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On Twisted Odd Graphs

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The twisted odd graphs are obtained from the well-known odd graphs through an involutive automorphism. As expected, the twisted odd graphs share some of the interesting properties of the odd graphs but, in general, they seem to have a more involved structure. Here we study some of their basic properties, such as their automorphism group, diameter, and spectrum. They turn out to be examples of the so-called boundary graphs, which are graphs satisfying an extremal property that arises from a bound for the diameter of a graph in terms of its distinct eigenvalues.

1. Introduction

The odd graphs O_k are a well-known family of graphs, independently introduced by Balaban, Farcusiu and Banica [1] and Biggs [2]. Their vertices correspond to the $(k-1)$-subsets of a $(2k-1)$-set and adjacency is defined by void intersection. The odd graphs constitute a good example of the interaction between graph theory and other areas of combinatorics, such as coding theory and design theory. For a variety of results concerning them, we refer the reader to Biggs [5, 4], Godsil [16], and Zelinka [25]. Among their striking properties, we can mention, for instance, some of those with an algebraic nature. Thus, answering in the affirmative two questions posed by Biggs in [4], Moon [21] proved that the odd graphs are characterized by their intersection array (among distance-regular graphs) and, from this result, Huang [20] managed to show that, for small degrees ($\leqslant 6$), odd graphs are also characterized by their spectrum (among regular graphs). In fact, combining Moon's characterization and the result in [9] – characterizing distance-regular graphs by their spectrum and the number of vertices at maximum distance from each vertex – we have the following theorem.

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Theorem 1.1. *Any regular graph with the same spectrum, and the same number of vertices at distance $k-1$ from each vertex as the odd graph O_k is such a graph.* □

Other interesting properties of the odd graphs (which bear some resemblances to hypercubes) are those related to coding theory, such as the (non-)existence of some perfect e-codes. (In particular, 1-perfect codes in odd graphs turn out to be a kind of Steiner system.) See, for instance, Hammond and D. H. Smith [18] and Solé, Ghafoor and Sheikh [24]. Recently, the odd graphs have also been proposed as good models for fault-tolerant interconnection networks (see Ghafoor and Bashkow [15], and Simó and Yebra [22, 23]).

In this paper we study a new family of graphs derived from the odd graphs through an involutive automorphism. The new graphs have the same vertex set as the odd graphs they derive from. Some of the old adjacencies also remain unchanged, but some others become modified (the adjacency condition is still based on the void intersection of some subsets). As a consequence some of the original symmetries disappear, and the resulting structure turns out to be more involved. This prompted us to refer to these graphs as the 'twisted odd graphs'. Here we study some of their basic properties, such as metric properties, symmetries, and spectral properties. In particular, it is shown that the twisted odd graphs are examples of the recently considered 'boundary graphs', which are in the limit case of a spectral condition bounding above the diameter of a graph: see [11, 13, 12].

Before proceeding with our study, undertaken in the next sections, we will fix some of the basic terminology and notation used through the paper. For a (simple, finite, and connected) graph $G = (V, E)$ with vertex set V and edge set E, we will denote adjacency by writing $u \sim v$ (that is, $uv \in E$). If $\boldsymbol{A}$ is the adjacency matrix of G, its spectrum is represented by $\operatorname{sp} G := \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where superscripts stand for multiplicities, and its (set of) eigenvalues $\operatorname{ev} G := \operatorname{ev} \boldsymbol{A} = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$. We will suppose that matrices and vectors are indexed by the elements of V, that is, the vertices of G. The usual distance function between vertices will be denoted by ∂, so that the *eccentricity* (of a vertex), *radius*, and *diameter* are $\operatorname{ecc}(u) := \max_{v \in V} \partial(u, v)$, $r := \min_{u \in V} \operatorname{ecc}(u)$, and $D := \max_{u \in V} \operatorname{ecc}(u)$, respectively. We will also use the concepts of *even distance* and *odd distance* between vertices (see Bond and Delorme [6]), denoted by ∂^+ and ∂^- respectively. They are defined as the length of a shortest even (respectively odd) walk between the corresponding vertices. A *polarity* of a bipartite graph G is an involutive automorphism which exchanges its partite sets. The *quotient* graph of G by a polarity ρ is the graph $G^* = G/\rho$ with vertices the pairs $\{u, \rho u\}$ and adjacencies $\{u, \rho u\} \sim \{v, \rho v\}$ if and only if $u \sim v$ or $u \sim \rho v$ in G. (Equivalently, the vertices of G^* are the vertices of a partite set of G, say V_0, and $u_0 \sim v_0$ in G^* if and only if $u_1 := \rho u_0 \sim v_0$ in G.)

2. The twisted odd graphs

We recall that the *odd graph* O_k is the graph whose vertices can be labelled as (and we will identify with) the $(k-1)$-subsets of a $(2k-1)$-set Ω, so that two vertices $u, v \subset \Omega$ are adjacent if and only if $u \cap v = \emptyset$. Then, O_k has $n = \binom{2k-1}{k-1}$ vertices and it is regular with

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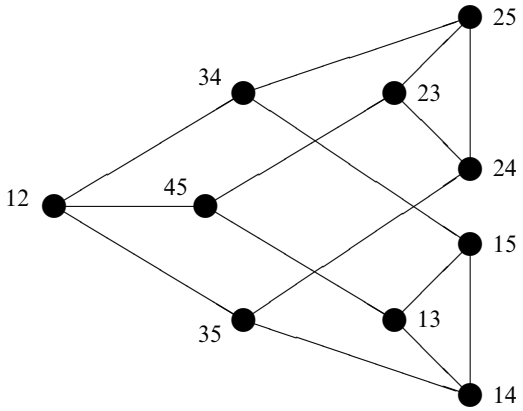


Figure 1 The twisted odd graph O_3^σ when $\sigma = (12)$

degree k. The following (less trivial) known properties of O_k will be used in our study (see, for instance, [5]).

P1. The even and odd distances between two vertices u, v of O_k are

$$\begin{aligned} \partial^-(u,v) &= 2|u \cap v| + 1; \\ \partial^+(u,v) &= 2(k-1-|u \cap v|). \end{aligned}$$

Therefore, the distance between two vertices u, v of O_k is determined by $\ell := |u \cap v|$ through the formula

$$\partial(u,v) = \min\{\partial^-(u,v), \partial^+(u,v)\} = \begin{cases} 2\ell + 1 & (\ell \leqslant \lfloor k/2 \rfloor - 1), \\ 2(k-1-\ell) & (\ell \geqslant \lfloor k/2 \rfloor). \end{cases}$$

In particular, u and v are at maximum distance when $\ell = \lfloor (k-1)/2 \rfloor$ and hence the diameter of O_k is $D = k-1$.

P2. The girth $g \equiv g(O_k)$ of the odd graph is $g = 3$ if $k = 2$, $g = 5$ if $k = 3$, and $g = 6$ if $k \geqslant 4$.

P3. The distinct eigenvalues of O_k are $\lambda_i = (-1)^i(k-i)$, with multiplicities

$$m(\lambda_i) = \binom{2k-1}{i} - \binom{2k-1}{i-1} = \frac{k-i}{k}\binom{2k}{i} \qquad (0 \leqslant i \leqslant k-1).$$

P4. The automorphism group of O_k, denoted by $\operatorname{aut} O_k$, is (isomorphic to) the symmetric group $\operatorname{Sym}(\Omega) \cong S_{2k-1}$.

Let $\Omega := \{1, 2, 3, \ldots, 2k-1\}$ and consider the involution of Ω, $\sigma := (12)(34)\cdots(2s{-}1\ 2s)$, where $|\sigma| := s < k-1$. If $X \subset \Omega$ we write $\sigma X := \{\sigma x, x \in X\}$ (we shall use the letters x, y, z to denote generic elements of Ω) and $\overline{X} := \Omega \setminus X$. We now define the *twisted odd graph*, denoted by O_k^σ or O_k^s, as the graph with vertices those of O_k and where two vertices u and v are adjacent in O_k^σ whenever the vertices u and σv (or, equivalently, σu and v) are adjacent in O_k. The twisted odd graph O_3^σ when $\sigma = (12)$, derived from the Petersen graph $P = O_3$, is shown in Figure 1.

The graph O_k^σ can also be seen as the quotient graph of the double cover of O_k by a polarity. In the double cover DO_k (see [5] or [7]), each vertex u of O_k gives rise to two

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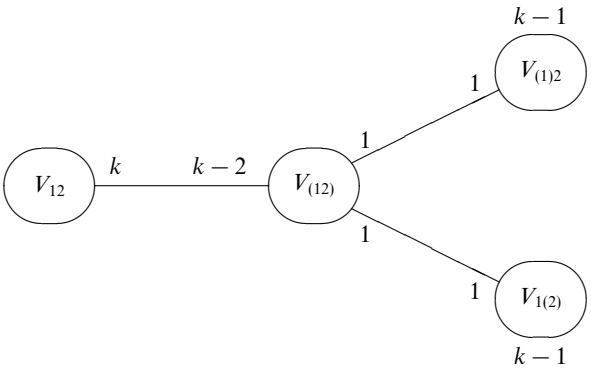


Figure 2 Decomposition of O_k^σ

vertices u_0 and u_1 of DO_k, and each edge uv of O_k gives rise to two edges u_0v_1 and u_1v_0 of DO_k. This is a bipartite distance-regular 2-antipodal graph with distinct eigenvalues $\{\pm k, \pm(k-1), \pm(k-2), \ldots, \pm 1\}$ and automorphism group $\mathrm{Sym}(\Omega) \times \mathbb{Z}_2$. The graph O_k^σ is then obtained from DO_k through the polarity ρ defined by $\rho u_0 := (\sigma u)_1$ and $\rho u_1 := (\sigma u)_0$.

Let $\boldsymbol{A}$ be the adjacency matrix of O_k, and let $\boldsymbol{\sigma}$ represent the permutation matrix associated to σ acting on the vertices of O_k. Then, since σ is an involutive automorphism we have $\boldsymbol{\sigma}^2 = \boldsymbol{I}$ and $\boldsymbol{\sigma A} = \boldsymbol{A\sigma}$. Moreover, if $\boldsymbol{A}_\sigma$ denotes the adjacency matrix of O_k^σ, it is readily checked that $\boldsymbol{A}_\sigma = \boldsymbol{\sigma A}$.

In order to derive some properties of the twisted odd graphs, it is useful to consider the following partition of its vertex set. Let us single out the first transposition by writing $\sigma = (12)\cdots(2s-1, 2s) = (12)\tau$, and decompose the set of vertices of O_k^σ as $V = V_{12} \cup V_{(12)} \cup V_{1(2)} \cup V_{(1)2}$, where V_{12} is the set of vertices with 1 and 2, $V_{(12)}$ is the set of vertices without 1 or 2, $V_{1(2)}$ is the set of vertices with 1 but not 2, and $V_{(1)2}$ is the set of vertices with 2 but not 1. Notice that all vertices of a given set have the same number of adjacent vertices belonging to the other sets, as the diagram of Figure 2 illustrates (compare with the drawing of O_3^σ in Figure 1). For instance, each vertex of $V_{(12)}$ is adjacent to $k-2$ vertices of V_{12}, one vertex of $V_{(1)2}$, and one vertex of $V_{1(2)}$. In other words, such a decomposition is a *regular* (or *equitable*) *partition* with the depicted *distribution diagram* (also called *quotient graph*).

The adjacency matrix $\boldsymbol{B}$ of this quotient (di)graph is referred to as the *quotient matrix* of the partition. In our case, we have

$$\boldsymbol{B} = \begin{pmatrix} 0 & k & 0 & 0 \\ k-2 & 0 & 1 & 1 \\ 0 & 1 & k-1 & 0 \\ 0 & 1 & 0 & k-1 \end{pmatrix}. \tag{2.1}$$

Some basic results about regular partitions can be found in Godsil [17] or Brouwer, Cohen and Neumaier [7]. As some simple consequences of our regular partition and such general results, we can derive, for instance, the following properties of the twisted odd graphs.

Q1. The set $V_{(12)}$ is a maximal independent set (and hence a vertex-dominating set) of O_{k-1}^σ, with cardinality $\binom{2k-3}{k-1}$.

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Q2. Since the induced subgraphs in $V_{1(2)}$ and $V_{(1)2}$ are isomorphic to O_{k-1}^{τ} through $\{1, i_1, \ldots, i_{k-2}\} \mapsto \{i_1, \ldots, i_{k-2}\}$ and $\{2, i_1, \ldots, i_{k-2}\} \mapsto \{i_1, \ldots, i_{k-2}\}$, we conclude, reasoning by induction, that the twisted odd graph O_k^s contains, as disjoint induced subgraphs, 2^t copies of O_{k-t}^{s-t} for any $0 \leqslant t \leqslant s$. In particular, O_k^s contains 2^s disjoint induced subgraphs isomorphic to the odd graph O_{k-s}.

Q3. Since all eigenvalues of $\boldsymbol{B}$ are also eigenvalues of $\boldsymbol{A}_\sigma$ (see Godsil [17]), we have that $\operatorname{ev} \boldsymbol{B} = \{k, k-1, k-2, -k+1\} \subset \operatorname{ev} O_k^\sigma$. A more precise analysis of the spectrum of O_k^σ, also using the above regular partition, is carried out in Section 4.

Q4. Let $\tau \in S_{2k-1}$ be a permutation of Ω which commutes with σ. Then, since $\boldsymbol{\tau A}_\sigma = \boldsymbol{\tau\sigma A} = \boldsymbol{A\sigma\tau} = \boldsymbol{A}_\sigma \boldsymbol{\tau}$, we have that τ is an automorphism of O_k^σ and hence we conclude that $\operatorname{aut} O_k^s$ has a subgroup isomorphic to $\mathbb{Z}_2^s \times S_s \times S_{2(k-s)-1}$, where $\mathbb{Z}_2^s := \mathbb{Z}_2 \times \overset{s}{\cdots} \times \mathbb{Z}_2$. (In fact, we conjecture that this is the whole group of automorphisms of O_k^s. As one referee noted, this could be worth investigating in the light of the proof of **P4**, where the Erdős–Ko–Rado theorem is used; see [4].)

3. Some metric properties

In this section we study some of the metric parameters of the twisted odd graphs, such as the radius, diameter, girth, *etc*. To begin with, note that, since $\boldsymbol{A}_\sigma^k = (\boldsymbol{\sigma A})^k = \boldsymbol{A}^k$ for even k, and $\boldsymbol{A}_\sigma^k = \boldsymbol{\sigma A}^k$ for odd k, we get

$$(\boldsymbol{A}_\sigma^k)_{uv} = \begin{cases} (\boldsymbol{A}^k)_{uv} & (k \text{ even}), \\ (\boldsymbol{A}^k)_{(\sigma u)v} & (k \text{ odd}). \end{cases} \tag{3.1}$$

Therefore, the distance ∂_σ between two vertices of O_k^σ can be given in terms of the 'parity distances' ∂^+ and ∂^- in O_k as

$$\partial_\sigma(u, v) = \min\{\partial^+(u, v), \partial^-(\sigma u, v)\}. \tag{3.2}$$

Let us see some simple consequences of this expression. First, if two vertices u, v of O_k are at even distance, then their distance in O_k^σ satisfies $\partial_\sigma(u, v) \leqslant \partial(u, v)$. Moreover, if σ fixes u, $\sigma u = u$, formula (3.2) becomes $\partial_\sigma(u, v) = \partial(u, v)$. Hence, since σ always fixes some vertices of O_k and the radius of O_k equals its diameter, $r = D = k - 1$, we have proved the following result.

Lemma 3.1. *The twisted odd graph O_k^σ has radius $r_\sigma = k - 1$.* □

We next compute the diameter of O_k^σ.

Theorem 3.2.

(a) *Let O_k^σ be the twisted odd graph with $|\sigma| = s$, and let $S := \{1, 2, \ldots 2s\}$. Consider two generic vertices u, v such that $u = u_1 \cup u_2$ and $v = v_1 \cup v_2$, with $u_1, v_1 \subset S$ and $u_2, v_2 \subset \overline{S}$. Then,*

$$\partial_\sigma(u, v) = \min\{2(k - 1 - |u_1 \cap v_1| - |u_2 \cap v_2|), 2(|\sigma u_1 \cap v_1| + |u_2 \cap v_2|) + 1\} \tag{3.3}$$

where ∂_σ denotes distance in O_k^σ.

(b) *The twisted odd graph O_k^σ has diameter $D_\sigma = k - 1 + s$.*

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Proof. From the decomposition of the vertices u and v, we trivially have $|u \cap v| = |u_1 \cap v_1| + |u_2 \cap v_2|$ and $|\sigma u \cap v| = |\sigma u_1 \cap v_1| + |u_2 \cap v_2|$. Then by (3.1) and Property **P1** we have the following even and odd distances:

$$\partial^+_\sigma(u,v) = \partial^+(u,v) = 2(k-1-|u_1 \cap v_1| - |u_2 \cap v_2|), \tag{3.4}$$

$$\partial^-_\sigma(u,v) = \partial^-(\sigma u,v) = 2(|\sigma u_1 \cap v_1| + |u_2 \cap v_2|) + 1, \tag{3.5}$$

and (3.3) follows from (3.2). Moreover, by adding the above two equalities, we get

$$\partial^+_\sigma(u,v) + \partial^-_\sigma(u,v) = 2(k + |\sigma u_1 \cap v_1| - |u_1 \cap v_1|) - 1.$$

But, decomposing u_1 and v_1 in terms of their subsets $u_{1i}, v_{1i} \subseteq \{2i-1, 2i\}$, $1 \leqslant i \leqslant s$, we realize that

$$|\sigma u_1 \cap v_1| - |u_1 \cap v_1| = \sum_{i=1}^{s} \left(|\sigma u_{1i} \cap v_{1i}| - |u_{1i} \cap v_{1i}|\right) \leqslant s,$$

with equality if and only if u_1 and v_1 constitute an equipartition of S (that is, $|u_1| = |v_1| = s$) satisfying $\sigma u_1 = v_1$. Consequently,

$$\partial^+_\sigma(u,v) + \partial^-_\sigma(u,v) \leqslant 2k + 2s - 1$$

and at least one of the distances, $\partial^+_\sigma(u,v)$ or $\partial^-_\sigma(u,v)$, must be not greater than $k+s-1$, which proves that $D \leqslant k+s-1$. Finally, to show that equality is attained, we only need to exhibit some pairs of vertices at maximum distance. Now, let $u = u_1 \cup u_2$ such that $|u_1| = s$ and $u_1 \cup \sigma u_1 = S$, and take $v = \sigma u_1 \cup v_2$, with $|u_2 \cap v_2| = (k-1-s)/2$ if k and s have different parity, and $|u_2 \cap v_2| = (k-s)/2 - 1$ otherwise. Then, in the first case we get $\partial^+_\sigma(u,v) = \partial^-_\sigma(u,v) - 1 = k+s-1$ and, in the second case, $\partial^+_\sigma(u,v) - 1 = \partial^-_\sigma(u,v) = k+s-1$. This completes the proof of the theorem. □

Notice that the above u_2 and v_2 can be seen as vertices of O_{k-s}, constructed now from the set $\Omega = \{2s+1, 2s+2, \ldots, 2k-1\}$. Moreover, as $|u_2 \cap v_2| = \lfloor (k-s-1)/2 \rfloor$, Property **P1** states that they are vertices at maximum distance $k-s-1$. This fact is used in [14] to derive an inductive proof (on k) of $D_\sigma = k+s-1$.

Let us now study the girth of our graphs which, by Property **Q2**, clearly satisfies $g(O^s_k) \leqslant g(O_{k-s})$. Thus, Property **P2** yields explicit upper bounds for $g(O^s_k)$. In order to show that equalities are attained we require, however, a more detailed analysis.

Lemma 3.3. *The girth of the twisted odd graphs is* $g(O^s_k) = 3$ *if* $k-s = 2$, $g(O^s_k) = 5$ *if* $k-s = 3$, *and* $g(O^s_k) = 6$ *if* $k-s \geqslant 4$.

Proof. In the case $k-s = 2$ there is nothing to prove. For the other cases, note first that the girth cannot be 4 since, otherwise, there would be two different paths of length 2 between two vertices of O^s_k and hence, by (3.1), the same would be true in O_k, a contradiction. This completes the proof of case $k-s = 3$ since, according to the quotient graph of Figure 2, any 3-cycle must have all its vertices in either $V_{1(2)}$ or $V_{(1)2}$, and then induction would lead to the contradiction $g(O_3) = 3$. Finally, suppose that in O^s_k, $k-s \geqslant 4$, there is a 5-cycle C_5. Again using induction, we can suppose that it is not contained in

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$V_{1(2)}$ nor in $V_{(1)2}$. Then, the only possibility is that a vertex $w \in V_{12}$ be at distance two of two adjacent vertices $u, v \in V_{1(2)}$ (or $u, v \in V_{(1)2}$). Using the same notation as in the proof of Theorem 3.2, let $u = u_1 \cup u_2$, $v = v_1 \cup v_2$, and $w = w_1 \cup w_2$. Since $u \sim v$, it must be $\sigma u_1 \cap v_1 = u_2 \cap v_2 = \emptyset$ and hence either $|\sigma u_1| = |u_1| \leqslant s$ or $|v_1| \leqslant s$. Moreover, the existence of the two 2-paths between w and u, v leads to the structures $u = (w \setminus \{2\}) \cup \{x\}$ and $v = (w \setminus \{2\}) \cup \{y\}$ for some $x, y \in \Omega \setminus \{2\}$, $x \neq y$. Therefore $|u \cap v| = k - 2$ and hence $k - 2 = |u_1 \cap v_1| \leqslant s$. (Note that this proves that, even in the case $k - s = 3$, the 5-cycles are in $V_{1(2)}$ or $V_{(1)2}$, that is, by induction, in the subgraphs isomorphic to the Petersen graph.) □

4. The spectrum

The next result determines the eigenvalues of the twisted odd graphs O_k^σ, as well as a recursive formula giving their multiplicities.

Theorem 4.1. *The twisted odd graph O_k^σ, with $|\sigma| = s$, has eigenvalues*

$$\operatorname{ev} O_k^\sigma = \{k, \pm(k-1), \ldots, \pm 2, \pm 1\}, \tag{4.1}$$

with multiplicities $m_k^s(\lambda)$ satisfying the following recurrence formula

$$m_k^s(\lambda) = m_{k-1}^{s-1}(\lambda+1) + m_{k-1}^{s-1}(\lambda) + m_{k-1}^{s-1}(\lambda-1) + m_{k-1}^{s-1}(-\lambda) \qquad (k \geqslant 3, s < k-1). \tag{4.2}$$

(*Of course, $m_k^s(\lambda) := 0$ when λ is not an eigenvalue of O_k^σ.*)

Proof. Note first that, since $\boldsymbol{A}_\sigma^2 = \boldsymbol{A}^2$, for every eigenvalue λ of O_k either λ or $-\lambda$ must be an eigenvalue of O_k^σ. That is, the set of eigenvalues of O_k^σ is a subset of $\pm \operatorname{ev} O_k = \{\pm k, \pm(k-1), \pm(k-2). \ldots, \pm 1\}$. Moreover, the multiplicities satisfy $m_k^s(\lambda) + m_k^s(-\lambda) = m_k(\lambda)$, where $m_k(\cdot)$ denotes the multiplicity in O_k.

Consider again the regular partition $V = V_{12} \cup V_{(12)} \cup V_{1(2)} \cup V_{(1)2}$. Now, in (the vertices of) $V_{(12)}$ we set the lexicographic order and carry it over to both $V_{1(2)}$ and $V_{(1)2}$ (recall that each vertex of these sets is adjacent to just one vertex of $V_{(12)}$). Then, the adjacency matrix of O_k^σ may be written as

$$\boldsymbol{A}_\sigma = \begin{pmatrix} 0 & \boldsymbol{C} & 0 & 0 \\ \boldsymbol{C}^\top & 0 & \boldsymbol{I} & \boldsymbol{I} \\ 0 & \boldsymbol{I} & \boldsymbol{B}_\tau & 0 \\ 0 & \boldsymbol{I} & 0 & \boldsymbol{B}_\tau \end{pmatrix},$$

where $\boldsymbol{B}_\tau$ is the adjacency matrix of O_{k-1}^τ (to be compared with the quotient matrix in (2.1)). Now we claim that the above (sub)matrices $\boldsymbol{B}_\tau$ and $\boldsymbol{C}$ satisfy

$$\boldsymbol{B}_\tau^2 = \boldsymbol{C}^\top \boldsymbol{C} + \boldsymbol{I}. \tag{4.3}$$

Indeed, note first that $(\boldsymbol{C}^\top \boldsymbol{C})_{uv}$ is the number of walks of length 2 between vertices u and v of $V_{(12)}$ in the (bipartite) subgraph G induced in $V_{12} \cup V_{(12)}$. All (u,u)-entries of $\boldsymbol{C}^\top \boldsymbol{C}$ are equal to $k-2$ and all (u,u)-entries of B_τ^2 are equal to $k-1$, so that the result holds for the main diagonals of the matrices. Consider now two vertices $u, v \in V_{(12)}$, $u \neq v$. The

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only paths of length 2 between u and v should be in G and their number is the number of $(k-1)$-subsets in $\overline{\sigma u \cup \sigma v}$. But, since $u \neq v$, we have $|u \cap v| \leqslant k-2$ and hence

$$|\overline{\sigma u \cup \sigma v}| = |\overline{\sigma(u \cup v)}| = 2k - 1 - |u \cup v| = |u \cap v| + 1 \leqslant k-1,$$

so that we necessarily have

$$(\boldsymbol{C}^\top \boldsymbol{C})_{uv} = \begin{cases} 1 & \text{if} \quad |u \cap v| = k-2, \\ 0 & \text{if} \quad |u \cap v| < k-2, \end{cases} \qquad (u \neq v).$$

Now, let u' and v' be the vertices of $V_{1(2)}$ adjacent to u and v, respectively. Then, $u' = \{1\} \cup u_1$ where $u_1 = \overline{\sigma u} \setminus \{1,2\}$, and analogously $v' = \{1\} \cup v_1$ with $v_1 = \overline{\sigma v} \setminus \{1,2\}$. Considering u_1, v_1 as vertices of O_{k-1}^τ and reasoning as before, we have

$$(\boldsymbol{B}_\tau^2)_{u_1 v_1} = \begin{cases} 1, & \text{if} \quad |u_1 \cap v_1| = k-3, \\ 0, & \text{if} \quad |u_1 \cap v_1| < k-3. \end{cases}$$

Now, (4.3) follows from the equivalences

$$|u \cap v| = k-2 \iff |\overline{u_1} \cap \overline{v_1}| = k \iff |u_1 \cup v_1| = k-1 \iff |u_1 \cap v_1| = k-3.$$

We next aim at the computation of the recursive formula for the multiplicities of the eigenvalues of O_k^σ. If $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$ is an eigenvector of O_k^σ corresponding to the eigenvalue λ, from the equation $\boldsymbol{A}_\sigma \boldsymbol{v} = \lambda \boldsymbol{v}$ we obtain the system

$$\begin{array}{lrcl} \text{(a)} & \boldsymbol{C}\boldsymbol{v}_2 & = & \lambda \boldsymbol{v}_1, \\ \text{(b)} & \boldsymbol{C}^\top \boldsymbol{v}_1 + \boldsymbol{v}_3 + \boldsymbol{v}_4 & = & \lambda \boldsymbol{v}_2, \\ \text{(c)} & \boldsymbol{v}_2 + \boldsymbol{B}_\tau \boldsymbol{v}_3 & = & \lambda \boldsymbol{v}_3, \\ \text{(d)} & \boldsymbol{v}_2 + \boldsymbol{B}_\tau \boldsymbol{v}_4 & = & \lambda \boldsymbol{v}_4. \end{array}$$

Since $\lambda \neq 0$, from (a), (b) and (d), we have, in terms of $\boldsymbol{w} := \boldsymbol{v}_4$,

$$\begin{aligned} \boldsymbol{v}_1 &= -\frac{1}{\lambda} \boldsymbol{C}(\boldsymbol{B}_\tau - \lambda \boldsymbol{I})\boldsymbol{w}, \\ \boldsymbol{v}_2 &= -(\boldsymbol{B}_\tau - \lambda \boldsymbol{I})\boldsymbol{w}, \\ \boldsymbol{v}_3 &= \frac{1}{\lambda} \left[\boldsymbol{C}^\top \boldsymbol{C}(\boldsymbol{B}_\tau - \lambda \boldsymbol{I}) - \lambda^2 (\boldsymbol{B}_\tau - \lambda \boldsymbol{I}) - \lambda \boldsymbol{I}\right] \boldsymbol{w}, \end{aligned}$$

and equation (c) becomes

$$(\boldsymbol{B}_\tau - \lambda \boldsymbol{I}) \left[(\boldsymbol{C}^\top \boldsymbol{C} - \lambda^2 \boldsymbol{I})(\boldsymbol{B}_\tau - \lambda \boldsymbol{I}) - 2\lambda \boldsymbol{I}\right] \boldsymbol{w} = \boldsymbol{0}.$$

Since $\boldsymbol{C}^\top \boldsymbol{C} = \boldsymbol{B}_\tau^2 - \boldsymbol{I}$ from (4.3), by factorizing we obtain

$$(\boldsymbol{B}_\tau - (\lambda+1)\boldsymbol{I})(\boldsymbol{B}_\tau - \lambda \boldsymbol{I})(\boldsymbol{B}_\tau - (\lambda-1)\boldsymbol{I})(\boldsymbol{B}_\tau + \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{0}.$$

The recursion in (4.2) now follows by noting that the required multiplicity is just the sum of the dimensions of the eigenspaces involved.

Finally, it is easily checked, by induction, that $m_k^s(\pm 1)$, …, $m_k^s(\pm(k-1))$ are always positive (but $m_k^s(-k) = 0$, which agrees with O_k^σ being non-bipartite). Hence $\operatorname{ev} O_k^\sigma$ is as claimed and we conclude the proof of the theorem. □

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Now the following remarks are in order.

R1. The recurrence relation (4.2) allows the computation of the multiplicities $m_k^s(\lambda)$ of the eigenvalues of $O_k^s \equiv O_k^\sigma$, from the multiplicities of the eigenvalues of O_{k-s}, as the following example shows:

G						sp G				
O_2				2^1			-1^2			
O_3^1			3^1	2^1	1^3		-1^2	-2^3		
O_4^2		4^1	3^2	2^8	1^6		-1^8	-2^6	-3^4	
O_5^3	5^1	4^3	3^{15}	2^{22}	1^{22}		-1^{20}	-2^{26}	-3^{12}	-4^5.
⋮						⋮				

R2. As a by-product, we can also obtain a recurrence for the multiplicities $m_k(\lambda)$ of the eigenvalues of the odd graphs. Indeed, it suffices to apply (4.2) to both λ and $-\lambda$, and add up to obtain $m_k(\lambda) = m_k^s(\lambda) + m_k^s(-\lambda)$. Besides, taking into account that, if O_k has the eigenvalue λ, then O_{k-1} does not have the eigenvalues $-(\lambda+1)$, λ, and $-(\lambda-1)$, we finally get

$$m_k(\lambda) = m_{k-1}(\lambda+1) + 2m_{k-1}(-\lambda) + m_{k-1}(\lambda-1) \qquad (k \geqslant 3). \tag{4.4}$$

5. Twisted odd graphs as boundary graphs

Recently, some results bounding the diameter D of a (regular) graph G from its distinct eigenvalues $\operatorname{ev} G = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$ have been given by Haemers and Van Dam [8] and the authors [10]. Roughly speaking, such results state that, if the value at λ_0 of a polynomial P is large enough, then the diameter D is at most $k := \operatorname{dgr} P$. In particular, optimal results are obtained when P is chosen to be the so-called 'k-alternating polynomial' which, in the case of degree $k = d-1$, is characterized by $P(\lambda_i) = (-1)^{i+1}$, $1 \leqslant i \leqslant d$, and satisfies $P(\lambda_0) = \sum_{i=1}^d (\pi_0/\pi_i)$, where $\pi_i := \prod_{j=0, j\neq i}^d |\lambda_i - \lambda_j|$ (see [10] for more details). For such a polynomial, which from now on will be referred to as the alternating polynomial, the above-mentioned results have the form

$$P(\lambda_0) > n-1 \quad \Longrightarrow \quad D \leqslant d-1,$$

where n is the order of G. *Boundary graphs*, studied by the authors in some previous papers [11, 12], are those graphs satisfying equality in the above left-hand side expression and so they are characterized by

$$P(\lambda_0) + 1 = \sum_{i=0}^{d} \frac{\pi_0}{\pi_i} = n. \tag{5.1}$$

In [11], the authors showed that those extremal ($D = d$) boundary graphs where each vertex have maximum eccentricity are, in fact, 2-antipodal distance-regular graphs. This

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is the case, for instance, of the above-mentioned double covers DO_k, which have the antipodal pairs of vertices (u_0, u_1). In the case $D < d$, however, the boundary graphs seem to have a much more involved structure and, up to now, a complete characterization has eluded our efforts. Fortunately, the known examples belonging to these 'degenerate' cases still show some kind of 'algebraic antipodality'. This is made apparent by the the presence of some distinguished pairs of (not necessarily different) vertices u, v called 'conjugate', which are characterized by annihilating the (u, v)-entry of the matrix $P(\boldsymbol{A})$ (just as the antipodal vertices trivially do in the case $D = d$; recall that $\operatorname{dgr} P = d - 1$). As was shown in [13], conjugate vertices seem to be crucial to settling the whole structure of boundary graphs in general. In this section we will use the previously derived properties of the twisted boundary graphs to show that they satisfy (5.1) and, consequently, constitute examples of non-extremal boundary graphs. Moreover, their pairs of conjugate vertices will be shown to derive, through a polarity, from the antipodal pairs of DO_k.

We begin with two results which are interesting by themselves in the study of boundary graphs. To state the first one, recall that the Hoffman polynomial H of a regular graph G with n vertices and $\operatorname{ev} G = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$ is defined by $H(\lambda_i) = 0$, $1 \leqslant i \leqslant d$, and $H(\lambda_0) = n/\pi_0$, and satisfies $H(\boldsymbol{A}) = \boldsymbol{J}$ (the all-1 matrix). See [19].

Lemma 5.1. *Let G be a bipartite regular graph with eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d = -\lambda_0$, and assume that it has a polarity ρ such that the quotient graph of G by ρ, G^*, has eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_{d-1}$. Then*

(a) *G^* is a boundary graph if and only if G is.*

In this case, we have the following.

(b) *If P and H are the alternating and Hoffman polynomials of G, the alternating and Hoffman polynomials of G^* are given by*

$$P^*(x) = \frac{P(x) + (-1)^d P(-x)}{2} = \begin{cases} P^+(x), & \text{if } d \text{ is even,} \\ P^-(x), & \text{if } d \text{ is odd;} \end{cases}$$

$$H^*(x) = \frac{P(x) + (-1)^{d-1} P(-x)}{2} = \begin{cases} P^-(x), & \text{if } d \text{ is even,} \\ P^+(x), & \text{if } d \text{ is odd.} \end{cases}$$

(c) *The vertices $\{u, \rho u\}$ and $\{v, \rho v\}$ are conjugate in G^* if and only if the vertices u and v are conjugate in G.*

Proof. In proving (a) and (b) we will only deal with the case of even d, since the case of odd d is analogous. Let n and $n^* = n/2$ be the orders of G and G^*, respectively. If G is a boundary graph with alternating polynomial P, the polynomial P^+ has degree $(d-1)-1$ and satisfies

$$P^+(\lambda_0) = \frac{P(\lambda_0) + P(-\lambda_0)}{2} = \frac{(n-1)-1}{2} = n^* - 1,$$

$$P^+(\lambda_i) = \frac{P(\lambda_i) + P(-\lambda_i)}{2} = P(\lambda_i) = (-1)^{i+1}, \quad (1 \leqslant i \leqslant d-1).$$

Therefore G^* is a boundary graph with alternating polynomial $P^* = P^+$. Moreover, P^-

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is a polynomial of degree $d-1$ which satisfies

$$\begin{aligned} P^-(\lambda_0) &= \frac{P(\lambda_0)-P(-\lambda_0)}{2} = \frac{(n-1)+1}{2} = n^*, \\ P^-(\lambda_i) &= \frac{P(\lambda_i)-P(-\lambda_i)}{2} = 0, \quad (1 \leqslant i \leqslant d-1), \end{aligned}$$

so that P^- is the Hoffman polynomial of G^*.

Reciprocally, if G^* is a boundary graph with alternating and Hoffman polynomials P^* and H^*, respectively, the polynomial $P = P^* + H^*$ has degree $d-1$ and satisfies

$$\begin{aligned} P(\lambda_0) &= P^*(\lambda_0) + H^*(\lambda_0) = (n^*-1) + n^* = n-1, \\ P(\lambda_i) &= P^*(\lambda_i) + H^*(\lambda_i) = (-1)^{i+1} \qquad (1 \leqslant i \leqslant d-1), \\ P(\lambda_d) &= P^*(\lambda_d) + H^*(\lambda_d) = (n^*-1) - n^* = -1, \end{aligned}$$

so that G is a boundary graph with alternating polynomial P.

Let us now turn our attention to (c) and assume now that d is odd (as above, the other case is proved similarly). Let us see G as the double cover of G^*, where each vertex $u^* := \{u, \rho u\}$ of G^* gives rise to the vertices u_0^* and u_1^* of G (thus, $\{u, \rho u\} = \{u_0^*, u_1^*\}$). Then, if $\boldsymbol{A}$ is the adjacency matrix of G^* with rows $u^*, v^*, \ldots$ its double cover $DG^* \cong G$ has adjacency matrix

$$\boldsymbol{B} = \left(\begin{array}{c|c} \boldsymbol{O} & \boldsymbol{A} \\ \hline \boldsymbol{A} & \boldsymbol{O} \end{array}\right),$$

with rows $u_0^*, v_0^*, \ldots, u_1^*, v_1^*, \ldots$ Then, by using the results of (b), we get

$$P(\boldsymbol{B}) = \left(\begin{array}{c|c} P^+(\boldsymbol{A}) & P^-(\boldsymbol{A}) \\ \hline P^-(\boldsymbol{A}) & P^+(\boldsymbol{A}) \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{J} & P^*(\boldsymbol{A}) \\ \hline P^*(\boldsymbol{A}) & \boldsymbol{J} \end{array}\right).$$

Hence, any pair u, v of conjugate vertices of G, satisfying $(P(\boldsymbol{B}))_{uv} = 0$, must necessarily belong to different partite sets and hence, seen as vertices of DG^*, they correspond to u_0^*, v_1^* or u_1^*, v_0^*. With this notation, the above matrix equation gives

$$(P(\boldsymbol{B}))_{u_0^* v_1^*} = (P(\boldsymbol{B}))_{u_1^* v_0^*} = 0 \iff (P^*(\boldsymbol{A}))_{u^* v^*} = 0$$

and the result follows. □

When applying these results to the twisted odd graphs, the following simple lemma will provide an alternative proof of their boundary character.

Lemma 5.2. *Let G be a regular connected graph of order n.*

(a) *If its distinct eigenvalues are $k, \pm(k-1), \pm(k-2), \ldots, \pm 1$, G is a boundary graph if and only if $n = \binom{2k-1}{k-1}$.*

(b) *If its distinct eigenvalues are $\pm k, \pm(k-1), \pm(k-2), \ldots, \pm 1$, G is a boundary graph if and only if $n = 2\binom{2k-1}{k-1} = \binom{2k}{k}$.*

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Proof. Since both results are analogous, we only prove (a). Let the $(2k-1)$ eigenvalues of G be indexed as $\mu_0 = k$, $\mu_i = i$, $i = \pm 1, \pm 2, \ldots, \pm(k-1)$. Then

$$\pi_0 = \frac{(2k-1)!}{k} \quad \text{and} \quad \pi_i = \frac{(k-i)!(k+i-1)!}{i}, \quad \pi_{-i} = \frac{(k+i)!(k-i-1)!}{i},$$

for $i = 1, 2, \ldots, k-1$, so that

$$\frac{\pi_0}{\pi_i} = \frac{i}{k}\binom{2k-1}{k-i}, \qquad \frac{\pi_0}{\pi_{-i}} = \frac{i}{k}\binom{2k-1}{k+i} = \frac{i}{k}\binom{2k-1}{k-i-i},$$

and therefore

$$\sum_{i=1}^{k-1}\left(\frac{\pi_0}{\pi_i} + \frac{\pi_0}{\pi_{-i}}\right) = \sum_{i=1}^{k-1}\frac{i}{k}\binom{2k}{k-i} = \sum_{i=1}^{k-1}\frac{k-i}{k}\binom{2k}{i} = \sum_{i=1}^{k-1}\binom{2k}{i} - \sum_{i=1}^{k-1}\frac{i}{k}\binom{2k}{i}.$$

But

$$\begin{aligned}
\sum_{i=1}^{k-1}\binom{2k}{i} &= \frac{1}{2}\left(2^{2k} - \binom{2k}{k}\right) - 1 = 2^{2k-1} - \binom{2k-1}{k-1} - 1, \\
\sum_{i=1}^{k-1}\frac{i}{k}\binom{2k}{i} &= \sum_{i=1}^{k-1}\frac{i}{k}\frac{(2k)!}{i!(2k-i)!} = 2\sum_{i=1}^{k-1}\binom{2k-1}{i-1} \\
&= 2\sum_{i=0}^{k-2}\binom{2k-1}{i} = 2^{2k-1} - 2\binom{2k-1}{k-1},
\end{aligned}$$

so that we finally get

$$1 + \sum_{i=1}^{k-1}\left(\frac{\pi_0}{\pi_i} + \frac{\pi_0}{\pi_{-i}}\right) = \binom{2k-1}{k-1},$$

and therefore G is a boundary graph if and only if $n = \binom{2k-1}{k-1}$. □

As a consequence of the above results we now have the following result.

Theorem 5.3. *The twisted odd graph O_k^σ is a (non-extremal) boundary graph, with pairs of conjugate vertices $(u, \sigma u)$.*

Proof. The first part of the statement follows from either Theorem 4.1 and Lemma 5.2(a), or Lemmas 5.1(a) and 5.2(b). (In fact, in the latter approach we have already seen that the double cover DO_k is a 2-antipodal distance-regular graph and, hence, a boundary graph.) To prove the second part recall that the antipodal pairs of vertices of DO_k are (u_0, u_1) (coming from vertex u of O_k). Since this graph is extremal such pairs are also conjugate. Therefore, we can apply Lemma 5.1(c) to infer that the corresponding conjugate vertices of $O_k^\sigma \cong DO_k/\rho$ (where $\rho u_0 = (\sigma u)_1$ and $\rho u_1 = (\sigma u)_0$) are

$$\{u_0, \rho u_0\} = \{u_0, (\sigma u)_1\} \quad \text{and} \quad \{u_1, \rho u_1\} = \{u_1, (\sigma u)_0\},$$

which correspond to the vertices u and σu of O_k^σ, as claimed. □

By way of example, the graph $O_3^{(12)}$ of Figure 1 has the pairs of conjugate vertices

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(12, 12), (34, 34), (35, 35), (45, 45), formed by 'selfconjugate' vertices, and (13, 23), (14, 24), and (15, 25).

As another simple consequence of the previous results, notice that, given any rational number $p/q \in (\frac{1}{2}, 1)$, the twisted odd graph O_k^σ with $k = q + 1$ and $s = |\sigma| = 2p - q$ satisfies

$$\frac{D}{d} = \frac{k + s - 1}{2(k - 1)} = \frac{p}{q}.$$

Thus, we see that there are boundary graphs with any rational ratio $\frac{1}{2} < \frac{D}{d} < 1$ (as was also evidenced by some other examples given in [13]). This suggests ending the paper with the following question: Is the diameter D of a boundary graph always greater than $d/2$?

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