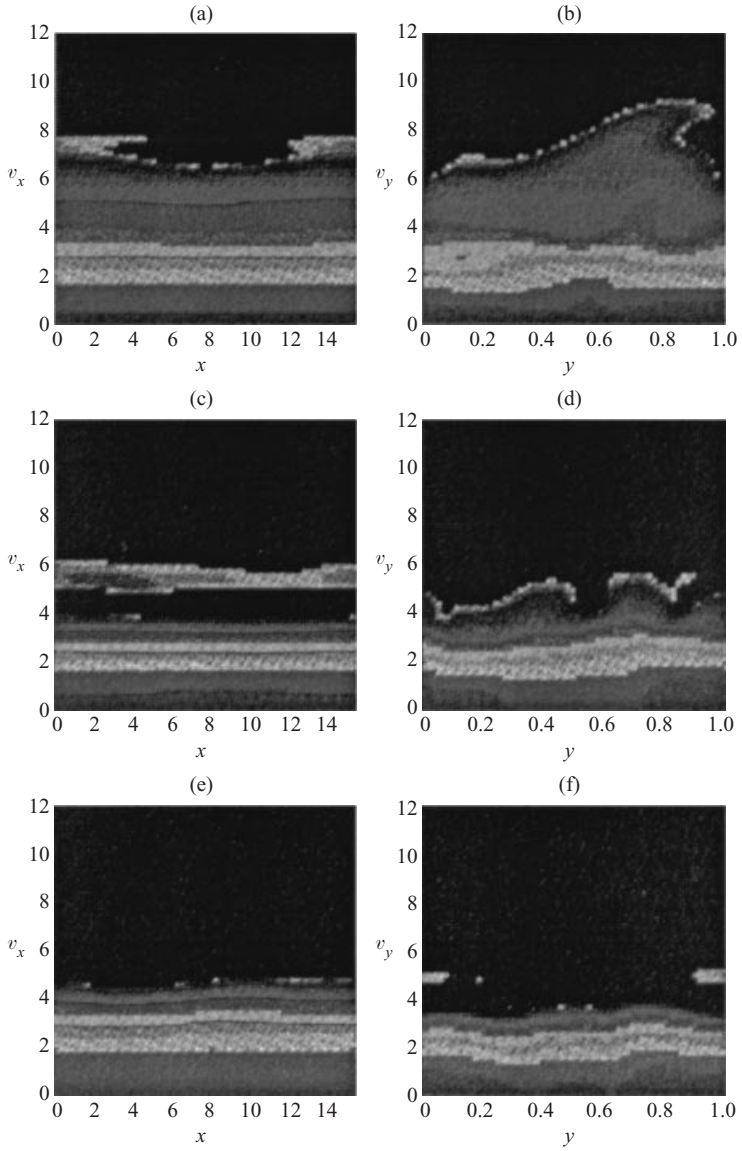


Figure 5. View of the phase-space projections at $t = 100\omega_p^{-1}$. The gyrofrequency values are as in Fig. 4.

for $t = 12\omega_p^{-1}$ and Fig. 5 for $t = 100\omega_p^{-1}$. For ω_{c_1} at $t = 12\omega_p^{-1}$, we observe, as in the case $\omega_c = 0.1\omega_p$, a competition between two effects : an attempt, due to the nonlinear Landau-damping behaviour, to create a phase-space hole structure, and a very strong mixing of the high-velocity particles, due to the gyrokinetic effect. These two effects cancel out after a large time (Fig. 5). In the ω_{c_2} case the gyrofrequency is nearly the same as the plasma frequency, and no effects appear at all. In the case $\omega_c > \omega_p$ an oscillation appears after a short time (Fig. 4), but is again totally destroyed after a large time (Fig. 5).

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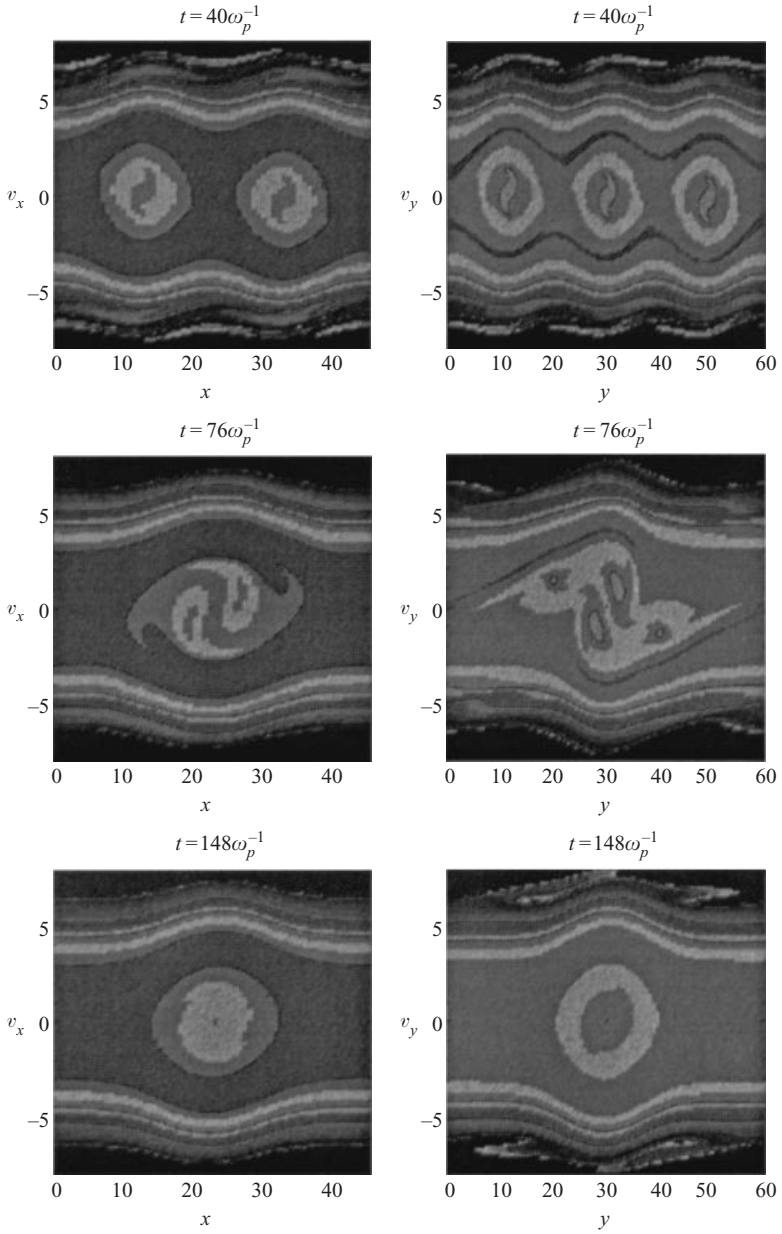


Figure 6. Phase space for a double-stream problem evolving to BGK-type holes, without a magnetic field. There is hole formation at $t = 40\omega_p^{-1}$, and coalescence for longer time. The velocity varies from −8 to +8 (in thermal velocity units) in both x and y.

4. Time-independent (BGK) holes

To study the static-holes problem (zero-mean-velocity holes), we start with the initial condition (6). The perturbation coefficients Q_x and Q_y both have the value 0.02. The choice of a computational box length in x and y is now a difficult problem. In the one-dimensional case, the Poisson equation is an ordinary differential

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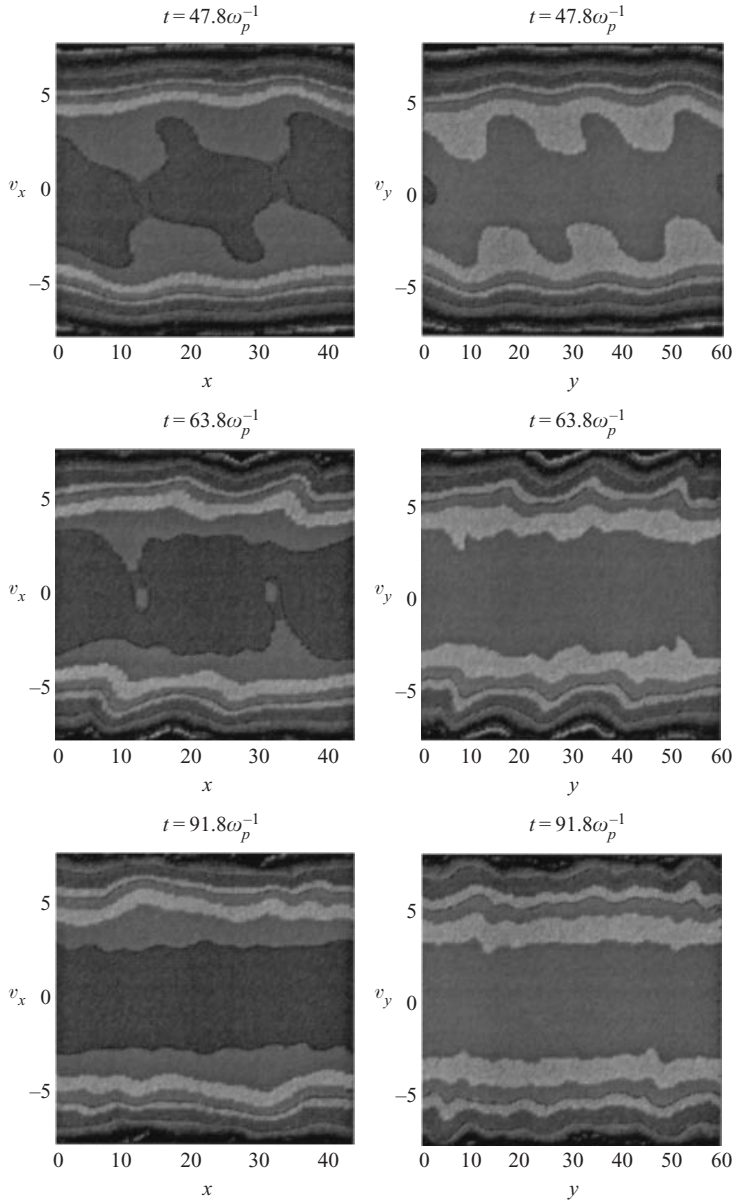


Figure 7. The same double-stream problem as in Fig. 6 until $t = 44\omega_p^{-1}$; then a magnetic field with a gyrofrequency value of $\omega_c = 0.3\omega_p$ is turned on.

equation. Since we are looking for a BGK solution of the Vlasov equation, i.e. a solution that is only energy-dependent, and with an exponential behaviour in the v variable, the electon density takes the form

$$n(x) = n_0 e^{\phi(x)}, \tag{7}$$

with $\phi(x)$ the electric potential. We can now solve the given initial-value problem above, starting with $\phi(x = 0) = 0$, until we again obtain a zero value for the potential. The value of x where this happens is the length of the box for one hole.

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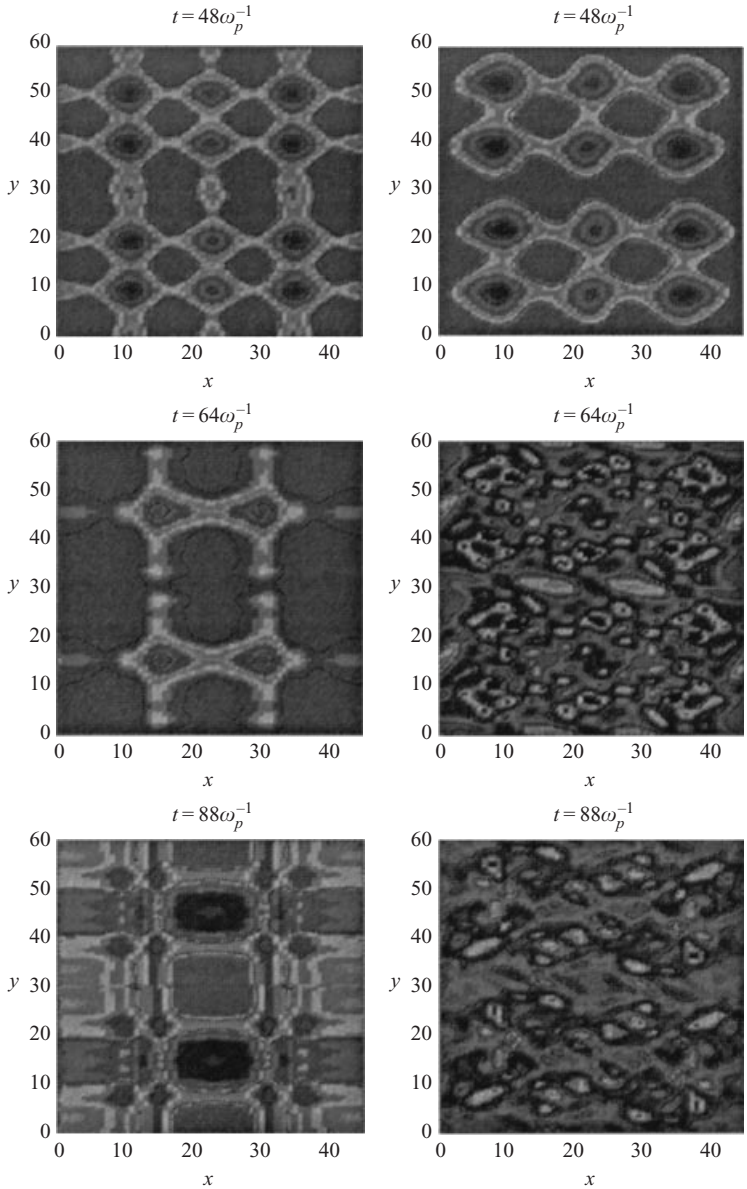


Figure 8. The density for the problem shown in Figs 6 and 7. The left column is without a magnetic field; the right column is with a magnetic field.

In two dimensions such a process is impossible, since the Poisson equation is now a partial differential equation of elliptic type, whose solution can be found only on the whole domain, accordingly to given boundary conditions.

The only way is to proceed by trial and error. In the present work, a choice of $x_L = y_L = 46.17\lambda_D$ was taken. This choice gives the possibility to create two or three holes with null mean velocity. The result is shown in Fig. 6. At $t = 40\omega_p^{-1}$, two holes can be seen in the (x, v_x) subspace and three in the (y, v_y) phase-space projection.

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At $t = 76\omega_p^{-1}$, the two holes have collapsed to a single hole with two centres, and the three holes to two holes in a mixing phase. At $t = 148\omega_p^{-1}$, all of the holes have collapsed to a single hole, as in the classical 1D case.

To study the behaviour of this type of hole under the influence of magnetic field, there are two approaches. The first is to start the numerical integration with a magnetic field present from the beginning. In this case the holes are not created. A second way is to create the holes as they appear in Fig. 6 for a value of $t = 40\omega_p^{-1}$, and then to turn on the magnetic field. This is the way chosen to obtain the results shown in Fig. 7. The field was turned on at $t = 44\omega_p^{-1}$. At $t = 47.8\omega_p^{-1}$, we can observe a partial destruction of the holes in both subspaces. At $t = 63.8\omega_p^{-1}$, the hole structure has disappeared in the (y, v_y) subspace, but not totally in (x, v_x). At $t = 91.8\omega_p^{-1}$, no sign of a hole remains.

To finish this study of the behaviour of phase-space holes, it is interesting to compare the evolution in time of the density, with and without a magnetic field, for the case of holes with zero mean velocity. This is done in Fig. 8.

At $t = 48\omega_p^{-1}$, a coherent structure remains. We can count four minima in the x direction, and six in y. At $t = 64\omega_p^{-1}$, without the magnetic field, the holes collapse in a good way, and we see only two minima. If the plasma evolves under the influence of a magnetic field, the density space is totally chaotic.

5. Conclusions

The use of an Eulerian numerical code to solve the Vlasov–Poisson system in two dimensions allows the hole structures to be found in a four-dimensional phase space. It has been shown that two types of hole structures are solution of the Vlasov–Poisson system, in the absence of external magnetic fields: time-dependent holes (moving with a constant velocity in phase space), and time-independent holes, which are a function of the energy only:

$$f(x, y, v_x, v_y) = f(\tfrac{1}{2}(v_x^2 + v_y^2) + \phi(x, y)). \tag{8}$$

In the presence of a magnetic field, these structures are no longer possible solutions of the Vlasov–Poisson system. Actually, no coherent structures in phase space have been found that are solutions of the Vlasov equation for a plasma in a magnetic field. The present work is a pure 2D case. In such a configuration, the magnetic field is perpendicular to the plasma. In the 3D case, other solutions exist (Berk et al. 1997; Terry et al. 1990). These 3D solutions need a velocity component parallel to the magnetic field. We cannot find these solutions with our present code. A 3D+2D gyrokinetic code, currently being developed, can find these solutions.

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