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RETRACTIONS OF REVERSIBLE STRUCTURES

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Abstract. A relational structure is called reversible iff each bijective endomorphism (condensation) of that structure is an automorphism. We show that reversibility is an invariant of some forms of $L_{\infty\omega}$−bi-interpretability, implying that the condensation monoids of structures are topologically isomorphic. Applying these results, we prove that, in particular, all orbits of ultrahomogeneous tournaments and reversible ultrahomogeneous m-uniform hypergraphs are reversible relations and that the same holds for the orbits of reversible ultrahomogeneous digraphs definable by formulas which are not R-negative.

§1. Introduction. A structure is called reversible iff each bijective endomorphism of that structure is an automorphism and the relevance of investigation of that phenomenon is provided by the fact that several distinguished structures are reversible. For example, the class of reversible structures includes compact Hausdorff and many other prominent topological spaces [4, 19], tournaments, linear orders and Boolean lattices [9], and Henson graphs [12] and digraphs (see Example 4.4). In addition, classes of reversible structures have several nice properties; for example, the Cantor-Schröder-Bernstein property for condensations (bijective morphisms).

The first part of this article is concerned with the general question: which similarities of structures preserve reversibility? Namely, the reversibility of relational structures is an invariant of isomorphism and condensational equivalence, while it is not preserved under bi-embeddability and elementary equivalence [11, 12]. Here we show that reversibility is an invariant of some forms of bi-interpretability (although bi-definability does not preserve it in general, see Example 3.6).

The second part of the article deals with the task of detecting new (relevant) reversible structures. Some methods of detection were previously developed. First, extreme elements of definable classes of structures are reversible [12] (for example, the Henson graphs $\mathbb{H}_n$ (see [6]) are maximal $\mathbb{K}_n$-free graphs of infinite degree). Second, the reversibility of first-order definable relations in structures was considered in [10] and, in particular, all relations definable by the formulas of the empty language (in any structure), all monomorphic structures,[1] and all reducts of dense linear orders without end points are reversible (for example, the relations

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[1] An infinite relational structure $\mathbb{X}$ is *monomorphic* iff each two finite substructures of $\mathbb{X}$ of the same size are isomorphic, iff there is a linear order $<$ on its domain, X, such that the relations of $\mathbb{X}$ are $\emptyset$-definable in the structure $\langle X, <\rangle$ by quantifier-free formulas, see [5].

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characterizing the reducts of the rational line: the betweeness relation, the cyclic order and the separation relation, see [2]). Third, the reversibility of disconnected structures was characterized in [14] and, for example, an equivalence relation is reversible iff the number of equivalence classes of the same size is finite or all equivalence classes are finite and their sizes form a reversible sequence of integers. Here we show that, under some syntactical restrictions, retractions of reversible structures are reversible and apply these results. In particular we show that all orbits of reversible ultrahomogeneous tournaments and m-uniform hypergraphs are reversible relations and that the orbits of reversible ultrahomogeneous digraphs, which are not definable by R-negative formulas are reversible relations as well.

§2. **Preliminaries.** Let $L = \langle R_i : i \in I \rangle$ be a relational language, where $\mathrm{ar}(R_i) = n_i \in \mathbb{N}$, for $i \in I$. If X is a nonempty set and $\mathrm{Int}_L(X) = \prod_{i \in I} P(X^{n_i})$ the set of all interpretations of the language L over the domain X, then, abusing notation, for interpretations $\rho = \langle \rho_i : i \in I \rangle, \sigma = \langle \sigma_i : i \in I \rangle \in \mathrm{Int}_L(X)$ we will write: $\rho \subset \sigma$ iff $\rho_i \subset \sigma_i$, for all $i \in I$; $\rho \cong \sigma$ iff $\langle X, \rho \rangle \cong \langle X, \sigma \rangle$; and ρ^c will denote the interpretation $\langle X^{n_i} \setminus \rho_i : i \in I \rangle$. By $L_\emptyset$ we denote the empty language and by $\mathcal{D}_{L_\emptyset, X, \emptyset, n}$ the set of subsets of X^n which are $\emptyset$-definable (i.e., without parameters) by $L_\emptyset$-formulas.

An L-structure $\mathbb{X}$ is called *reversible* iff $\mathrm{Cond}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X})$, where $\mathrm{Cond}(\mathbb{X})$ denotes the set of all bijective endomorphisms (condensations) of $\mathbb{X}$. $\mathbb{X}$ is called *strongly reversible* iff $\mathrm{Aut}(\mathbb{X}) = \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the set of all permutations of the set X. An interpretation $\rho = \langle \rho_i : i \in I \rangle \in \mathrm{Int}_L(X)$ will be called reversible iff $\langle X, \rho \rangle$ is a reversible structure. By [13] we have

FACT 2.1. *Let $\mathbb{X} = \langle X, \rho \rangle$ be an L-structure, where $\rho = \langle \rho_i : i \in I \rangle$. Then*
(a) *$\mathbb{X}$ is strongly reversible iff $\rho_i \in \mathcal{D}_{L_\emptyset, X, \emptyset, n_i}$, for all $i \in I$;*
(b) *$\mathbb{X}$ is reversible iff $\sigma \ncong \rho$, for each $\sigma \subsetneq \rho$, iff $\sigma \ncong \rho$, for each $\sigma \supsetneq \rho$;*
(c) *ρ is reversible iff ρ^c is reversible.*

2.1. Infinitary languages. Let $\kappa \geq \omega$ be a cardinal, $\mathrm{Var} = \{v_\alpha : \alpha \in \kappa\}$ a set of variables and At_L the corresponding set of *atomic formulas*, that is, the formulas $v_\alpha = v_\beta$, for $\alpha, \beta \in \kappa$, and $R_i(v_{\alpha_1}, \ldots, v_{\alpha_{n_i}})$, where $i \in I$ and $\langle \alpha_1, \ldots, \alpha_{n_i} \rangle \in \kappa^{n_i}$. The class Form_L of *$L_{\infty\omega}$-formulas* is the smallest class containing all atomic formulas and closed under negation, finite quantification, and arbitrary conjunctions and disjunctions. So, if $\mathbb{X}$ is an L-structure and $\vec{x} = \langle x_\alpha : \alpha \in \kappa \rangle \in {}^\kappa X$ a valuation, then, for example, $\mathbb{X} \models (\bigvee \Phi)[\vec{x}]$ iff $\mathbb{X} \models \varphi[\vec{x}]$, for some $\varphi \in \Phi$, etc.

$L_{\infty\omega}$-formulas φ and ψ are *logically equivalent*, in notation $\varphi \leftrightarrow \psi$ iff for each L-structure $\mathbb{X}$ and each valuation $\vec{x} \in {}^\kappa X$ we have: $\mathbb{X} \models \varphi[\vec{x}]$ iff $\mathbb{X} \models \psi[\vec{x}]$. If $\mathbb{X}$ and $\mathbb{Y}$ are L-structures, a mapping $f : X \to Y$ *preserves an $L_{\infty\omega}$-formula φ* iff

$$\forall \vec{x} \in {}^\kappa X \quad \Big(\mathbb{X} \models \varphi[\vec{x}] \Rightarrow \mathbb{Y} \models \varphi[f\vec{x}] \Big), \tag{1}$$

where $f\vec{x} = \langle f(x_\alpha) : \alpha \in \kappa \rangle$ and φ is *absolute under f* iff we have "$\Leftrightarrow$" in (1).

The class of *R-positive $L_{\infty\omega}$-formulas*, $\mathcal{P}$ (resp. *R-negative $L_{\infty\omega}$-formulas*, $\mathcal{N}$) is the smallest class containing all atomic formulas and the formulas $\neg v_\alpha = v_\beta$ (resp. all negations of atomic formulas and the formulas $v_\alpha = v_\beta$) for $\alpha, \beta \in \kappa$, and which is closed under finite quantification and arbitrary conjunctions and disjunctions. In order to make a correspondence between the classes $\mathcal{P}$ and $\mathcal{N}$, by recursion, to each $L_{\infty\omega}$-formula φ we adjoin a formula $\varphi^\neg$ as follows:

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– $(v_\alpha = v_\beta)^\neg := \neg v_\alpha = v_\beta$ and $(R_i(v_{\alpha_1}, \dots, v_{\alpha_{n_i}}))^\neg := \neg R_i(v_{\alpha_1}, \dots, v_{\alpha_{n_i}})$;
– $(\neg\varphi)^\neg := \varphi$, $(\forall v_\alpha\ \varphi)^\neg := \exists v_\alpha\ \varphi^\neg$ and $(\exists v_\alpha\ \varphi)^\neg := \forall v_\alpha\ \varphi^\neg$, if $\varphi^\neg$ is defined;
– $(\bigwedge \Phi)^\neg := \bigvee \Phi^\neg$ and $(\bigvee \Phi)^\neg := \bigwedge \Phi^\neg$, where $\Phi^\neg := \{\varphi^\neg : \varphi \in \Phi\}$, if $\Phi \subset \mathrm{Form}_L$ and $\varphi^\neg$ is defined for all $\varphi \in \Phi$.

The following folklore statement is easily provable by induction.

FACT 2.2. (a) *All $L_{\infty\omega}$-formulas are absolute under isomorphisms*;
(b) *All formulas from the class $\mathcal{P}$ are preserved under condensations*;
(c) *For each $\varphi \in \mathcal{N}$ we have $\varphi^\neg \in \mathcal{P}$ and $\varphi^\neg \leftrightarrow \neg\varphi$.*

2.2. $L_{\infty\omega}$-interpretations. In the sequel Form_L denotes the set of all $L_{\infty\omega}$-formulas of a language L. Two relational structures $\mathbb{X}$ and $\mathbb{Y}$ of languages $L_\mathbb{X}$ and $L_\mathbb{Y}$ are called *bi-interpretable* (without parameters), in notation $\mathbb{X} \leftrightsquigarrow \mathbb{Y}$, iff conditions (A)–(D) are satisfied:

(A) there is a 0-interpretation $\Gamma_{\mathbb{X}\mathbb{Y}} = \langle n, S, f, \sigma, \eta_f, \{\varphi_R : R \in L_\mathbb{X}\}\rangle$ of $\mathbb{X}$ in $\mathbb{Y}$, where $n \in \mathbb{N}$, $S \subset Y^n$, $f : S \to X$ is a surjection and (denoting an n-tuple $v_1, \dots, v_n$ by $\bar{v}$) $\sigma(\bar{v}), \eta_f(\bar{v}^1, \bar{v}^2), \varphi_R(\bar{v}^1, \dots, \bar{v}^k) \in \mathrm{Form}_{L_\mathbb{Y}}$, for $R \in L_\mathbb{X}$ of arity k, where $\bar{v}^i$, $1 \le i \le k$ are disjoint n-tuples of distinct variables and

$$\forall \bar{y} \in Y^n \ \Big(\bar{y} \in S \Leftrightarrow \mathbb{Y} \models \sigma[\bar{y}]\Big), \tag{2}$$

$$\forall \bar{y}^1, \bar{y}^2 \in S \ \Big(f(\bar{y}^1) = f(\bar{y}^2) \ \Leftrightarrow\ \mathbb{Y} \models\ \eta_f[\bar{y}^1, \bar{y}^2]\Big), \tag{3}$$

$$\forall \bar{y}^1, \dots, \bar{y}^k \in S \ \Big(\langle f(\bar{y}^1), \dots, f(\bar{y}^k)\rangle \in R^\mathbb{X} \Leftrightarrow \mathbb{Y} \models \varphi_R[\bar{y}^1, \dots, \bar{y}^k]\Big). \tag{4}$$

(B) there is a 0-interpretation $\Gamma_{\mathbb{Y}\mathbb{X}} = \langle m, T, g, \tau, \eta_g, \{\psi_P : P \in L_\mathbb{Y}\}\rangle$ of $\mathbb{Y}$ in $\mathbb{X}$, where $m \in \mathbb{N}$, $T \subset X^m$, $g : T \to Y$ is a surjection and (denoting an m-tuple $v_1, \dots, v_m$ by $\tilde{v}$) $\tau(\tilde{v}), \eta_g(\tilde{v}^1, \tilde{v}^2), \psi_P(\tilde{v}^1, \dots, \tilde{v}^l) \in \mathrm{Form}_{L_\mathbb{X}}$, for $P \in L_\mathbb{Y}$ of arity l, where $\tilde{v}^i$, $1 \le i \le l$ are disjoint m-tuples of distinct variables and

$$\forall \tilde{x} \in X^m \ \Big(\tilde{x} \in T \Leftrightarrow \mathbb{X} \models \tau[\tilde{x}]\Big), \tag{5}$$

$$\forall \tilde{x}^1, \tilde{x}^2 \in T \ \Big(g(\tilde{x}^1) = g(\tilde{x}^2) \ \Leftrightarrow\ \mathbb{X} \models\ \eta_g[\tilde{x}^1, \tilde{x}^2]\Big), \tag{6}$$

$$\forall \tilde{x}^1, \dots, \tilde{x}^l \in T \ \Big(\langle g(\tilde{x}^1), \dots, g(\tilde{x}^l)\rangle \in P^\mathbb{Y} \Leftrightarrow \mathbb{X} \models \psi_P[\tilde{x}^1, \dots, \tilde{x}^l]\Big). \tag{7}$$

(C) The composition $\Gamma_{\mathbb{X}\mathbb{Y}} * \Gamma_{\mathbb{Y}\mathbb{X}} = \langle mn, (g^n)^{-1}[S], f \circ (g^n \restriction (g^n)^{-1}[S])\rangle$ is homotopic to $1_X = \langle 1, X, \mathrm{id}_X\rangle$, that is there is $\chi_X(\tilde{v}^1, \dots, \tilde{v}^n, v) \in \mathrm{Form}_{L_\mathbb{X}}$ such that for each $x \in X$ and each $\langle \tilde{x}^1, \dots, \tilde{x}^n\rangle \in (g^n)^{-1}[S]$

$$f(g(\tilde{x}^1), \dots, g(\tilde{x}^n)) = x \Leftrightarrow \mathbb{X} \models \chi_X[\tilde{x}^1, \dots, \tilde{x}^n, x]. \tag{8}$$

(D) The composition $\Gamma_{\mathbb{Y}\mathbb{X}} * \Gamma_{\mathbb{X}\mathbb{Y}} = \langle nm, (f^m)^{-1}[T], g \circ (f^m \restriction (f^m)^{-1}[T])\rangle$ is homotopic to $1_Y = \langle 1, Y, \mathrm{id}_Y\rangle$, that is there is $\chi_Y(\bar{w}^1, \dots, \bar{w}^m, w) \in \mathrm{Form}_{L_\mathbb{Y}}$ such that for each $y \in Y$ and each $\langle \bar{y}^1, \dots, \bar{y}^m\rangle \in (f^m)^{-1}[T]$

$$g(f(\bar{y}^1), \dots, f(\bar{y}^m)) = y \Leftrightarrow \mathbb{Y} \models \chi_Y[\bar{y}^1, \dots, \bar{y}^m, y]. \tag{9}$$

If condition (A) is satisfied, then the structure $\mathbb{X}$ is called *interpretable* in $\mathbb{Y}$. If conditions (A), (B), and (C) are satisfied, the structure $\mathbb{X}$ is called a *retraction* of $\mathbb{Y}$ and condition (C) will be abbreviated as $f * g \sim 1_\mathbb{X}$.

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Let $\mathcal{F}_{\Gamma_{\mathbb{XY}}} := \{\sigma, \eta_f\} \cup \{\varphi_R : R \in L_{\mathbb{X}}\}$ and $\mathcal{F}_{\Gamma_{\mathbb{YX}}} := \{\tau, \eta_g\} \cup \{\psi_P : P \in L_{\mathbb{Y}}\}$. If Σ is a class of formulas, we will write $\mathbb{X} \leftrightsquigarrow_\Sigma \mathbb{Y}$ iff there are interpretations $\Gamma_{\mathbb{XY}}$ and $\Gamma_{\mathbb{YX}}$ and homotopies χ_X and χ_Y such that $\mathcal{F}_{\Gamma_{\mathbb{YX}}} \cup \{\chi_X\} \subset \Sigma_{L_{\mathbb{X}}}$ and $\mathcal{F}_{\Gamma_{\mathbb{XY}}} \cup \{\chi_Y\} \subset \Sigma_{L_{\mathbb{Y}}}$. If the interpretations are *unrelativized*, that is $S = Y^n$ and $T = X^m$, we will write $\mathbb{X} \leftrightsquigarrow^u \mathbb{Y}$, and, if they are *injective*, that is if f and g are injections, we will write $\mathbb{X} \leftrightsquigarrow^i \mathbb{Y}$.

If $\mathbb{X} \leftrightsquigarrow_\Sigma \mathbb{Y}$, where $m = n = 1$, $S = T = Y$, and $f = g = \mathrm{id}_Y$, the structures $\mathbb{X}$ and $\mathbb{Y}$ will be called *bi-definable* and if, in addition, Σ is a set of first-order formulas, then $\mathbb{X}$ and $\mathbb{Y}$ will be called *first-order bi-definable*.

§3. Reversibility and interpretability. Working with endomorphism monoids (resp. automorphism groups) having the topology of pointwise convergence we will write $\Phi : \mathrm{Cond}(\mathbb{X}) \cong_t \mathrm{Cond}(\mathbb{Y})$ (resp. $\Phi : \mathrm{Aut}(\mathbb{X}) \cong_t \mathrm{Aut}(\mathbb{Y})$ iff the mapping Φ is both an isomorphism of monoids (resp. groups) and a homeomorphism. In [1] Ahlbrandt and Ziegler proved that (1) if relational structures $\mathbb{X}$ and $\mathbb{Y}$ are first-order bi-interpretable, then $\mathrm{Aut}(\mathbb{X}) \cong_t \mathrm{Aut}(\mathbb{Y})$, and (2) the converse holds if $\mathbb{X}$ and $\mathbb{Y}$ are countable ω-categorical structures. First, working with $L_{\infty\omega}$-formulas and condensations, we prove a claim similar to (1).

In the sequel we assume that $\mathbb{X}$ and $\mathbb{Y}$ are relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$. If $\Gamma_{\mathbb{XY}} = \langle n, S, f\rangle$ is an interpretation of $\mathbb{X}$ in $\mathbb{Y}$, for a condensation $G \in \mathrm{Cond}(\mathbb{Y})$ let the function $G_f : X \to X$ be defined in the following way: for $x \in X$ we choose $\bar{y} = \langle y_1, \ldots, y_n\rangle \in S$ satisfying $x = f(y_1, \ldots, y_n)$ and define

$$G_f(x) = f(G\bar{y}). \tag{10}$$

We will show that, under some restrictions, the function G_f is well defined and that $\Phi_f : \mathrm{Cond}(\mathbb{Y}) \to \mathrm{Cond}(\mathbb{X})$, where

$$\Phi_f(G) = G_f, \text{ for } G \in \mathrm{Cond}(\mathbb{Y}). \tag{11}$$

Similarly, if $\Gamma_{\mathbb{YX}} = \langle m, T, g\rangle$ is an interpretation of $\mathbb{Y}$ in $\mathbb{X}$, for a condensation $F \in \mathrm{Cond}(\mathbb{X})$ let the function $F_g : Y \to Y$ be defined in the following way: for $y \in Y$ we choose $\tilde{x} = \langle x_1, \ldots, x_m\rangle \in T$ satisfying $y = g(x_1, \ldots, x_m)$ and define

$$F_g(y) = g(F\tilde{x}). \tag{12}$$

In this way we obtain a mapping $\Phi_g : \mathrm{Cond}(\mathbb{X}) \to \mathrm{Cond}(\mathbb{Y})$, where

$$\Phi_g(F) = F_g, \text{ for } F \in \mathrm{Cond}(\mathbb{X}). \tag{13}$$

THEOREM 3.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$. Then*

If $\Gamma_{\mathbb{XY}} = \langle n, S, f\rangle$ is an interpretation *of $\mathbb{X}$ in $\mathbb{Y}$ and the formulas φ_R, $R \in L_{\mathbb{X}}$, [resp. σ and η_f] are preserved by [resp. absolute under] each condensation $G \in \mathrm{Cond}(\mathbb{Y})$, then*

(a) *Φ_f maps $\mathrm{Cond}(\mathbb{Y})$ into $\mathrm{Cond}(\mathbb{X})$ and $\Phi_f[\mathrm{Aut}(\mathbb{Y})] \subset \mathrm{Aut}(\mathbb{X})$;*

(b) *Φ_f is a continuous homomorphism of monoids.*

If, in addition, $\mathbb{X}$ is a retraction *of $\mathbb{Y}$, where $\Gamma_{\mathbb{YX}} = \langle m, T, g\rangle$ is an interpretation of $\mathbb{Y}$ in $\mathbb{X}$ such that $f * g \sim 1_{\mathbb{X}}$ and the formulas ψ_P, $P \in L_{\mathbb{Y}}$, and χ_X [resp. τ and η_g] are preserved [resp. absolute] under each condensation $F \in \mathrm{Cond}(\mathbb{X})$, then*

(c) *$\Phi_f \circ \Phi_g = \mathrm{id}_{\mathrm{Cond}(\mathbb{X})}$ and $\Phi_f[\mathrm{Aut}(\mathbb{Y})] = \mathrm{Aut}(\mathbb{X})$;*

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(d) $\Phi_f : \mathrm{Cond}(\mathbb{Y}) \to \mathrm{Cond}(\mathbb{X})$ *is a continuous epimorphism*;
(e) $\Phi_g : \mathrm{Cond}(\mathbb{X}) \to \mathrm{Cond}(\mathbb{Y})$ *is a continuous embedding*;
(f) $\mathbb{Y}$ *is reversible* $\Rightarrow$ $\mathbb{X}$ *is reversible.*

If, in addition, $\mathbb{X}$ *and* $\mathbb{Y}$ *are* bi-interpretable *structures and the formula* χ_Y *is preserved under each condensation* $G \in \mathrm{Cond}(\mathbb{Y})$*, then*
(g) $\Phi_f : \mathrm{Cond}(\mathbb{Y}) \cong_t \mathrm{Cond}(\mathbb{X})$ *and* $\Phi_f^{-1} = \Phi_g$;
(h) $\mathbb{X}$ *is reversible* $\Leftrightarrow$ $\mathbb{Y}$ *is reversible.*

PROOF. (a) Let $G \in \mathrm{Cond}(\mathbb{Y})$. In order to prove that the function $G_f : X \to X$ is well defined by (10), first we show that $G^n[S] \subset S$, that is

$$\forall \bar{y} \in S \;\; G\bar{y} \in S. \tag{14}$$

So, if $\bar{y} \in S$, then, by (2), $\mathbb{Y} \models \sigma[\bar{y}]$ and, since σ is preserved by G we have $\mathbb{Y} \models \sigma[G\bar{y}]$, that is, by (2) again, $G\bar{y} \in S$ and (14) is true.

Now we show that the value $G_f(x)$ given by (10) is well defined. First, since the function $f : S \to X$ is onto, for $x \in X$ there is $\bar{y} = \langle y_1, \ldots, y_n \rangle \in S$ satisfying $x = f(y_1, \ldots, y_n)$. Second, for a proof that the value of $G_f(x)$ does not depend on the choice of the coordinates y_i for x, we show that for each $\bar{y}^1, \bar{y}^2 \in S$

$$f(\bar{y}^1) = f(\bar{y}^2) \Rightarrow f(G\bar{y}^1) = f(G\bar{y}^2). \tag{15}$$

So, if $f(\bar{y}^1) = f(\bar{y}^2)$, then, by (3), $\mathbb{Y} \models \eta_f[\bar{y}^1, \bar{y}^2]$ and, since η_f is preserved by G, $\mathbb{Y} \models \eta_f[G\bar{y}^1, G\bar{y}^2]$ so, by (3) and (14), $f(G\bar{y}^1) = f(G\bar{y}^2)$ and (15) is true.

G_f is a homomorphism. For $R \in L_{\mathbb{X}}$ of arity k and $x_1, \ldots, x_k \in X$, where $x_i = f(\bar{y}^i)$ and $\bar{y}^i \in S$, for $i \in \{1, \ldots, k\}$, we show that

$$\langle x_1, \ldots, x_k \rangle \in R^{\mathbb{X}} \Rightarrow \langle G_f(x_1), \ldots, G_f(x_k) \rangle \in R^{\mathbb{X}}. \tag{16}$$

So, if $\langle f(\bar{y}^1), \ldots, f(\bar{y}^k) \rangle \in R^{\mathbb{X}}$, then, by (4), $\mathbb{Y} \models \varphi_R[\bar{y}^1, \ldots, \bar{y}^k]$ and, since φ_R is preserved by G we have $\mathbb{Y} \models \varphi_R[G\bar{y}^1, \ldots, G\bar{y}^k]$ which by (14) and (4) implies $\langle f(G\bar{y}^1), \ldots, f(G\bar{y}^k) \rangle \in R^{\mathbb{X}}$, that is, by (10), $\langle G_f(x_1), \ldots, G_f(x_k) \rangle \in R^{\mathbb{X}}$.

G_f is a surjection. Let $x \in X$ and $\bar{y} \in S$, where $x = f(\bar{y})$. Since G is onto there is $\bar{y}' \in Y^n$ such that $\bar{y} = G\bar{y}'$. By (2) we have $\mathbb{Y} \models \sigma[\bar{y}]$, that is, $\mathbb{Y} \models \sigma[G\bar{y}']$, and, since σ is absolute under G, $\mathbb{Y} \models \sigma[\bar{y}']$, which by (2) implies $\bar{y}' \in S$. Now, for $x' := f(\bar{y}')$ by (10) we have $G_f(x') = f(G\bar{y}') = f(\bar{y}) = x$.

G_f is an injection. Let $x_1, x_2 \in X$, and $G_f(x_1) = G_f(x_2)$. If $\bar{y}^1, \bar{y}^2 \in S$, where $x_1 = f(\bar{y}^1)$ and $x_2 = f(\bar{y}^2)$, then by (10) we have $f(G\bar{y}^1) = f(G\bar{y}^2)$ and, by (14) and (3), $\mathbb{Y} \models \eta_f[G\bar{y}^1, G\bar{y}^2]$ which, since the formula η_f is absolute under G gives $\mathbb{Y} \models \eta_f[\bar{y}^1, \bar{y}^2]$. So, by (3), $f(\bar{y}^1) = f(\bar{y}^2)$, that is $x_1 = x_2$.

If $G \in \mathrm{Aut}(\mathbb{Y})$ and $R \in L_{\mathbb{X}}$, then the formula φ_R is absolute under G and as above we show that we have "$\Leftrightarrow$" in (16); thus G_f is a strong homomorphism and, hence, $\Phi_f(G) = G_f \in \mathrm{Aut}(\mathbb{X})$.

(b) For $G', G'' \in \mathrm{Cond}(\mathbb{Y})$ we prove that $\Phi_f(G' \circ G'') = \Phi_f(G') \circ \Phi_f(G'')$, which means that for each $x \in X$ we have $(G' \circ G'')_f(x) = G'_f(G''_f(x))$. Let $\bar{y} \in S$, where $x = f(\bar{y})$. By (14) $G''\bar{y} \in S$ and, by (10), $x' := f(G''\bar{y}) = G''_f(x)$ so, by (10), and since the value of $G'_f(x')$ does not depend on the choice of the coordinates for x', we have: $G'_f(G''_f(x)) = G'_f(x') = f(G'G''\bar{y}) = (G' \circ G'')_f(x)$.

We prove the continuity of the mapping Φ_f at a point $G \in \mathrm{Cond}(\mathbb{Y})$. Let $B = \bigcap_{i=1}^{k} \pi_{a_i}^{-1}[\{b_i\}] \cap \mathrm{Cond}(\mathbb{X})$ be a neighborhood of $\Phi_f(G) = G_f \in \mathrm{Cond}(\mathbb{X}) \subset X^X$

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and, for $i \leq k$, let $\bar{y}^i, \bar{z}^i \in S$, where $a_i = f(\bar{y}^i)$ and $b_i = f(\bar{z}^i)$, be the corresponding coordinates. Then, since $G_f(a_i) = b_i$, for $i \leq k$, we have

$$\forall i \leq k \quad f(G\bar{y}^i) = f(\bar{z}^i). \tag{17}$$

Clearly $B' = \bigcap_{i\leq k} \bigcap_{j\leq n} p^{-1}_{y^i_j}[\{G(y^i_j)\}] \cap \mathrm{Cond}(\mathbb{Y})$ is a neighborhood of G and, if $H \in B'$, then $H(y^i_j) = G(y^i_j)$, for all i and j, so by (17) for $i \leq k$ we have $\Phi_f(H)(a_i) = H_f(a_i) = f(H\bar{y}^i) = f(G\bar{y}^i) = f(\bar{z}^i) = b_i$, thus $\Phi_f[B'] \subset B$.

(c) By (a) we have $\Phi_f \circ \Phi_g : \mathrm{Cond}(\mathbb{X}) \to \mathrm{Cond}(\mathbb{X})$. For $F \in \mathrm{Cond}(\mathbb{X})$ we prove that $\Phi_f(\Phi_g(F)) = (F_g)_f = F$, that is

$$\forall x \in X \quad (F_g)_f(x) = F(x). \tag{18}$$

For $x \in X$ let

$$\bar{y} \in S, \text{ where } x = f(\bar{y}), \tag{19}$$

$$\tilde{x}^i \in T, \text{ where } y_i = g(\tilde{x}^i), \text{ for } i \leq n. \tag{20}$$

Since $F_g \in \mathrm{Cond}(\mathbb{Y})$, by (19), (10), and the independence of coordinates proved in (a) we have $(F_g)_f(x) = f(F_g\bar{y})$ and, similarly, since $F \in \mathrm{Cond}(\mathbb{X})$, by (20) and (12), $F_g(y_i) = g(F\tilde{x}^i)$, for $i \leq n$, and thus

$$(F_g)_f(x) = f(g(F\tilde{x}^1), \ldots, g(F\tilde{x}^n)). \tag{21}$$

By (19) and (20) we have $\langle g(\tilde{x}^1), \ldots, g(\tilde{x}^n)\rangle = \bar{y} \in S$, so

$$\langle \tilde{x}^1, \ldots, \tilde{x}^n\rangle \in (g^n)^{-1}[S], \tag{22}$$

and we show that

$$\langle F\tilde{x}^1, \ldots, F\tilde{x}^n\rangle \in (g^n)^{-1}[S]. \tag{23}$$

First, since $F \in \mathrm{Cond}(\mathbb{X})$, by (20) we have $\tilde{x}^i \in T$, for $i \leq n$, and, as in the proof of (a) we obtain $F\tilde{x}^i \in T = \mathrm{dom}\, g$. Now by (20) and (a) we have $F_g(y_i) = g(F\tilde{x}^i)$. Since $\Phi_g[\mathrm{Cond}(\mathbb{X})] \subset \mathrm{Cond}(\mathbb{Y})$, we have $F_g \in \mathrm{Cond}(\mathbb{Y})$ and, by (a) applied to F_g, from $\bar{y} \in S$ it follows that $F_g\bar{y} \in S$ and hence we have $\langle g(F\tilde{x}^1), \ldots, g(F\tilde{x}^n)\rangle \in S$ which implies (23).

Since $f * g \sim 1_{\mathbb{X}}$ there is a formula $\chi_X(\tilde{v}^1, \ldots, \tilde{v}^n, v) \in \mathrm{Form}_{L_{\mathbb{X}}}$ such that for each $\langle \tilde{x}^1, \ldots, \tilde{x}^n\rangle \in (g^n)^{-1}[S]$ and each $x \in X$ the equivalence (8) holds.

Now, by (19) and (20) we have $f(g(\tilde{x}^1), \ldots, g(\tilde{x}^n)) = x$, which by (8) and (22) implies $\mathbb{X} \models \chi_X[\tilde{x}^1, \ldots, \tilde{x}^n, x]$ and, since χ_X is preserved by F, $\mathbb{X} \models \chi_X[F\tilde{x}^1, \ldots, F\tilde{x}^n, F(x)]$. By (8) and (23) we have $f(g(F\tilde{x}^1), \ldots, g(F\tilde{x}^n)) = F(x)$, which by (21) gives $(F_g)_f(x) = F(x)$ and (18) is proved.

By (a) we have $\Phi_f[\mathrm{Aut}(\mathbb{Y})] \subset \mathrm{Aut}(\mathbb{X})$ and $\Phi_g[\mathrm{Aut}(\mathbb{X})] \subset \mathrm{Aut}(\mathbb{Y})$. So, since $\Phi_f \circ \Phi_g = \mathrm{id}_{\mathrm{Cond}(\mathbb{X})}$, we have $\mathrm{Aut}(\mathbb{X}) = \Phi_f[\Phi_g[\mathrm{Aut}(\mathbb{X})]] \subset \Phi_f[\mathrm{Aut}(\mathbb{Y})] \subset \mathrm{Aut}(\mathbb{X})$, which implies that $\Phi_f[\mathrm{Aut}(\mathbb{Y})] = \mathrm{Aut}(\mathbb{X})$.

(d) and (e). By (b), Φ_f and Φ_g are continuous homomorphisms of monoids and by (c) Φ_f is a surjection and Φ_g an injection.

(f) If the structure $\mathbb{Y}$ is reversible, then $\mathrm{Cond}(\mathbb{Y}) = \mathrm{Aut}(\mathbb{Y})$. So, by (c) and (d) we have $\mathrm{Cond}(\mathbb{X}) = \Phi_f[\mathrm{Cond}(\mathbb{Y})] = \Phi_f[\mathrm{Aut}(\mathbb{Y})] = \mathrm{Aut}(\mathbb{X})$ and $\mathbb{X}$ is a reversible structure too.

(g) By (c) and symmetry we obtain $\Phi_g \circ \Phi_f = \mathrm{id}_{\mathrm{Cond}(\mathbb{Y})}$. Thus $\Phi_f^{-1} = \Phi_g$ and Φ_f is a continuous isomorphism having a continuous inverse.

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(h) If $\mathbb{X}$ and $\mathbb{Y}$ are bi-interpretable structures then $\mathbb{X}$ is a retraction of $\mathbb{Y}$ and vice versa and we apply (f). ⊣

THEOREM 3.2. *Reversibility is an invariant of the relation* $\leftrightsquigarrow_{\mathcal{P}}^{u,\,i}$.

PROOF. Assuming that $\mathbb{X}$ is a reversible structure and $\mathbb{X} \leftrightsquigarrow_{\mathcal{P}}^{u,\,i} \mathbb{Y}$ we show that $\mathbb{Y}$ is reversible too. By Fact 2.2(b) the formulas from $\mathcal{F}_{\Gamma_{\mathbb{YX}}} \cup \{\chi_X\} \cup \mathcal{F}_{\Gamma_{\mathbb{XY}}} \cup \{\chi_Y\}$ are preserved under condensations. By the assumption $S = Y^n$, $T = X^m$ and $f : Y^n \to X$ and $g : X^m \to Y$ are bijections. Since $S = Y^n$ we have $\mathbb{Y} \models \sigma[\bar{y}]$, for *all* $\bar{y} \in Y^n$, which implies that σ is absolute under all mappings from $\mathrm{Cond}(\mathbb{Y})$. Similarly, τ is absolute under all mappings from $\mathrm{Cond}(\mathbb{X})$.

Since f is an injection, the formula η_f is (modulo $\mathrm{Th}(\mathbb{Y})$) equivalent to the formula $\eta'_f = \bigwedge_{i=1}^n v_i^1 = v_i^2$. Since for each $G \in \mathrm{Cond}(\mathbb{Y})$ and $\bar{y}^1, \bar{y}^2 \in Y^n$ we have $\bigwedge_{i=1}^n y_i^1 = y_i^2 \Leftrightarrow \bigwedge_{i=1}^n G(y_i^1) = G(y_i^2)$, the formula η_f is absolute under the maps from $\mathrm{Cond}(\mathbb{Y})$. Similarly, η_g is absolute under the maps from $\mathrm{Cond}(\mathbb{X})$.

By Theorem 3.1(h) the structure $\mathbb{Y}$ is reversible. ⊣

In order to make a correspondence between interpretations and their complements to each $L_{\infty\omega}$-formula φ we adjoin the formula φ^c defined by recursion in the following way:

- $(v_\alpha = v_\beta)^c := v_\alpha = v_\beta$ and $(R_i(v_{\alpha_1}, \ldots, v_{\alpha_{n_i}}))^c := \neg R_i(v_{\alpha_1}, \ldots, v_{\alpha_{n_i}})$,
- $(\neg\varphi)^c := \neg\varphi^c$, $(\forall v_\alpha\, \varphi)^c := \forall v_\alpha\, \varphi^c$ and $(\exists v_\alpha\, \varphi)^c := \exists v_\alpha\, \varphi^c$, if φ^c is defined,
- $(\bigwedge \Phi)^c := \bigwedge \Phi^c$ and $(\bigvee \Phi)^c := \bigvee \Phi^c$, where $\Phi^c := \{\varphi^c : \varphi \in \Phi\}$, if $\Phi \subset \mathrm{Form}_L$ and φ^c is defined for all $\varphi \in \Phi$.

The following fact is easily provable by induction (see [12]).

FACT 3.3. *For each L-structure* $\langle X, \rho\rangle$ *and each formula* $\varphi \in \mathrm{Form}_{L_{\infty\omega}}$ *we have*

$$\forall \vec{x} \in {}^{\kappa}X \left(\langle X, \rho^c\rangle \models \varphi^c[\vec{x}] \Leftrightarrow \langle X, \rho\rangle \models \varphi[\vec{x}] \right). \tag{24}$$

THEOREM 3.4. *Let a structure* $\mathbb{X}$ *be a retraction of a reversible structure* $\mathbb{Y}$, *where*

(i) *the formulas* τ *and* η_g *are absolute under each* $F \in \mathrm{Cond}(\mathbb{X})$,
(ii) *the formula* χ_X *is preserved by each* $F \in \mathrm{Cond}(\mathbb{X})$.

Then each of the following conditions implies that $\mathbb{X}$ *is a reversible structure too*

(iii) *the formulas* ψ_P, $P \in L_{\mathbb{Y}}$, *are preserved by each* $F \in \mathrm{Cond}(\mathbb{X})$,
(iii') *the formulas* $\neg\psi_P$, $P \in L_{\mathbb{Y}}$, *are preserved by each* $F \in \mathrm{Cond}(\mathbb{X})$.

PROOF. Since $\mathrm{Cond}(\mathbb{Y}) = \mathrm{Aut}(\mathbb{Y})$, by Fact 2.2(a) the formulas σ, η_f and φ_R, $R \in L_{\mathbb{X}}$, are absolute under each $G \in \mathrm{Cond}(\mathbb{Y})$. So, if conditions (i)–(iii) are satisfied, we apply Theorem 3.1(f).

Suppose that conditions (i), (ii), and (iii') are satisfied. We show that

$$\Gamma_{\mathbb{XY}^c} = \langle n, S, f, \sigma^c, \eta_f^c, \{\varphi_R^c : R \in L_{\mathbb{X}}\}\rangle$$

is an interpretation of $\mathbb{X}$ in $\mathbb{Y}^c$. By Fact 3.3 and (2) for each $\bar{y} \in Y^n$ we have $\bar{y} \in S$ iff $\mathbb{Y} \models \sigma[\bar{y}]$ iff $\mathbb{Y}^c \models \sigma^c[\bar{y}]$; thus the set $S \subset Y^n$ is definable in $\mathbb{Y}$ by the formula σ^c. By (3), for each $\bar{y}^1, \bar{y}^2 \in S$ we have $f(\bar{y}^1) = f(\bar{y}^2)$ iff $\mathbb{Y} \models \eta_f[\bar{y}^1, \bar{y}^2]$ iff $\mathbb{Y}^c \models \eta_f^c[\bar{y}^1, \bar{y}^2]$ and the set $\ker(f) \subset (Y^n)^2$ is definable in $\mathbb{Y}$ by the formula η_f^c. Finally, by (4), for each $\bar{y}^1, \ldots, \bar{y}^k \in S$ we have $\langle f(\bar{y}^1), \ldots, f(\bar{y}^k)\rangle \in R^{\mathbb{X}}$ iff $\mathbb{Y} \models \varphi_R[\bar{y}^1, \ldots, \bar{y}^k]$ iff $\mathbb{Y}^c \models \varphi_R^c[\bar{y}^1, \ldots, \bar{y}^k]$.

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Now we show that

$$\Gamma_{\mathbb{Y}^c\mathbb{X}} = \langle m, T, g, \tau, \eta_g, \{\neg\psi_P : P \in L_{\mathbb{Y}}\}\rangle$$

is an interpretation of $\mathbb{Y}^c$ in $\mathbb{X}$. Again the formulas τ and η_g define the sets $T \subset X^m$ and $\ker(g)$. If $P \in L_{\mathbb{Y}}$, where $\mathrm{ar}(P) = l$, then $P^{\mathbb{Y}^c} = X^l \setminus P^{\mathbb{Y}}$ and, by (7), for each $\tilde{x}^1, \dots, \tilde{x}^l \in T$ we have $\langle g(\tilde{x}^1), \dots, g(\tilde{x}^l)\rangle \in P^{\mathbb{Y}^c}$ iff it is not true that $\mathbb{X} \models \psi_P[\tilde{x}^1, \dots, \tilde{x}^l]$ iff $\mathbb{X} \models \neg\psi_P[\tilde{x}^1, \dots, \tilde{x}^l]$. Thus the relations of the structure $\mathbb{Y}^c$ are defined by the formulas $\neg\psi_P$, $P \in L_{\mathbb{Y}}$.

By Fact 2.1(c), $\mathbb{Y}^c$ is a reversible structure and, hence, $\mathrm{Cond}(\mathbb{Y}^c) = \mathrm{Aut}(\mathbb{Y}^c)$ so, by Fact 2.2(a), the formulas σ^c, η_f^c, and φ_R^c, $R \in L_{\mathbb{X}}$, are absolute under each condensation $G \in \mathrm{Cond}(\mathbb{Y}^c)$. By (i) the formulas τ and η_g are absolute under each $F \in \mathrm{Cond}(\mathbb{X})$, and by (ii) the formula χ_X is preserved by each $F \in \mathrm{Cond}(\mathbb{X})$. Since by (iii') the formulas $\neg\psi_P$, $P \in L_{\mathbb{Y}}$, are preserved by each $F \in \mathrm{Cond}(\mathbb{X})$, by Theorem 3.1(f) the structure $\mathbb{X}$ is reversible. ⊣

COROLLARY 3.5. *If $\mathbb{X}$ and $\mathbb{Y}$ are bi-definable relational structures, $\mathbb{Y}$ is reversible and the formulas ψ_P, $P \in L_{\mathbb{Y}}$, or the formulas $\neg\psi_P$, $P \in L_{\mathbb{Y}}$, are preserved by each condensation $F \in \mathrm{Cond}(\mathbb{X})$, then the structure $\mathbb{X}$ is reversible.*

This holds if, in particular, $\{\psi_P : P \in L_{\mathbb{Y}}\} \subset \mathcal{P}$ or $\{\psi_P : P \in L_{\mathbb{Y}}\} \subset \mathcal{N}$.

PROOF. By the assumption we can take $m = n = 1$, $S = T = Y$, $f = g = \mathrm{id}_Y$, $\tau(v_1) := v_1 = v_1$, $\eta_g(v_1^1, v_1^2) := v_1^1 = v_1^2$ and $\chi_X(v_1^1, v) := v_1^1 = v$. Then these formulas are absolute under all bijections and we apply Theorem 3.4.

If $\{\psi_P : P \in L_{\mathbb{Y}}\} \subset \mathcal{N}$, then, by Fact 2.2(c), for each $P \in L_{\mathbb{Y}}$ the formula $\neg\psi_P$ is logically equivalent to an R-positive formula and by Fact 2.2(b) it is preserved under each condensation $F \in \mathrm{Cond}(\mathbb{X})$. ⊣

EXAMPLE 3.6. Reversibility is not an invariant of bi-definability. If Y is a nonempty set, $n \in \mathbb{N}$ and ρ is a nonreversible n-ary relation on Y, then, by [13], the structure $\mathbb{Y} = \langle Y, \langle \rho, \rho^c\rangle\rangle$ is reversible, while $\mathbb{X} = \langle Y, \rho\rangle$ is not a reversible structure, although these two structures are bi-definable.

The following example shows that the assumption of Theorem 3.1 and its consequences—that the formulas of the considered interpretations are preserved by condensations of the corresponding structures—is weaker than the condition that these formulas are equivalent (modulo the corresponding theories) to some formulas from the class $\mathcal{P}$.[2]

EXAMPLE 3.7. A first-order formula $\varphi(\bar{v})$ is preserved by (and, moreover, absolute under) each condensation $F \in \mathrm{Cond}(\mathbb{X})$, but there is no $\psi(\bar{v}) \in \mathcal{P}$ such that $\mathbb{X} \models \forall\bar{v}\ (\varphi(\bar{v}) \Leftrightarrow \psi(\bar{v}))$ and, hence, there is no formula $\psi(\bar{v}) \in \mathcal{P}$ such that $\mathrm{Th}(\mathbb{X}) \vdash \varphi(\bar{v}) \Leftrightarrow \psi(\bar{v})$.

Let $L = \langle R\rangle$, where $\mathrm{ar}(R) = 2$, let $\mathcal{T}$ be the L-theory of one equivalence relation having exactly one equivalence class of size n, for each $n \in \mathbb{N}$, and, for a cardinal $\kappa \leq \omega$, let $\mathbb{E}_\kappa$ be a countable model of $\mathcal{T}$ having exactly κ-many infinite equivalence classes. It is known that $\mathcal{T}$ is a complete theory and that each countable model of

[2] By Lyndon's theorem [17] (see also [8], p. 498), if a first-order formula $\varphi(\bar{v})$ is preserved by each condensation between *any* two models of a theory $\mathcal{T}$, then it is, modulo $\mathcal{T}$, equivalent to a first-order formula from $\mathcal{P}$ (see [8]).

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$\mathcal{T}$ is isomorphic to some $\mathbb{E}_\kappa$. In addition (see [14]) the structures $\mathbb{E}_n$, $n < \omega$, are reversible, while $\mathbb{E}_\omega$ is not. So, since the structure $\mathbb{X} := \mathbb{E}_0$ is reversible, we have $\mathrm{Cond}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X})$ and, by Fact 2.2(a), the formula $\varphi(v_1, v_2) := \neg R(v_1, v_2)$ is absolute (and, thus, preserved) under each condensation $F \in \mathrm{Cond}(\mathbb{X})$.

Suppose that $\mathbb{X} \models \eta$, where $\eta := \forall \bar{v}\ (\varphi(\bar{v}) \Leftrightarrow \psi(\bar{v}))$ and $\psi(\bar{v}) \in \mathcal{P}$. Then $\eta \in \mathrm{Th}(\mathbb{X}) = \mathcal{T}$ and, since $\mathbb{E}_\omega$ is a model of $\mathcal{T}$, we have $\mathbb{E}_\omega \models \eta$. So, if $\mathbb{E}_\omega = \langle Y, \sim \rangle$ and $Y = \bigcup_{n\in\mathbb{N}} Y_n \cup \bigcup_{n\in\mathbb{N}} Z_n$ is the corresponding partition of Y, where $|Y_n| = n$ and $|Z_n| = \omega$, for $n \in \mathbb{N}$, we have

$$\forall y_1, y_2 \in Y \ \Big(y_1 \not\sim y_2 \Leftrightarrow \mathbb{E}_\omega \models \psi[y_1, y_2]\Big). \tag{25}$$

Let $Z_1 = Z_1' \cup Z_1''$, where Z_1' and Z_1'' are disjoint infinite sets and let $G : Y \to Y$ be a bijection such that $G[Y_n] = Y_n$, for all $n \in \mathbb{N}$, $G[Z_1] = Z_1'$, $G[Z_2] = Z_1''$, and $G[Z_n] = Z_{n-1}$, for $n \geq 3$. Then $G \in \mathrm{Cond}(\mathbb{E}_\omega)$. For $y_1 \in Z_1'$ and $y_2 \in Z_2$ we have

$$y_1 \not\sim y_2 \wedge G(y_1) \sim G(y_2)$$

and, by (25), $\mathbb{E}_\omega \models \psi[y_1, y_2]$, which, since $\psi(\bar{v}) \in \mathcal{P}$, by Fact 2.2(b) implies that $\mathbb{E}_\omega \models \psi[G(y_1), G(y_2)]$ and, by (25) again, $G(y_1) \not\sim G(y_2)$, which is a contradiction.

Example 3.8. For an ordinal $1 \leq \alpha \leq \omega$ let $\mathbb{L}_\alpha$ denote the linear digraph $\langle \alpha, \rho_a \rangle$ where $\rho_a = \{\langle k, k+1\rangle : 1 \leq k+1 < \alpha\}\rangle$ is the consecutivity relation. Let J be a nonempty set, $1 \leq \alpha_j \leq \omega$, for $j \in J$, and let $\mathbb{Y} = \langle Y, \to \rangle$ be a disjoint union $\bigcup_{j\in J} \mathbb{Y}_{\alpha_j}$, where $\mathbb{Y}_{\alpha_j} \cong \mathbb{L}_{\alpha_j}$, for all $j \in J$. Then $L_\mathbb{Y} = \langle P \rangle$, where $\mathrm{ar}(P) = 2$ and $P^\mathbb{Y} = \to$. Let $\mathbb{X}$ be the structure $\langle Y, \langle \rho_n : n \in \omega \rangle\rangle$, where ρ_0 is the connectivity (equivalence) relation of the digraph $\mathbb{Y}$ and, for $n \in \mathbb{N}$, ρ_n is the n-th level of $\mathbb{Y}$. Here we have $L_\mathbb{X} = \langle R_n : n \in \omega \rangle$, $\mathrm{ar}(R_0) = 2$ and $\mathrm{ar}(R_n) = 1$, for $n \in \mathbb{N}$. Clearly, the formulas

$$\varphi_{R_0}(u, v) := u = v \vee \bigvee_{n\geq 2} \ \exists v_1 \dots v_n \ \Big[\bigwedge_{1\leq i<n} P(v_i, v_{i+1}) \ \wedge \Big((u = v_1 \wedge v = v_n) \vee (v = v_1 \wedge u = v_n)\Big)\Big],$$

$$\varphi_{R_n}(v) := \exists v_1 \dots v_n \ \Big[\bigwedge_{1\leq i<n} P(v_i, v_{i+1}) \ \wedge \ v = v_n \ \wedge \forall v_0 \ \neg P(v_0, v_1)\Big]$$

define the relations $R_n^\mathbb{X} = \rho_n$ in $\mathbb{Y}$ and the formula

$$\psi_P(u, v) := R_0(u, v) \wedge \bigvee_{n\in\mathbb{N}} R_n(u) \wedge R_{n+1}(v)$$

defines the relation $P^\mathbb{Y} = \to$ in $\mathbb{X}$ and $\psi_P \in \mathcal{P}$. So, by Corollary 3.5, if the structure $\mathbb{Y}$ is reversible, then $\mathbb{X}$ is reversible as well. In particular, by the results of [14], each of the following conditions implies that the structure $\mathbb{Y}$ is reversible: 1. $|J| < \omega$, 2. for each $\alpha \leq \omega$ the set $\{j \in J : \alpha_j = \alpha\}$ is finite, 3. for each $j \in J$ and $K \subset J$ of size > 1 we have $\alpha_j \neq \sum_{i\in K} \alpha_i$.

Example 3.9. Let $\mathbb{Y} = \langle Y, \rho \rangle$ be a reversible structure, $L_\mathbb{Y} = \langle P_i : i \in I \rangle$ and $\mathrm{ar}(P_i) = n_i$. Then, by Corollary 3.5, its complement, the structure $\mathbb{X} = \langle Y, \rho^c \rangle$, is reversible too since the formulas $\varphi_{R_i}(\bar{v}) := \neg P_i(\bar{v})$ define the relations $R_i^\mathbb{X}$ in $\mathbb{Y}$ and the formulas $\psi_{P_i}(\bar{v}) := \neg R_i(\bar{v})$ define the relations $P_i^\mathbb{Y}$ in $\mathbb{X}$ and $\psi_{P_i} \in \mathcal{N}$.

Example 3.10. Let $\mathbb{Y} = \langle Y, \rho \rangle$ be a reversible structure, $L_\mathbb{Y} = \langle P \rangle$ and $\mathrm{ar}(P) = n$.

1. Adding or deleting a strongly reversible relation. If $\theta(v_1, \dots, v_n) \in \mathrm{Form}_{L_\emptyset}$, then by Fact 2.1, $D_\theta = \{\bar{y} \in Y^n : \mathbb{Y} \models \theta[\bar{y}]\}$ is a strongly reversible n-ary relation.

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If $\rho \cap D_\theta = \emptyset$, then, by Corollary 3.5, the structure $\mathbb{X} = \langle Y, D_\theta \cup \rho\rangle$ is reversible, namely the formula $\varphi_R(\bar v) := \theta(\bar v) \vee P(\bar v)$ defines the relation $R^{\mathbb{X}}$ in $\mathbb{Y}$ and the formula $\psi_P(\bar v) := \neg\theta(\bar v) \wedge R(\bar v)$ defines the relation $P^{\mathbb{Y}}$ in $\mathbb{X}$ and $\psi_P \in \mathcal{P}$.

If $D_\theta \subset \rho$, then, by Corollary 3.5, the structure $\mathbb{X} = \langle Y, \rho \setminus D_\theta\rangle$ is reversible, since the formula $\varphi_R(\bar v) := \neg\theta(\bar v) \wedge P(\bar v)$ defines the relation $R^{\mathbb{X}}$ in $\mathbb{Y}$ and the formula $\psi_P(\bar v) := \theta(\bar v) \vee R(\bar v)$ defines the relation $P^{\mathbb{Y}}$ in $\mathbb{X}$ and $\psi_P \in \mathcal{P}$.

2. Symmetries induced by permutations. If π is a permutation of the set $\{1, \dots, n\}$ and $\rho_\pi = \{\langle y_{\pi(1)}, \dots, y_{\pi(1)}\rangle : \langle y_1, \dots, y_n\rangle \in \rho\}$, then $\mathbb{X} = \langle Y, \rho_\pi\rangle$ is a reversible structure since the formula $\varphi_R(v_1, \dots, v_n) := P(v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(n)})$ defines the relation $R^{\mathbb{X}}$ in $\mathbb{Y}$ and the formula $\psi_P(v_1, \dots, v_n) := R(v_{\pi(1)}, \dots, v_{\pi(n)})$ defines the relation $P^{\mathbb{Y}}$ in $\mathbb{X}$ and $\psi_P \in \mathcal{P}$.

3. Products with full relations. If $k, l \in \mathbb{N}$, then $\mathbb{X} = \langle Y, Y^k \times \rho \times Y^l\rangle$ is a reversible structure of the language $L_{\mathbb{X}} = \langle R\rangle$, where $\mathrm{ar}(R) = k + n + l$, since the $L_{\mathbb{Y}}$-formula $\varphi_R(v_1, \dots, v_{k+n+l}) := \bigwedge_{j=1}^{k} v_j = v_j \wedge P(v_{k+1}, \dots, v_{k+n}) \wedge \bigwedge_{j=k+n+1}^{k+n+l} v_j = v_j$ defines the relation $R^{\mathbb{X}}$ in $\mathbb{Y}$ and the formula

$$\psi_P(v_{k+1}, \dots, v_{k+n}) := \exists v_1, \dots, v_k, v_{k+n+1}, \dots, v_{k+n+l} \;\; R(v_1, \dots, v_{k+n+l})$$

defines the relation $P^{\mathbb{Y}}$ in $\mathbb{X}$ and $\psi_P \in \mathcal{P}$.

In particular, if $\mathbb{Y} = \langle Y, \rho\rangle$ is a reversible irreflexive binary structure, then, since for each binary relation σ we have $(\sigma^c)^c = (\sigma^{-1})^{-1} = \sigma$ and $(\sigma^{-1})^c = (\sigma^c)^{-1}$, using inverses, complementation, reflexivization, and irreflexivization we obtain the following reversible relations: ρ, ρ^{-1}, ρ^c, $(\rho^{-1})^c$, $\rho \cup \Delta_Y$, $\rho^{-1} \cup \Delta_Y$, $\rho^c \setminus \Delta_Y$, and $(\rho^{-1})^c \setminus \Delta_Y$.

§4. Orbits of reversible ultrahomogeneous structures. For a relational structure $\mathbb{Y}$ by $\mathrm{Pi}(\mathbb{Y})$ we denote the set of all isomorphisms between its finite substructures. We recall that $\mathbb{Y}$ is called ultrahomogeneous iff each finite partial isomorphism $p \in \mathrm{Pi}(\mathbb{Y})$ can be extended to an automorphism of $\mathbb{Y}$ (for more information on this topic we recommend the survey [18]). It is known that the set of orbits of the automorphism group $\mathrm{Aut}\,\mathbb{Y}$ on Y^n, that is, the collection of the sets $\mathrm{orb}_{\mathrm{Aut}\,\mathbb{Y}}(\bar x) := \{\bar y \in Y^n : \exists f \in \mathrm{Aut}\,\mathbb{Y}\ \bar y = f\bar x\}$, where $\bar x = \langle x_1, \dots, x_n\rangle \in Y^n$, is a finite partition of the set Y^n.

In the sequel we assume that $\mathbb{Y} = \langle Y, \rho\rangle$ is a reversible ultrahomogeneous structure, where $L_{\mathbb{Y}} = \langle P\rangle$ and $\mathrm{ar}(P) = m$.

FACT 4.1. *For each n-tuple $\bar x = \langle x_1, \dots, x_n\rangle \in Y^n$ there are unique functions $a : \{1, \dots, n\}^2 \to \{0, 1\}$ and $b : \{1, \dots, n\}^m \to \{0, 1\}$ such that for the formula*

$$\varepsilon_{ab}(v_1, \dots, v_n) := \bigwedge_{\langle i,j\rangle \in \{1,\dots,n\}^2} (v_i = v_j)^{a_{ij}} \wedge \bigwedge_{\langle i_1,\dots,i_m\rangle \in \{1,\dots,n\}^m} P(v_{i_1}, \dots, v_{i_m})^{b_{i_1\dots i_m}}, \tag{26}$$

where, by definition, $\varphi^0 := \varphi$ and $\varphi^1 := \neg\varphi$, we have

$$\bar x \in D_{\varepsilon_{ab}} := \{\langle y_1, \dots, y_n\rangle \in Y^n : \mathbb{Y} \models \varepsilon_{ab}[y_1, \dots, y_n]\} = \mathrm{orb}_{\mathrm{Aut}\,\mathbb{Y}}(\bar x).$$

PROOF. Defining $a_{ij} = 0 \Leftrightarrow x_i = x_j$ and $b_{i_1\dots i_m} = 0 \Leftrightarrow \langle x_{i_1}, \dots, x_{i_m}\rangle \in \rho$, we have $\mathbb{Y} \models \varepsilon_{ab}[\bar x]$, that is $\bar x \in D_{\varepsilon_{ab}}$. The uniqueness of a and b is clear.

If $\bar y \in \mathrm{orb}^n_{\mathrm{Aut}\,\mathbb{Y}}(\bar x)$, then there is $f \in \mathrm{Aut}\,\mathbb{Y}$ such that $\bar y = f\bar x$ and, since $\mathbb{Y} \models \varepsilon_{ab}[\bar x]$, we have $\mathbb{Y} \models \varepsilon_{ab}[f\bar x]$, that is, $\mathbb{Y} \models \varepsilon_{ab}[\bar y]$; thus $\bar y \in D_{\varepsilon_{ab}}$. If $\bar y \in D_{\varepsilon_{ab}}$,

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then $\mathbb{Y} \models \varepsilon_{ab}[\bar{y}]$, so, since $\mathbb{Y} \models \varepsilon_{ab}[\bar{x}]$, defining $p = \{\langle x_i, y_i\rangle : i \leq n\}$ we have $x_i = x_j \Leftrightarrow p(x_i) = p(x_j)$ and $\langle x_{i_1}, \ldots, x_{i_m}\rangle \in \rho \Leftrightarrow \langle p(x_{i_1}), \ldots, p(x_{i_m})\rangle \in \rho$ which means that $p \in \mathrm{Pi}(\mathbb{Y})$. By the ultrahomogeneity of $\mathbb{Y}$ there is an automorphism $f \in \mathrm{Aut}\,\mathbb{Y}$ such that $p \subset f$. So $\bar{y} = f\bar{x} \in \mathrm{orb}^n_{\mathrm{Aut}\,\mathbb{Y}}(\bar{x})$. ⊣

For $l \leq k \in \mathbb{N}$, let $\Delta^{<l}_{Y^k} := \{\langle x_1, \ldots, x_k\rangle \in Y^k : |\{x_1, \ldots, x_k\}| < l\}$.

THEOREM 4.2. *If $\mathbb{Y} = \langle Y, \rho\rangle$ is a reversible ultrahomogeneous structure, where*

(i) $\rho \subset Y^m \setminus \Delta^{<m}_{Y^m}$ *and*
(ii) $\forall \tilde{x}, \tilde{y} \in \rho\ \ \{\langle x_i, y_i\rangle : i \leq m\} \in \mathrm{Pi}(\mathbb{Y})$,

then, for each $n \in \mathbb{N}$, each orbit $\mathrm{orb}_{\mathrm{Aut}\,\mathbb{Y}}(\bar{x}) = D_{\varepsilon_{ab}} \subset Y^n$ such that $\bar{x} \in \Delta^{<m}_{Y^n}$ or $\varepsilon_{ab} \notin \mathcal{N}$, is a reversible n-ary relation on Y.

PROOF. If $\bar{x} \in \Delta^{<m}_{Y^n}$, that is, $|\{x_1, \ldots, x_n\}| < m$, then, since $\rho \subset Y^m \setminus \Delta^{<m}_{Y^m}$ and $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{x}]$ we have $\varepsilon_{ab} = \theta \wedge \eta$, where $\theta := \bigwedge_{\langle i,j\rangle \in \{1,\ldots,n\}^2} (v_i = v_j)^{a_{ij}}$ and $\eta(v_1, \ldots, v_n) := \bigwedge_{\langle i_1,\ldots,i_m\rangle \in \{1,\ldots,n\}^m} \neg P(v_{i_1}, \ldots, v_{i_m})$. Thus, the formula ε_{ab} is, modulo $\mathrm{Th}(\mathbb{Y})$, equivalent to the $L_\emptyset$-formula θ and, by Fact 2.1, the relation $\mathrm{orb}_{\mathrm{Aut}_{\mathbb{Y}}}(\bar{x}) = D_{\varepsilon_{ab}}$ is (strongly) reversible.

If $\varepsilon_{ab} \notin \mathcal{N}$ there is a tuple $\langle i_1, \ldots, i_m\rangle \in \{1, \ldots, n\}^m$ such that $P(v_{i_1}, \ldots, v_{i_m})$ is a subformula of the formula ε_{ab}. Since $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{x}]$ we have $\langle x_{i_1}, \ldots, x_{i_m}\rangle \in \rho$ and, by (i), for each $r < s \leq m$ we have $x_{i_r} \neq x_{i_s}$ and, hence, $i_r \neq i_s$. Let $\pi \in \mathrm{Sym}(m)$, where $i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(m)}$. Let $\bar{v}^1 = \langle v_1, \ldots, v_{i_{\pi(1)}-1}\rangle$, $\bar{v}^k = \langle v_{i_{\pi(k-1)}+1}, \ldots, v_{i_{\pi(k)}-1}\rangle$, for $2 \leq k \leq m$, and $\bar{v}^{m+1} := \langle v_{i_{\pi(m)}+1}, \ldots, v_n\rangle$ be (possibly empty) tuples of variables and let the formula $\psi_P(v_{i_1}, \ldots, v_{i_m}) \in \mathrm{Form}_{\langle R\rangle}$ be defined by

$$\psi_P(v_{i_1}, \ldots, v_{i_m}) := \exists \bar{v}^1, \ldots, \bar{v}^{m+1}\ \ R(\bar{v}^1, v_{i_{\pi(1)}}, \bar{v}^2, v_{i_{\pi(2)}}, \ldots, v_{i_{\pi(m)}}, \bar{v}^{m+1}).$$

We prove that

$$\forall \tilde{y} \in Y^m\ \ \Big(\tilde{y} \in \rho \Leftrightarrow \langle Y, D_{\varepsilon_{ab}}\rangle \models \psi_P[\tilde{y}]\Big). \tag{27}$$

Let $\tilde{y} = \langle y_1, \ldots, y_m\rangle \in Y^m$. Then $\langle Y, D_{\varepsilon_{ab}}\rangle \models \psi_P[\tilde{y}]$ if and only if there are (possibly empty) tuples $\bar{z}^1 := \langle z_1, \ldots, z_{i_{\pi(1)}-1}\rangle$, $\bar{z}^k = \langle z_{i_{\pi(k-1)}+1}, \ldots, z_{i_{\pi(k)}-1}\rangle$, for $2 \leq k \leq m$, and $\bar{z}^{m+1} := \langle z_{i_{\pi(m)}+1}, \ldots, z_n\rangle$ of elements of Y such that $\langle Y, D_{\varepsilon_{ab}}\rangle \models R[\bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}]$, which means that the n-tuple $\langle \bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}\rangle$ belongs to the set $D_{\varepsilon_{ab}}$, that is, in other words, $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}]$. Thus we show that

$$\tilde{y} \in \rho \Leftrightarrow \exists \bar{z}^1, \ldots, \bar{z}^{m+1} \in Y\ \ \langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}].$$

If $\tilde{y} = \langle y_1, \ldots, y_m\rangle \in \rho$, then, by (ii) and since $\langle x_{i_1}, \ldots, x_{i_m}\rangle \in \rho$, the mapping $p := \{\langle x_{i_k}, y_k\rangle : k \leq m\}$ is a partial isomorphism and, since the structure $\langle Y, \rho\rangle$ is ultrahomogeneous, there is an automorphism $f \in \mathrm{Aut}\langle Y, \rho\rangle$ such that $f(x_{i_k}) = y_k$, for all $k \leq m$. Now, since $\bar{x} \in D_{\varepsilon_{ab}}$, defining tuples $\bar{x}^1 := \langle x_1, \ldots, x_{i_{\pi(1)}-1}\rangle$, $\bar{x}^k = \langle x_{i_{\pi(k-1)}+1}, \ldots, x_{i_{\pi(k)}-1}\rangle$, for $2 \leq k \leq m$, and $\bar{x}^{m+1} := \langle x_{i_{\pi(m)}+1}, \ldots, x_n\rangle$ we have $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{x}^1, x_{i_{\pi(1)}}, \bar{x}^2, x_{i_{\pi(2)}}, \ldots, x_{i_{\pi(m)}}, \bar{x}^{m+1}]$ and, hence, $\langle Y, \rho\rangle \models \varepsilon_{ab}[f\bar{x}^1, f(x_{i_{\pi(1)}}), f\bar{x}^2, f(x_{i_{\pi(2)}}), \ldots, f(x_{i_{\pi(m)}}), f\bar{x}^{m+1}]$ so, defining $\bar{z}^1 := f\bar{x}^1$, $\bar{z}^k := f\bar{x}^k$, for $2 \leq k \leq m$, and $\bar{z}^{m+1} := f\bar{x}^{m+1}$, we have $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}]$.

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Conversely, let $\langle Y,\rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}]$, where, as above, $\bar{z}^1,\ldots,\bar{z}^{m+1} \in Y$. Then $\langle \bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}\rangle \in D_{\varepsilon_{ab}}$ and since $\bar{x} \in D_{\varepsilon_{ab}}$ and, by Fact 4.1 we have $D_{\varepsilon_{ab}} = \mathrm{orb}^n_{\mathrm{Aut}\langle Y,\rho\rangle}(\bar{x})$, there is $f \in \mathrm{Aut}\langle Y,\rho\rangle$ such that $f\bar{x} = \langle \bar{z}^1, y_{\pi(1)}, \bar{z}^2, y_{\pi(2)}, \ldots, y_{\pi(m)}, \bar{z}^{m+1}\rangle$ and, hence, $f(x_{i_{\pi(k)}}) = y_{\pi(k)}$, for $k \le m$, which means that $f(x_{i_k}) = y_k$, for $k \le m$. So, since $\langle x_{i_1},\ldots,x_{i_m}\rangle \in \rho$, we have $\tilde{y} \in \rho$.

By (27), the relation ρ is definable in the structure $\langle Y, D_{\varepsilon_{ab}}\rangle$ by the R-positive $\langle R\rangle$-formula ψ_P and, by Corollary 3.5, $\langle Y, D_{\varepsilon_{ab}}\rangle$ is a reversible structure. ⊣

COROLLARY 4.3. *If $\mathbb{Y} = \langle Y,\to\rangle$ is a reversible ultrahomogeneous digraph, then each orbit of* Aut $\mathbb{Y}$ *defined by a formula $\varepsilon_{ab} \notin \mathcal{N}$ is a reversible relation on Y.*

PROOF. We show that all the assumptions of Theorem 4.2 are satisfied. By the irreflexivity of $\to$ we have $\to \subset Y^2 \setminus \Delta^{<2}_{Y^2}$, so (i) is true. If $x_1 \to x_2$ and $y_1 \to y_2$ then $\{\langle x_1,y_1\rangle, \langle x_2,y_2\rangle\} \in \mathrm{Pi}(\mathbb{Y})$ thus (ii) is true as well. ⊣

EXAMPLE 4.4. The assumption $\varepsilon_{ab} \notin \mathcal{N}$ can not be removed from Corollary 4.3. If $\mathcal{T}$ is a set of finite tournaments of size ≥ 3, which are pairwise incomparable under embedding, then the class $\mathcal{C}_{\mathcal{T}}$ of all finite digraphs which do not embed any member of $\mathcal{T}$ is a free amalgamation class and its Fraïssé limit $\mathbb{Y}_{\mathcal{T}} = \langle Y,\to\rangle$ is an ultrahomogeneous digraph. Such digraphs are called the *Henson digraphs* and, using Fact 2.1(b), we show first that all of them are reversible structures. On the contrary, suppose that $\to \subset \rho$, $\langle a,b\rangle \in \rho \setminus \to$ and $\langle Y,\to\rangle \cong \langle Y,\rho\rangle$. Then, clearly,

$$a \ne b \wedge a \not\to b \wedge b \not\to a. \tag{28}$$

Let $\mathbb{T} = \langle T,\tau\rangle$ be an element of $\mathcal{T}$ of minimal size, say $T = \{t_1,\ldots,t_n\}$, and w.l.o.g. suppose that $\langle t_1,t_2\rangle \in \tau$. Then $\mathbb{T}' = \langle T,\tau'\rangle$, where $\tau' = \tau \setminus \{\langle t_1,t_2\rangle\}$, is a digraph, $t_1 \ne t_2$ and $\langle t_2,t_1\rangle \notin \tau'$. In addition, $\mathbb{T}'$ does not contain any copy of any member of $\mathcal{T}$ and, hence, there is an embedding $f : \mathbb{T}' \hookrightarrow \mathbb{Y}_{\mathcal{T}}$, which implies

$$f(t_1) \ne f(t_2) \wedge f(t_1) \not\to f(t_2) \wedge f(t_2) \not\to f(t_1). \tag{29}$$

By (28) and (29) we have $p := \{\langle f(t_1),a\rangle, \langle f(t_2),b\rangle\} \in \mathrm{Pi}(\mathbb{Y}_{\mathcal{T}})$ and, since the structure $\mathbb{Y}_{\mathcal{T}}$ is ultrahomogeneous, there is $F \in \mathrm{Aut}(\mathbb{Y}_{\mathcal{T}})$ such that $p \subset F$. Now $\langle F[f[T]], \to\restriction F[f[T]]\rangle$ is a copy of $\mathbb{T}'$ in $\mathbb{Y}_{\mathcal{T}}$ and, since $\langle a,b\rangle \in \rho$, the structure $\langle F[f[T]], \rho \restriction F[f[T]]\rangle$ is a copy of $\mathbb{T}$ in $\langle Y,\rho\rangle \cong \mathbb{Y}_{\mathcal{T}}$, which is impossible.

In [7] Henson constructed a countable family $\mathcal{T}^*$ of finite tournaments of size ≥ 7, which are pairwise incomparable under embedding, and such that for each subset $\mathcal{T}$ of $\mathcal{T}^*$, the corresponding ultrahomogeneous digraph $\mathbb{Y}_{\mathcal{T}} = \langle Y,\to\rangle$ has the additional property—that its symmetrization, $(\mathbb{Y}_{\mathcal{T}})_s = \langle Y, \to \cup \to^{-1}\rangle$, is isomorphic to the Rado graph (the countable homogeneous universal graph), which is not reversible, since deleting an edge produces a graph isomorphic to the initial graph (see [3]). Now, let $\bar{x} = \langle x_1,x_2\rangle \in Y^2$, where $x_1 \ne x_2$, $x_1 \not\to x_2$, and $x_2 \not\to x_1$. Then $\mathrm{orb}_{\mathrm{Aut}\,\mathbb{Y}_{\mathcal{T}}}(\bar{x}) = D_{\varepsilon_{ab}}$, where the formula $\varepsilon_{ab} \in \mathcal{N}$ is given by

$$\varepsilon_{ab} := v_1 \ne v_2 \wedge \neg P(v_1,v_2) \wedge \neg P(v_2,v_1).$$

Also the binary relation $\mathrm{orb}_{\mathrm{Aut}\,\mathbb{Y}_{\mathcal{T}}}(\bar{x}) = Y^2 \setminus (\Delta_Y \cup \to \cup \to^{-1})$ is the graph-complement of the relation of the Rado graph, which is isomorphic to the Rado graph and, hence, not reversible.

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This example also shows that the union of two reversible orbits of a reversible ultrahomogeneous structure can be a nonreversible relation.

THEOREM 4.5. *If $\mathbb{Y} = \langle Y, \rho\rangle$ is a reversible ultrahomogeneous structure, where*

(i) $\rho \subset Y^m \setminus \Delta^{<m}_{Y^m}$,
(ii) $\forall \tilde{x}, \tilde{y} \in \rho \ \{\langle x_i, y_i\rangle : i \leq m\} \in \mathrm{Pi}(\mathbb{Y})$,
(iii) $\forall \tilde{x}, \tilde{y} \in \rho^c \setminus \Delta^{<m}_{Y^m} \ \{\langle x_i, y_i\rangle : i \leq m\} \in \mathrm{Pi}(\mathbb{Y})$,

then all orbits of $\mathrm{Aut}\,\mathbb{Y}$ *are reversible relations.*

PROOF. Let $\bar{x} = \langle x_1, \dots, x_n\rangle \in Y^n$ and let $\varepsilon_{ab}(v_1, \dots, v_n)$ be the $L_{\mathbb{Y}}$-formula of the form (26) such that $\mathrm{orb}_{\mathrm{Aut}_{\mathbb{Y}}}(\bar{x}) = D_{\varepsilon_{ab}}$ (see Fact 4.1).

If $\bar{x} \in \Delta^{<m}_{Y^n}$ or $\varepsilon_{ab} \notin \mathcal{N}$ we apply Theorem 4.2.

If $\bar{x} \in Y^n \setminus \Delta^{<m}_{Y^n}$ and $\varepsilon_{ab} \in \mathcal{N}$, then $n \geq m$ and, using Corollary 3.5 we show that the $\langle R\rangle$-structure $\langle Y, D_{\varepsilon_{ab}}\rangle$ is reversible. Since $\bar{x} \in Y^n \setminus \Delta^{<m}_{Y^n}$, there are $i_1 < \cdots < i_m \leq n$ such that $|\{x_{i_1}, \dots, x_{i_m}\}| = m$ and $\varepsilon_{ab} \in \mathcal{N}$ implies that $\neg P(v_{i_1}, \dots, v_{i_m})$ is a subformula of $\varepsilon_{ab}(\bar{v})$. So, since $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{x}]$ we have $\langle x_{i_1}, \dots, x_{i_m}\rangle \in \rho^c$ and thus

$$\tilde{x} := \langle x_{i_1}, \dots, x_{i_m}\rangle \in \rho^c \setminus \Delta^{<m}_{Y^m}. \tag{30}$$

Let $\bar{v}^1 = \langle v_1, \dots, v_{i_1-1}\rangle$, $\bar{v}^k = \langle v_{i_{k-1}+1}, \dots, v_{i_k-1}\rangle$, for $2 \leq k \leq m$, and $\bar{v}^{m+1} := \langle v_{i_m+1}, \dots, v_n\rangle$ be (possibly empty) tuples of variables. We prove that

$$\forall \tilde{y} \in Y^m \ \Big(\tilde{y} \in \rho \Leftrightarrow \langle Y, D_{\varepsilon_{ab}}\rangle \models \psi_P[\tilde{y}]\Big), \tag{31}$$

where the formula $\psi_P \in \mathrm{Form}_{\langle R\rangle}$ is defined by

$$\psi_P(v_{i_1}, \dots, v_{i_m}) := \bigwedge\nolimits_{1\leq r<s\leq m} v_{i_r} \neq v_{i_s} \\ \wedge \ \forall \bar{v}^1, \dots, \bar{v}^{m+1} \ \neg R(\bar{v}^1, v_{i_1}, \bar{v}^2, v_{i_2}, \dots, v_{i_m}, \bar{v}^{m+1}).$$

Let $\tilde{y} = \langle y_1, \dots, y_m\rangle \in Y^m$. Then $\langle Y, D_{\varepsilon_{ab}}\rangle \not\models \psi_P[\tilde{y}]$ iff $y_r = y_s$, for some $r \neq s$, or for some (possibly empty) tuples $\bar{z}^1 = \langle z_1, \dots, z_{i_1-1}\rangle$, $\bar{z}^k = \langle z_{i_{k-1}+1}, \dots, z_{i_k-1}\rangle$, for $2 \leq k \leq m$, and $\bar{z}^{m+1} := \langle z_{i_m+1}, \dots, z_n\rangle$ in Y we have that $\langle Y, D_{\varepsilon_{ab}}\rangle \models R[\bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}]$, that is $\langle \bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}\rangle \in D_{\varepsilon_{ab}}$, which means that $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}]$. Thus we prove that

$$\tilde{y} \in \rho^c \Leftrightarrow \bigvee\nolimits_{1\leq r<s\leq m} y_r = y_s \\ \vee \ \exists \bar{z}^1, \dots, \bar{z}^{m+1} \in Y \ \langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}].$$

Let $\tilde{y} = \langle y_1, \dots, y_m\rangle \in \rho^c$. If $y_r = y_s$, for some $r \neq s$, we are done. Otherwise we have $\tilde{y} \in \rho^c \setminus \Delta^{<m}_{Y^m}$ and, by (30) and (iii), $p := \{\langle x_{i_k}, y_k\rangle : k \leq m\}$ is a partial isomorphism. So, since the structure $\langle Y, \rho\rangle$ is ultrahomogeneous, there is an automorphism $f \in \mathrm{Aut}\langle Y, \rho\rangle$ such that $f(x_{i_k}) = y_k$, for all $k \leq m$. Now, since $\bar{x} \in D_{\varepsilon_{ab}}$, defining $\bar{x}^1 := \langle x_1, \dots, x_{i_1-1}\rangle$, $\bar{x}^k = \langle x_{i_{k-1}+1}, \dots, x_{i_k-1}\rangle$, for $2 \leq k \leq m$, and $\bar{x}^{m+1} := \langle x_{i_m+1}, \dots, x_n\rangle$ we have $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{x}^1, x_{i_1}, \bar{x}^2, x_{i_2}, \dots, x_{i_m}, \bar{x}^{m+1}]$ and, hence, $\langle Y, \rho\rangle \models \varepsilon_{ab}[f\bar{x}^1, f(x_{i_1}), f\bar{x}^2, f(x_{i_2}), \dots, f(x_{i_m}), f\bar{x}^{m+1}]$ so, defining $\bar{z}^1 := f\bar{x}^1$, $\bar{z}^k := f\bar{x}^k$, for $2 \leq k \leq m$, and $\bar{z}^{m+1} := f\bar{x}^{m+1}$, we have $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}]$.

Conversely, if $y_r = y_s$, for some $r \neq s$, then $\tilde{y} \in \rho^c$, because $\rho \cap \Delta^{<m}_{Y^m} = \emptyset$. If $\langle Y, \rho\rangle \models \varepsilon_{ab}[\bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}]$, for some $\bar{z}^1, \dots, \bar{z}^{m+1} \in Y$, then $\langle \bar{z}^1, y_1, \bar{z}^2, y_2, \dots, y_m, \bar{z}^{m+1}\rangle \in D_{\varepsilon_{ab}}$ and since $\bar{x} \in D_{\varepsilon_{ab}}$ and, by Fact 4.1

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we have $D_{\varepsilon_{ab}} = \mathrm{orb}^n_{\mathrm{Aut}\langle Y,\rho\rangle}(\bar{x})$, there is $f \in \mathrm{Aut}\langle Y,\rho\rangle$ such that $f\bar{x} = \langle \bar{z}^1, y_1, \bar{z}^2, y_2, \ldots, y_m, \bar{z}^{m+1}\rangle$ and, hence, $f(x_{i_k}) = y_k$, for $k \leq m$, So, since $\langle x_{i_1}, \ldots, x_{i_m}\rangle \in \rho^c$, we have $\tilde{y} \in \rho^c$.

By (31), the relation ρ is definable in the structure $\langle Y, D_{\varepsilon_{ab}}\rangle$ by the R-negative $\langle R\rangle$-formula ψ_P and, by Corollary 3.5, $\langle Y, D_{\varepsilon_{ab}}\rangle$ is a reversible structure. ⊣

Corollary 4.6. *If $\mathbb{Y} = \langle Y, \to\rangle$ is an ultrahomogeneous tournament, then each orbit of* Aut $\mathbb{Y}$ *is a reversible relation on Y.*

Proof. We show that the structure $\mathbb{Y}$ satisfies all the assumptions of Theorem 4.5. First, by Fact 2.1(b), each tournament is a reversible structure. By the irreflexivity of $\to$ we have $\to \subset Y^2 \setminus \Delta^{<2}_{Y^2}$, so (i) is true. If $x_1 \to x_2$ and $y_1 \to y_2$ then $\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\} \in \mathrm{Pi}(\mathbb{Y})$ thus (ii) is satisfied. If $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in {\to}^c \setminus \Delta_Y$, then $x_1 \neq x_2$, $y_1 \neq y_2$, $x_1 \not\to x_2$ and $y_1 \not\to y_2$, which implies that $x_2 \to x_1$ and $y_2 \to y_1$ thus, again, $\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\} \in \mathrm{Pi}(\mathbb{Y})$ and (iii) is true as well. ⊣

Example 4.7. By the well known classification of Lachlan [15], each countable ultrahomogeneous tournament is isomorphic to one of the following three:

- $\mathbb{Q} = \langle Q, <\rangle$, the rational line,
- $\mathbb{T}^\infty$, the ω-universal-homogeneous tournament (the random tournament),
- $\mathbb{Q}^* = \langle D, \to\rangle$, the local order (the circular tournament), where D is a countable dense set of points of the unit circle, no two antipodal, and $x \to y$ iff $0 < \arg(x, y) < \pi$.

By Corollary 4.6, if $\mathbb{Y}$ is any of the countable ultrahomogeneous tournaments listed above, then for each $n \in \mathbb{N}$, each orbit of Aut $\mathbb{Y}$ on Y^n is a reversible relation on Y. By [10], all $\emptyset$-definable relations in a monomorphic structure with quantifier elimination are reversible so, if $\mathbb{Y} = \mathbb{Q}$, then all relations definable in $\mathbb{Y}$ are reversible. If $\mathbb{Y} = \mathbb{T}^\infty$ or $\mathbb{Y} = \mathbb{Q}^*$, then the structure $\mathbb{Y}$ is not 3-monomorphic (since it contains copies of the 3-cycle tournament, C_3, and copies of the 3-element linear order, L_3) so, the results of [10] can not be applied when we consider definable n-ary relations for $n \geq 3$. But, for example, by Corollary 4.6, for the formula $\varepsilon(v_1, v_2, v_3) := P(v_1, v_2) \wedge P(v_2, v_3) \wedge P(v_3, v_1)$, the structure $\langle Y, D_\varepsilon\rangle$, where

$$D_\varepsilon := \left\{\langle x_1, x_2, x_3\rangle \in Y^3 : x_1 \to x_2 \to x_3 \to x_1\right\}$$

is reversible. (D_ε is the set of triples determined by the copies of C_3.) In a similar way we obtain the reversibility of the n-ary relation $D_{\varepsilon'}$ on Y consisting of the n-tuples which are linearly ordered by the initial tournament relation $\to$,

$$D_{\varepsilon'} := \left\{\langle x_1, \ldots, x_n\rangle \in Y^n : \textstyle\bigwedge_{1\leq i<j\leq n} x_i \to x_j\right\}.$$

A structure $\mathbb{Y} = \langle Y, \rho\rangle$ is an m-uniform hypergraph iff $\rho \subset Y^m$ and for each $\langle y_1, \ldots y_n\rangle \in \rho$ we have (I) $y_r \neq y_s$, for $r \neq s$, and (S) $\langle y_{\pi(1)}, \ldots y_{\pi(n)}\rangle \in \rho$, for each $\pi \in \mathrm{Sym}(n)$.

Corollary 4.8. *If $\mathbb{Y} = \langle Y, \rho\rangle$ is a reversible ultrahomogeneous m-uniform hypergraph, then each orbit of* Aut $\mathbb{Y}$ *is a reversible relation on Y.*

Proof. We show that the structure $\mathbb{Y}$ satisfies all the assumptions of Theorem 4.5. By (I) we have $\rho \subset Y^m \setminus \Delta^{<m}_{Y^m}$, so (i) is true.

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If $\langle x_1, \dots, x_m\rangle, \langle y_1, \dots, y_m\rangle \in \rho$, then by (I) for the sets $A := \{x_1, \dots, x_m\}$ and $B := \{y_1, \dots, y_m\}$ we have $A, B \in [Y]^m$ and, by (S), $\rho \cap A^m = A^m \setminus \Delta^{<m}_{A^m}$ and $\rho \cap B^m = B^m \setminus \Delta^{<m}_{B^m}$; thus $\{\langle x_i, y_i\rangle : i \leq m\} \in \mathrm{Pi}(\mathbb{Y})$ and (ii) is satisfied.

If $\langle x_1, \dots, x_m\rangle, \langle y_1, \dots, y_m\rangle \in \rho^c \setminus \Delta^{<m}_{Y^m}$, then for the sets $A := \{x_1, \dots, x_m\}$ and $B := \{y_1, \dots, y_m\}$ we have $A, B \in [Y]^m$ and, by (S), $\rho \cap A^m = \rho \cap B^m = \emptyset$; thus $\{\langle x_i, y_i\rangle : i \leq m\} \in \mathrm{Pi}(\mathbb{Y})$ again and (iii) is true. ⊣

EXAMPLE 4.9. Clearly, graphs are 2-uniform hypergraphs and, by the well known theorem of Lachlan and Woodrow [16], each countable ultrahomogeneous graph is isomorphic to one of the following:

- $\mathbb{G}_{\mu\nu}$, the union of μ disjoint copies of $\mathbb{K}_\nu$, where $\mu\nu = \omega$,
- $\mathbb{G}_{\mathrm{Rado}}$, the Rado graph, the countable homogeneous universal graph,
- $\mathbb{H}_n$, the Henson graph, the countable homogeneous universal $\mathbb{K}_n$-free graph, for $n \geq 3$,
- the graph-complements of these graphs.

It is easy to check (see [12]) that the reversible countable ultrahomogeneous graphs are $\mathbb{G}_{\mu\nu}$, for $\mu < \omega$ or $\nu < \omega$, the Henson graphs $\mathbb{H}_n$, $n \geq 3$, and the complements of these graphs, and, by Corollary 4.8, all their orbits are reversible relations. For example if $\mathbb{H}_3 = \langle X, \sim\rangle$ is the Henson triangle-free graph, then the set of "empty triangles", $\rho := \{\langle x_1, x_2, x_3\rangle \in X^3 : x_1 \neq x_2 \neq x_3 \neq x_1 \wedge x_1 \not\sim x_2 \not\sim x_3 \not\sim x_1\}$, is a reversible ternary relation on the set X, etc.

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