Math. Proc. Camb. Phil. Soc. (2004), **136**, 75 © 2004 *Cambridge Philosophical Society*
DOI: 10.1017/S0305004103006996 *Printed in the United Kingdom*

Projective unitary groups are totally N-determined p-compact groups

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(*Received* 20 *December* 2000; *revised* 24 *June* 2002)

Abstract

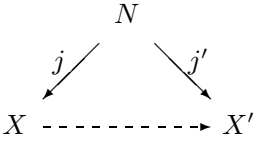
We show that projective unitary groups, or more generally, central quotients of products of unitary groups are totally N-determined when considered as p-compact groups at odd primes p.

1. *Introduction*

A finite loop space X consists of a pair of spaces (X, BX) together with a homotopy equivalence $X \simeq \Omega BX$ where X has the homotopy type of a finite CW-complex. The concept was introduced in the early 1970s by Rector as a homotopy theoretic substitute for the concept of compact Lie group. Indeed, any compact Lie group G together with its classifying space is a finite loop space. The concept of p-compact group has been introduced by W. G. Dwyer and C. W. Wilkerson [**5**] as the p-local analogue of finite loop space. A p-compact group X is a pair of spaces (X, BX) where $X \simeq \Omega BX$ has finite mod p homology and BX is p-complete in the sense of Bousfield–Kan [**1**]. Again the p-completion of a compact connected Lie group G and of its classifying space gives rise to a p-compact group $(G^\wedge_p, BG^\wedge_p)$. Among classical concepts that extend naturally to this new setting there are those of maximal torus, Weyl groups, normalizer of maximal torus or centralizers [**5**].

The concept of N-determinism was introduced by Møller [**15**] (see also [**19**]).

Definition 1·1. Let X be a p-compact group and $j\colon N \longrightarrow X$ its maximal torus normalizer. X is N-determined if any diagram



† Partially supported by DGES grant PB97-0203.
‡ Partially supported by DGES grant PB97-1095.

https://doi.org/10.1017/S0305004103006996 Published online by Cambridge University Press

where X' is another p-compact group with isomorphic maximal torus normalizer $j'\colon N \longrightarrow X'$, might be closed by an isomorphism of p-compact groups $X \longrightarrow X'$.

This essentially means that X is determined by its maximal torus normalizer in the sense that any other p-compact group with isomorphic maximal torus normalizer is actually isomorphic to X.

For a p-compact group X we denote $\mathrm{Out}\, X$ the group of homotopy classes of self homotopy equivalences of BX. Restriction to the maximal torus normalizer $j\colon N \longrightarrow X$ induces a homomorphism $\mathrm{Out}(X) \longrightarrow \mathrm{Out}(N)$ (see [**15**]).

Definition 1·2. A p-compact group X with maximal torus normalizer $j\colon N \longrightarrow X$ has N-determined automorphisms if $\mathrm{Out}(X) \longrightarrow \mathrm{Out}(N)$ is injective. A p-compact group is totally N-determined if it is N-determined and has N-determined automorphisms.

Conjecturally all p-compact groups are totally N-determined at odd primes. Known cases include torsion free p-compact groups, p odd [**4**, **17**]. Notice that there exist easy counterexamples at the prime two. For instance, $O(2)$ and $PU(2)$ are non isomorphic compact Lie groups or 2-compact groups but they have isomorphic maximal torus normalizer.

The aim of this paper is to show that the projective unitary groups $PU(n)$ considered as p-compact groups are N-determined at odd primes, so completing partial results that have appeared in [**2**, **14**, **22**]. Actually, this is to say that they are totally N-determined because all compact connected Lie groups have N-determined automorphisms according to results of Jackowski, McClure and Oliver [**12**].

THEOREM 1·3. *The p-compact group $PU(n)^\wedge_p$ is totally N-determined for any odd prime p.*

COROLLARY 1·4. *Let G be a central quotient of a direct product of a finite number of unitary groups, then $G^\wedge_p$ is totally N-determined for any odd prime p.*

Our method is mainly based on work of Møller [**15**, **16**] that is combined with the idea of [**2**] of using the Bockstein spectral sequence in order to determine the fundamental group of an adjoint form. In Section 2 we review the theory of preferred lifts due to Møller that provide some geometrical technical conditions on a compact connected Lie group for its mod p cohomology ring to be determined by the maximal torus normalizer (Proposition 2·4). Then it is shown in Theorem 2·7 that under such conditions $G^\wedge_p$ is N-determined if its universal cover $\tilde{G}^\wedge_p$ is N-determined. This result might be understood as reciprocal to [**15**, proposition 6·6]. The technical conditions we have referred to concern the rank one and rank two elementary abelian p-subgroups. In Section 3 we determine the different conjugacy classes of rank one and rank two elementary abelian p-subgroups of $PU(n)$ and its centralizers. We prove Theorem 1·3 and Corollary 1·4 in Section 4.

2. *N-determinism*

The concept of N-determinism has been developed by Møller in [**15**]. We first review some general results.

PROPOSITION 2·1 (Møller). *Let p be an odd prime and X a p-compact group.*

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(1) *If the connected component of X is N-determined, then X itself is N-determined.*
(2) *Assume that X is connected and let $Z(X)$ be the center of X. If the adjoint form $X/Z(X)$ is N-determined, then X itself is N-determined.*

In view of results of Dwyer–Wilkerson [**7**] and Notbohm [**18**] it is easy to conclude that products of N-determined connected p-compact groups are N-determined.

Proposition 2·2. *Let p be an odd prime. If X and Y are connected and N-determined p-compact groups, then $X \times Y$ is also N-determined.*

The rest of this section reviews some of the tools involved in these results and shows that under certain technical conditions a converse of Proposition 2·1(2) holds.

One main tool in the development of p-compact group theory has been the existence of various homology decompositions. One of those, due to Jackowski and McClure [**11**] is indexed on the Quillen category. The Quillen category of a group G at a prime p, namely $\mathscr{A}_p(G)$, is defined as the category whose objects are pairs $(V, \alpha) \in \mathscr{A}\mathrm{b} \times \mathrm{Mono}(V, G)$ such that V is a non-trivial elementary abelian p-group, and with morphisms $\mathrm{Mor}_{\mathscr{A}_p(G)}((V_1, \alpha_1), (V_2, \alpha_2))$, the set of group homomorphisms $f : V_1 \to V_2$ such that $(V_1, \alpha_1) = (V_1, \alpha_2 f)$. We will simply denote by α an object (V, α) of the Quillen category. The group of automorphisms in the Quillen category of an object α is denoted $\mathrm{Aut}_{\mathscr{A}_p(G)}(\alpha)$.

Let X be a p-compact group and $j : N \longrightarrow X$ its maximal torus normalizer. An elementary abelian p-subgroup of X $\nu : E \rightarrowtail X$, always factors through N [**6**, 2·14]

$$\begin{array}{ccc} & & N \\ & \overset{\mu}{\nearrow} & \big\downarrow j \\ E & \xrightarrow[\nu]{} & X\,. \end{array}$$

And any $E \overset{\mu}{\rightarrowtail} N$ induces a homomorphism $E \xrightarrow{\pi_0(\mu)} \pi_0(N) = W(X)$, where $W(X)$ denotes the Weyl group of X.

Definition 2·3. In the above situation, μ is a preferred lift of ν if the homomorphism $C_N(\mu) \longrightarrow C_X(\nu)$ is the inclusion of a maximal torus. If in addition, $\pi_0(\mu)$ has non-trivial kernel of maximal order, we say that μ is a special preferred lift of ν.

In [**15**, **16**] it is proved that special preferred lifts exist for every elementary abelian p-subgroup.

We will be interested in the case of rank one and two elementary abelian p-subgroups, E^1 and E^2, respectively. In fact, centralizers of rank two elementary abelian p-groups detect the mod p cohomology (cf. [**20**]).

Proposition 2·4 (Møller). *Let X be a p-compact group with maximal torus normalizer $j : N \longrightarrow X$ and X' another p-compact group with isomorphic maximal torus normalizer $j' : N \longrightarrow X'$. Suppose that the centralizer in X of any rank one or two elementary abelian p-subgroup is totally N-determined and for any rank two non-toral elementary abelian p-subgroup of X, $\nu : E^2 \longrightarrow X$ the following hold*:

(1) *the conjugacy class of $j'\mu$ is the same for all μ, special preferred lift of ν, and*

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(2) *the conjugacy class of the isomorphism* $f(\mu) : C_X(\nu) \longrightarrow C_{X'}(j'\mu)$ *under* $C_N(\mu)$ *does not depend on the choice of* μ, *special preferred lift of* ν.

Then, the images of Bj^* *and* Bj'^* *in the diagram*

$$H^*(BX) \xrightarrow{Bj^*} H^*(BN) \xleftarrow{Bj'^*} H^*(BX')$$

coincide.

Proof. Let $\mathscr{A}_p(X)_{\leqslant r}$ denote the full subcategory of $\mathscr{A}_p(X)$ generated by all the objects of rank $\leqslant r$. According to Oliver's cochain complex [**20**], $H^*(BX)$ is determined by the functor

$$H^*(BC_X) : \mathscr{A}_p(X)_{\leqslant 2} \longrightarrow \mathscr{A}\mathrm{b}$$

that takes the object (V, ν) of $\mathscr{A}_p(X)_{\leqslant 2}$ to $H^*(BC_X(\nu))$. Using special preferred lifts, we shall construct a functor $\Phi : \mathscr{A}_p(X)_{\leqslant 2} \longrightarrow \mathscr{A}_p(X')_{\leqslant 2}$, and a natural isomorphism of functors $H^*(BC_X) \longrightarrow H^*(BC_{X'}) \circ \Phi$.

Let $\Phi : \mathscr{A}_p(X)_{\leqslant 2} \longrightarrow \mathscr{A}_p(X')_{\leqslant 2}$ be the functor defined as:

(a) $\Phi(V, \nu) = (V, j'\mu)$ where μ is any preferred lift of ν. According to the first assumption of the proposition, the object $\Phi(V, \nu)$ is well defined.

(b) $\Phi(\alpha) = \alpha$, for any morphism $\alpha : (V_1, \nu_1) \longrightarrow (V_2, \nu_2)$ in $\mathscr{A}_p(X)$. According to [**16**, 4·9], there exists a special preferred lift μ_2 of ν_2 such that $\mu_2\alpha$ is a preferred lift of $\nu_1 = \nu_2\alpha$. Thus α is a morphism in $\mathscr{A}_p(X')_{\leqslant 2}$ from $\Phi(V_1, \nu_1) = (V_1, j'\mu_2\alpha)$ to $\Phi(V_2, \nu_2) = (V_2, j'\mu_2)$, and $\Phi(\alpha) = \alpha$ is well defined.

Let now

$$f(\nu) : C_X(\nu) \longrightarrow C_{X'}(\Phi(\nu))$$

denote the isomorphism under $C_N(\mu)$ where μ is any special preferred lift of ν. According to the second assumption of the lemma, $f(\nu)$ does not depend on the choice made.

In particular, for any rank two object (V, ν) of $\mathscr{A}_p(X)$, and any rank one subgroup L of V, there is a commutative diagram

$$\begin{array}{ccc} C_X(\nu|_L) & \longleftarrow & C_X(\nu) \\ f(\nu|_L)\downarrow\cong & & \cong\downarrow f(\nu) \\ C_{X'}(\Phi(\nu|_L)) & \longleftarrow & C_{X'}(\nu') \end{array}$$

of conjugacy classes of p-compact groups morphisms, where $\nu' = \Phi(\nu)$. This implies that we have another commutative diagram

$$\begin{array}{ccc} \prod_{\lambda \in Ob(\mathscr{A}_p(X)_{\leqslant 1})} (H^*BC_X(\lambda))^{\mathrm{Aut}_{\mathscr{A}_p(X)}(\lambda)} & \xrightarrow{\delta_X(\nu)} & \mathrm{Hom}(\mathrm{St}_V, H^*BC_X(\nu)) \\ \prod Bf(\lambda)^*\uparrow\cong & & \cong\uparrow Bf(\nu)^* \\ \prod_{\lambda' \in Ob(\mathscr{A}_p(X')_{\leqslant 1})} (H^*BC_{X'}(\lambda'))^{\mathrm{Aut}_{\mathscr{A}_p(X')}(\lambda')} & \xrightarrow{\delta_{X'}(\nu')} & \mathrm{Hom}(\mathrm{St}_V, H^*BC_{X'}(\nu')) \end{array}$$

where $\delta_X(\nu)$ and $\delta_{X'}(\nu')$ are coboundary operators in Oliver's cochain complex, and St_V is the Steinberg representation of $\mathrm{GL}(V)$. Since any rank two object of $\mathscr{A}_p(X')$

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is in the image of Φ, the vertical isomorphism on the left restricts to an isomorphism

$$H^*BX = \bigcap_{\nu\in Ob(\mathscr{A}_p(X)_{\leqslant 2})} \operatorname{Ker}\delta_X(\nu) \xleftarrow{\cong} \bigcap_{\nu'\in Ob(\mathscr{A}_p(X')_{\leqslant 2})} \operatorname{Ker}\delta_{X'}(\nu') = BX'$$

between the 0-dimensional cohomology groups of the cochain complex. This isomorphism of unstable algebras over the Steenrod algebra can be seen [**15**, p. 275] to respect the inclusions H^*Bj' and H^*Bj into H^*BN.

Since the restriction to the normalizer is always a monomorphism in mod p cohomology, we have in particular that BX and BX' have isomorphic mod p cohomology as algebras over the Steenrod algebra. We are now interested in conditions under which some other invariants of X and X' are forced to coincide. The idea is to use the Bockstein spectral sequence in order to characterize the fundamental group.

We will first show how the fundamental group of compact connected Lie group G, $\pi_1(G) \cong H_2(BG, \mathbb{Z})$ is hidden in the homology of its maximal torus normalizer.

Proposition 2·5. *Let G be a compact connected Lie group with Weyl group W and maximal torus normalizer N. There are isomorphisms*

(1) $H_1(BN;\mathbb{Z}) \cong H_1(W,\mathbb{Z})$, *and*

(2) $H_2(BN;\mathbb{Z}) \cong H_2(BG;\mathbb{Z}) \oplus H_2(W;\mathbb{Z})$,

naturally induced by the usual maps $BN \longrightarrow BG$ and $BN \longrightarrow BW$.

Proof. We analyze the homology Serre spectral sequence for the fibration

$$BT \longrightarrow BN \longrightarrow BW$$

with integral coefficients in low degrees. One observes that

$$H_1(BN;\mathbb{Z}) \cong H_1(W;\mathbb{Z})$$

and there is an exact sequence of abelian groups

$$\begin{aligned} H_4(W;\mathbb{Z}) \xrightarrow{d} H_1(W;L) \longrightarrow H_3(BN;\mathbb{Z}) \longrightarrow \\ \longrightarrow H_3(W;\mathbb{Z}) \xrightarrow{d} H_0(W;L) \xrightarrow{\hat{j}_*} H_2(BN;\mathbb{Z}) \longrightarrow H_2(W;\mathbb{Z}) \longrightarrow 0 \end{aligned}$$

where d is the differential in the E_3 page of the spectral sequence. The map $\hat{j}_*$ is the factorization of the natural map

$$j_* : L \cong H_2(BT;\mathbb{Z}) \longrightarrow H_2(BN;\mathbb{Z})$$

through the covariants $L_W \cong H_0(W,L)$.

Notice that j_* factors through $H_*(BG;\mathbb{Z})$, too. In fact, if

$$\tau_* : H_*(BG;\mathbb{Z}) \longrightarrow H_*(BN;\mathbb{Z})$$

denotes the transfer map in homology, according to [**3**] or [**9**, example IV·4]) there is a diagram

$$\begin{array}{ccc} H_*(BT;\mathbb{Z}) & \xrightarrow{j_{1*}} & H_*(BG;\mathbb{Z}) \\ & \searrow^{j_*} & \downarrow{\scriptstyle \tau_*} \uparrow{\scriptstyle j_{2*}} \\ & & H_*(BN;\mathbb{Z}) \end{array}$$

where $\tau_* \circ j_{2*} = Id_{H_*(BG;\mathbb{Z})}$ and $\tau_* \circ j_{1*} = j_*$.

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In degree two, both j_* and j_{1*} factor through $L_W \cong H_0(W, L)$ and

$$\hat{j}_{1*} : L_W \longrightarrow H_2(BG; \mathbb{Z}) \cong \pi_1(G)$$

is an epimorphism [**8**, lemma 9·3]. It follows that

$$\hat{j}_* : H_0(W; L) \longrightarrow H_2(BN; \mathbb{Z})$$

factors as an epimorphism followed by a monomorphism

$$H_0(W; L) \xrightarrow{\hat{j}_{1*}} H_2(BG; \mathbb{Z}) \xrightarrow{\tau_*} H_2(BN; \mathbb{Z})$$

and we obtain a split exact sequence of abelian groups

$$0 \longrightarrow H_2(BG; \mathbb{Z}) \xrightarrow{\tau_*} H_2(BN; \mathbb{Z}) \longrightarrow H_2(W; \mathbb{Z}) \longrightarrow 0$$

where the section of τ_* is j_{2*} hence there exists an isomorphism

$$H_2(BN; \mathbb{Z}) \cong H_2(BG; \mathbb{Z}) \oplus H_2(W; \mathbb{Z})$$

naturally induced by the maps $j_2 \colon BN \longrightarrow BG$ and the projection $BN \longrightarrow BW$.

The information about $\pi_1(G_p^\wedge)$ can also be read in the mod p cohomology of $BG_p^\wedge$ if one takes into consideration the higher order Bockstein operations from $H^2(BG; \mathbb{F}_p)$ to $H^3(BG; \mathbb{F}_p)$. The fact that the necessary information also embeds in the mod p cohomology of BN will lead to our main objective of this section. We will first state some well known facts on higher order Bockstein operations.

LEMMA 2·6. *Let Y be a space of finite type.*

(1) *If there are independent elements $x_1, \dots, x_r \in H^m(Y; \mathbb{F}_p)$ and each x_i is either a permanent cycle in the Bss or $\beta_{(k_i)}(x_i) = y_i$, with $y_1, \dots, y_r$ independent in $H^{m+1}(Y; \mathbb{F}_p)$, then there is a map*

$$Y \longrightarrow B^m A$$

where A is a finitely generated abelian group detecting exactly the classes x_i and y_i in dimensions m and $m+1$.

(2) *Conversely, assume that Y is $(m-1)$-connected and $m \geqslant 2$ or $m = 1$ and Y is simple. If $\pi_m(Y) = A$, then the map $Y \longrightarrow B^m A$ induces in mod p cohomology an isomorphism in dimension m and a monomorphism in dimension $m+1$.*

PROPOSITION 2·7. *Let G be a compact connected Lie group with maximal torus normalizer N. If X is a p-compact group with isomorphic maximal torus normalizer and the images of Bi^* and Bj^* in the diagram*

$$H^*(BG) \xrightarrow{Bi^*} H^*(BN) \xleftarrow{Bj^*} H^*(BX)$$

coincide, then

(1) *$\pi_1(G_p^\wedge) \cong \pi_1(X)$ and*

(2) *the universal covers of $G_p^\wedge$ and X have isomorphic maximal torus normalizer.*

Proof. The fundamental group of $G_p^\wedge$, $\pi_1(G_p^\wedge)$ is a finitely generated $\hat{\mathbb{Z}}_p$-module, $\pi_1(G_p^\wedge) \cong H_2(BG_p^\wedge; \hat{\mathbb{Z}}_p) \cong \mathbb{Z}/p^{k_1} \times \cdots \times \mathbb{Z}/p^{k_r} \times (\hat{\mathbb{Z}}_p)^s$. This group will be completely determined by looking at the higher order Bockstein operations on $H^*(BG; \mathbb{F}_p)$; more precisely, there is a base $\{x_1, \dots, x_r, \dots x_{r+s}\}$ of

$$\mathrm{Hom}(H_2(BG; \hat{\mathbb{Z}}_p), \mathbb{F}_p) \cong H^2(BG; \mathbb{F}_p) \tag{1}$$

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and $\{y_1, \dots, y_r\}$ of

$$\mathrm{Ext}(H_2(BG; \hat{\mathbb{Z}}_p), \mathbb{F}_p) \subseteq H^3(BG; \mathbb{F}_p)\,. \tag{2}$$

satisfying

$$\begin{cases} \beta_{(k_i)} x_i = y_i & \text{for } i = 1, \dots, r \text{ and} \\ \beta_{(l)}(x_j) = 0 & \text{for } j > r,\ l > 0. \end{cases} \tag{3}$$

We first show that these same factors and operations occur in $H^*(BN; \mathbb{F}_p)$. Actually the Bockstein spectral sequence for BN in low dimension turns out to be the sum of that of BG and W.

In fact, according to Proposition 2·5 and the Universal Coefficients formula, there are isomorphims

$$\begin{aligned} H^1(BN; \mathbb{F}_p) &\cong \mathrm{Hom}(H_1(BN; \mathbb{Z}), \mathbb{F}_p) \\ &\cong \mathrm{Hom}(H_1(W; \mathbb{Z}), \mathbb{F}_p) \\ &\cong H^1(W; \mathbb{F}_p) \end{aligned}$$

and

$$\begin{aligned} H^2(BN; \mathbb{F}_p) &\cong \mathrm{Ext}(H_1(BN; \mathbb{Z}), \mathbb{F}_p) \oplus \mathrm{Hom}(H_2(BN; \mathbb{Z}), \mathbb{F}_p) \\ &\cong \mathrm{Ext}(H_1(W; \mathbb{Z}), \mathbb{F}_p) \oplus \mathrm{Hom}(H_2(W; \mathbb{Z}), \mathbb{F}_p) \\ &\quad \oplus \mathrm{Hom}(H_2(BG; \mathbb{Z}), \mathbb{F}_p) \\ &\cong H^2(W; \mathbb{F}_p) \oplus H^2(BG; \mathbb{F}_p) \end{aligned}$$

and monomorphism

$$\begin{aligned} H^3(BN; \mathbb{F}_p) &\supseteq \mathrm{Ext}(H_2(BN; \mathbb{Z}), \mathbb{F}_p) \cong \\ &\cong \mathrm{Ext}(H_2(W; \mathbb{Z}), \mathbb{F}_p) \oplus \mathrm{Ext}(H_2(BG; \mathbb{Z}), \mathbb{F}_p) \\ &\subseteq H^3(W; \mathbb{F}_p) \oplus H^3(BG; \mathbb{F}_p). \end{aligned}$$

Hence the image of Bi^* in $H^*(BN; \mathbb{F}_p)$ contains the factors of equations (1) and (2) and reflects the same higher order Bockstein operations (3) as they occur in $H^*(BG; \mathbb{F}_p)$, while the other factors are affected by the same higher order Bocksteins as they are in $H^*(W; \mathbb{F}_p)$ from dimension one to two, and from two to three.

Let us turn now our attention to the map

$$H^*(BN; \mathbb{F}_p) \xleftarrow{Bj^*} H^*(BX; \mathbb{F}_p).$$

Since the image coincides with that of Bi^* and this carries all the information about higher order Bockstein relations between two and three dimensional classes, by naturality, this same higher Bockstein relations are reproduced in $H^*(BX; \mathbb{F}_p)$ and this means, according to Lemma 2·6 that there is a map $BX \longrightarrow B^2(\mathbb{Z}/p^{k_1} \times \cdots \times \mathbb{Z}/p^{k_r} \times (\hat{\mathbb{Z}}_p)^s) \cong B^2\pi_1(G_p^\wedge)$ detecting the mentioned classes of dimension two and three. In particular,

$$\pi_1(G_p^\wedge) \cong \pi_1(X).$$

The same argument applies to BN, hence there is also a map

$$BN \longrightarrow B^2\pi_1(G_p^\wedge),$$

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that by naturality will make the following diagram homotopy commutative

$$\begin{array}{ccccc}
B\tilde{G}_p^\wedge & \longleftarrow & BN_{\tilde{G}_p^\wedge} & \longrightarrow & B\tilde{X} \\
\downarrow & & \downarrow & & \downarrow \\
BG_p^\wedge & \longleftarrow & BN & \longrightarrow & BX \\
\downarrow & & \downarrow & & \downarrow \\
B^2\pi_1(G_p^\wedge) & = & B^2\pi_1(G_p^\wedge) & = & B^2\pi_1(G_p^\wedge).
\end{array}$$

This shows that the universal covers of $G_p^\wedge$ and X have isomorphic maximal torus normalizer.

THEOREM 2·8. *Let G be a connected and compact Lie group such that*:

(1) *the centralizer in G of any rank one or two elementary abelian p-subgroup is a totally N-determined p-compact group*;
(2) *there exists at most one conjugacy class $\nu\colon E^2 \longrightarrow G$ of non-toral elementary abelian p-subgroups of G of rank two, moreover $C_G(\nu) = E^2 \times K$ where $K_p^\wedge$ is a center free completely reducible p-compact group; and*
(3) *the automorphism group of ν in the Quillen category of G, $\mathrm{Aut}_{\mathscr{A}_p(G)}(\nu)$, acts transitively on the set of lines of E^2, and any group morphism $\mathrm{Aut}_{\mathscr{A}_p(G)}(\nu) \longrightarrow \mathrm{Aut}(K_p^\wedge)$ is trivial.*

If the universal cover of $G_p^\wedge$ is N-determined then $G_p^\wedge$ is also N-determined.

Proof. Let $j\colon N \longrightarrow G_p^\wedge$ be the maximal torus normalizer of $G_p^\wedge$, and let X' be another p-compact group with isomorphic maximal torus normalizer $j'\colon N \longrightarrow X'$.

Let $\mu : E^2 \longrightarrow N$ be a monomorphism, then μ is a special preferred lift of $j\mu$ if and only if μ is a preferred lift of $j'\mu$ [**15**, lemma 7·13]. Moreover, any non-toral rank two object $(E^2, j\mu)$ or $(E^2, j'\mu)$ has exactly $p+1$ special preferred lifts indexed by the set of nontrivial proper subgroups of E^2 [**15**, proposition 6·2]. Therefore both $G_p^\wedge$ and X' have just one conjugacy class of rank two nontoral elementary abelian p-subgroups, and hypothesis (1) in Proposition 2·4 holds. Moreover, as $\mathrm{Aut}_{\mathscr{A}_p(G)}(\nu)$ acts transitively on the set of lines of E^2, all the other $p+1$ special preferred lifts of $\nu = j\mu$ or $j'\mu$ can be written as $\mu\alpha$ where $\alpha \in \mathrm{Aut}_{\mathscr{A}_p(G)}(\nu)$. Now, let $f(\mu) : C_G(\nu)_p^\wedge \longrightarrow C_{X'}(j'\mu)$ be an isomorphism under $C_N(\mu)$. Given another preferred lift $\mu\alpha$, we have that $f(\mu\alpha) = C_{X'}(\alpha) f(\mu) C_G(\alpha)^{-1}$ as $C_G(\mu)_p^\wedge$ is a totally N-determined p-compact group. Therefore $\pi_0 f(\mu\alpha) = Id_{E^2} = \pi_0 f(\mu)$ and $f(\mu\alpha)_0 = f(\mu)_0$ (as $C_G(\alpha)_0 = C_{X'}(\alpha)_0 = Id_{K_p^\wedge}$ by hypothesis (3)), and according to [**13**, corollary 5·3] $f(\mu\alpha) = f(\mu)$, and $f(\mu)$ does not depend on the choice of μ (hypothesis (2) in Proposition 2·4). Then the proof follows by Propositions 2·4 and 2·7.

3. *Centralizers of elementary abelian p-subgroups of $PU(n)$*

Let $\pi : U(n) \to PU(n)$ be the standard projection and $V \subset PU(n)$ be an elementary abelian p-subgroup of $PU(n)$. Naturally we have the following commutative

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diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & \pi^{-1}(V) & \longrightarrow & V \\ \| & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & U(n) & \xrightarrow{\pi} & PU(n) \end{array}$$

where $S^1 \rightarrowtail \pi^{-1}(V) \twoheadrightarrow V$ is a central extension.

If $V = \mathbb{Z}/p$, this extension is trivial because $H^3(\mathbb{Z}/p;\mathbb{Z}) = 0$, and then $\pi^{-1}(V) = S^1 \times V$. But if $\mathrm{rk}(V) \geqslant 2$ there are examples where such extension has no section. Nevertheless, if $\mathrm{rk}(V) = r \geqslant 2$, and $v_1, \ldots, v_r$ is a base of V, we can apply the previous argument to each v_i separately, so obtaining elements $\hat{v}_1, \ldots, \hat{v}_r \in U(n)$ of order p, such that $\pi(\hat{v}_i) = v_i$. Notice that in general the elements $\hat{v}_1, \ldots, \hat{v}_r$ will not commute with each other and will not define a section to the extension, however they will be, together with the central S^1 of $U(n)$, a system of generators for $\pi^{-1}(V)$.

Now, each $\hat{v}_i$ defines a group homomorphism $\pi^{-1}C_{PU(n)}(V) \xrightarrow{l_i} \mathbb{Z}/p$ determined by the formula $x\hat{v}_i x^{-1}\hat{v}_i^{-1} = \zeta^{l_i(x)}$, where $\zeta = e^{\frac{2\pi i}{p}}$, and all of them together

$$l = (l_1, \ldots, l_r) : \pi^{-1}C_{PU(n)}(V) \longrightarrow \mathbb{Z}/p \times \overset{r}{\cdots} \times \mathbb{Z}/p \tag{4}$$

with $\mathrm{Ker}(l) = C_{U(n)}(\pi^{-1}(V))$. Notice also that

$$C_{U(n)}(\pi^{-1}(V)) = \bigcap_{i=1}^{r} C_{U(n)}(\hat{v}_i).$$

Therefore, a strategy for calculating $C_{PU(n)}(V)$, being $V \subset PU(n)$ an elementary abelian p-subgroup would be to calculate first $C_{U(n)}(\pi^{-1}(V))$ using available information about $C_{U(n)}(\hat{v}_i)$ for $i = 1, \ldots, r$, and then to analyze the behavior of the homomorphism l in (4). We will proceed accordingly.

Lemma 3·1. *Let $V = \mathbb{Z}/p$ be a subgroup of $PU(n)$, then $C_{PU(n)}(V)$ is conjugated to one of the following Lie groups*:

(1) $\Lambda_1 = U(n_1) \times \cdots \times U(n_k)/S^1$, *with* $\sum_{i=1}^{k} n_i = n$ *and* $2 \leqslant k \leqslant p$.

(2) $\Lambda_2 = (U(m) \wr \mathbb{Z}/p)/S^1$ *with* $n = m \cdot p$.

where the inclusion of S^1 is always diagonal.

Proof. Let v be generator of V and let $\hat{v}$ be an element of $U(n)$ such that $\hat{v}^p = 1$ and $\pi(\hat{v}) = v$. Existence of such an element has been shown above. Now, up to conjugacy, $\hat{v}$ is a diagonal matrix consisting of k diagonal boxes $\hat{v} = \mathrm{Diag}(\zeta^{r_1} I_{n_1}, \ldots, \zeta^{r_k} I_{n_k})$ where I_{n_i} is the identity matrix of $U(n_i)$ and $r_1 < \cdots < r_k$. Therefore $C_{U(n)}(\hat{v}) = U(n_1) \times \cdots \times U(n_k)$. Here, we can assume that $k \geqslant 2$ because $k = 1$ would mean that $\hat{v}$ is central and then $\pi(\hat{v}) = 1$.

Now, we have to consider two different cases according to whether or not l is trivial. If l is the trivial homomorphism, then $C_{U(n)}(\hat{v}) = \pi^{-1}C_{PU(n)}(V)$ and $C_{PU(n)}(V)$ is conjugated to Λ_1 in the lemma. If l is non-trivial, then we have to classify the extension

$$C_{U(n)}(\hat{v}) \longrightarrow \pi^{-1}C_{PU(n)}(V) \xrightarrow{l} \mathbb{Z}/p\,. \tag{5}$$

Take an element $x \in \pi^{-1}C_{PU(n)}(V)$ with $l(x) \neq 0$. Write it as a matrix subdivided in boxes $x = (x_{ij})$, $x_{ij} \in M_{n_i \times n_j}(\mathbb{C})$. Then, the formula $x\hat{v}_i x^{-1}\hat{v}_i^{-1} = \zeta^{l(x)}$ yields $x_{ij}\zeta^{r_j + l(x)} = x_{ij}\zeta^{r_i}$. Hence either $x_{ij} = 0$ or $r_j + l(x) = r_i$.

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Since the matrix x is non-singular, for each r_j there should be an r_i with $r_j + l(x) = r_i$; that is, adding $l(x)$ is a bijection of the set $\{r_1, \dots, r_k\}$, hence $k = p$, $r_1 = 1, \dots, r_p = p$ and addition with $l(x)$ is a cyclic permutation of these elements.

In particular, $\hat{v}$ is conjugated to the diagonal matrix

$$A = \mathrm{Diag}(\zeta I, \zeta^2 I, \dots, \zeta^p I)$$

where I is the identity matrix in $U(m)$ and $n = pm$. Moreover, the matrix

$$B = \left(\delta_{\sigma(i)j} I\right),$$

where σ is the cycle $(1, 2, \dots, p) \in \Sigma_p$, provides a section of the extension (5). Therefore $\pi^{-1}C_{PU(n)}(V) = (U(m) \times \cdots \times U(m)) \rtimes \mathbb{Z}/p$, and $C_{PU(n)}(V)$ is conjugated to Λ_2 in the lemma.

COROLLARY 3·2. *The subgroup $W_1 \subset PU(n)$ generated by the class of the matrix A, is, up to conjugacy, the only elementary abelian p subgroup of rank one in $PU(n)$ with non-connected centralizer. In particular, this can only happen if $p|n$.*

Now, if $V = \mathbb{Z}/p \oplus \mathbb{Z}/p$ generated by v_1 and v_2, being $\hat{v}_1$ and $\hat{v}_2$ their respective lifts to $\pi^{-1}(V) \subset U(n)$, then as $C_{PU(n)}(V) = C_{C_{PU(n)}(v_1)}(v_2)$, similar arguments to those above give rise to the exact sequence

$$1 \longrightarrow C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) \longrightarrow \pi^{-1}(C_{PU(n)}(V)) \xrightarrow{l_2} \mathbb{Z}/p\,.$$

l_1 that it is equally defined on $\pi^{-1}(C_{PU(n)}(V))$ can be restricted to $C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2)$ and we will have another exact sequence

$$1 \longrightarrow C_{U(n)}(\pi^{-1}(V)) \longrightarrow C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) \xrightarrow{l_1} \mathbb{Z}/p\,.$$

Those fit together with the map of equation (4) to give the following commutative diagram

$$\begin{array}{ccccc} C_{U(n)}(\pi^{-1}(V)) & = & C_{U(n)}(\pi^{-1}(V)) & & \\ \downarrow & & \downarrow & & \\ C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) & \rightarrowtail & \pi^{-1}(C_{PU(n)}(V)) & \xrightarrow{l_2} & \mathbb{Z}/p \\ \downarrow l_1 & & \downarrow l & & \| \\ \mathbb{Z}/p & \rightarrowtail & \mathbb{Z}/p \oplus \mathbb{Z}/p & \twoheadrightarrow & \mathbb{Z}/p \end{array} \tag{6}$$

Thus, we can use Lemma 3·1 to identify $\pi^{-1}(C_{PU(n)}(v_1))$, and proceed to calculate $C_{PU(n)}(V)$.

LEMMA 3·3. *Let $V = \mathbb{Z}/p \oplus \mathbb{Z}/p$ be a subgroup of $PU(n)$, then $C_{PU(n)}(V)$ is conjugated to one of the following Lie groups*:

(1) $\Upsilon_1 \cong U(n_1) \times \cdots \times U(n_k)/S^1$ *where* $\sum_{i=1}^k n_i = n$, $2 \leqslant k \leqslant p^2$,
(2) $\Upsilon_2 \cong ((\Pi_{i=1}^k U(m_j)) \wr \mathbb{Z}/p)/S^1$ *where* $n = (\sum_{i=1}^k m_i) \cdot p$, $2 \leqslant k \leqslant p$,
(3) $\Upsilon_3 \cong (U(m) \wr (\mathbb{Z}/p \times \mathbb{Z}/p))/S^1$ *where* $n = m \cdot p^2$, *or*
(4) $\Upsilon_4 \cong PU(m) \times \mathbb{Z}/p \times \mathbb{Z}/p$ *where* $n = m \cdot p$,

where the inclusion of S^1 is diagonal and $U(0) = 1$.

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Proof. Let $V = \mathbb{Z}/p \oplus \mathbb{Z}/p$ be a subgroup of $PU(n)$. Let v_1 and v_2 be generators of V, and let $\hat{v}_1$ and $\hat{v}_2$ be respective lifts to $\pi^{-1}(V) \subset U(n)$, of order p. As we show above, all we have to do is to check the possibilities for $\pi^{-1}(C_{PU(n)}(v_1))$ and $C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2)$, and analyze the map l_2. According to Lemma 3·1 we only have to consider two cases, up to conjugacy, namely $C_{PU(n)}(v_1)$ is either Λ_1 or Λ_2.

Case 1: $\pi^{-1}(C_{PU(n)}(v_1)) = U(n_1) \times \cdots \times U(n_k)$. Then $\hat{v}_2$ is (up to conjugacy) a diagonal matrix $\mathrm{Diag}(x_1, \ldots, x_k)$, where $x_i \in U(n_i)$ is a diagonal element of order p. Therefore

$$C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) = \big(U(n_{1,1}) \times \cdots \times U\big(n_{1,q_1}\big)\big) \times \cdots \\ \cdots \times \big(U(n_{k,1}) \times \cdots \times U\big(n_{k,q_k}\big)\big)$$

where $\sum_{j=1}^{q_i} n_{i,j} = n_i$, and we have to consider two possibilities:

The homomorphism l_2 is trivial. In this case

$$\pi^{-1}\big(C_{PU(n)}(V)\big) = C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2)$$

and $C_{PU(n)}(V)$ is of the form Υ_1.

The homomorphism l_2 is non-trivial. An argument similar to that of Lemma 3·1 implies $p = q_1 = \cdots = q_k$, $m_i = n_{i,1} = \cdots = n_{i,p}$. We keep $2 \leqslant k \leqslant p$ and the action of $\mathbb{Z}/p$ permutes cyclically the p factors of each block $(U(m_i) \times \cdots \times U(m_i))$. Hence after a reordering of factors, $C_{PU(n)}(V)$ is of the form Υ_2.

Case 2: $\pi^{-1}(C_{PU(n)}(v_1)) = U(m) \wr \mathbb{Z}/p$. According to Corollary 3·2 we can assume that $\hat{v}_1 = A$, and we will have two possibilities for $\hat{v}_2$, either $\hat{v}_2 \in \Pi_{i=1}^{p} U(m)$ or $\hat{v}_2 \notin \Pi_{i=1}^{p} U(m)$.

Assume $\hat{v}_2 \in \Pi_{i=1}^{p} U(m)$. Then $\hat{v}_2$ is, up to conjugacy, a diagonal matrix $\mathrm{Diag}(x_1, \ldots, x_p)$, where $x_i \in U(m)$ is a diagonal element or order p. The centralizer $C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2)$ is either

(a) $C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) = (U(n_{1,1}) \times \cdots \times U(n_{1,k_1})) \times \cdots \times (U(n_{p,1}) \times \cdots \times U(n_{p,k_p}))$ where $\sum_{j=1}^{k_i} n_{i,j} = m$, or

(b) $C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) = (U(m_1) \times \cdots \times U(m_k)) \wr \mathbb{Z}/p$, if $\hat{v}_2 = \mathrm{Diag}(\zeta x, \ldots, \zeta^p x)$.

The case (a) was studied above and will lead to Υ_1 or Υ_2 according to whether or not l_2 is trivial. Again, in case (b), if the homomorphism l_2 is trivial, $\pi^{-1}(C_{PU(n)}(V)) = C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2)$ is a group of the form Υ_2.

Finally, if in the case (b) the homomorphism l_2 is non-trivial, we can find again a section for l_2 and show that

$$\pi^{-1}\big(C_{PU(n)}(V)\big) = \big(\big(\Pi_{j=1}^{p} U(m)\big) \wr \mathbb{Z}/p\big) \rtimes \mathbb{Z}/p$$

where the action of the second copy of $\mathbb{Z}/p$ permutes the factors of each copy of $\Pi_{j=1}^{p} U(m)$. We have therefore the groups Υ_3.

Assume $\hat{v}_2 \in \Pi_{i=1}^{p} U(m)$. Now, $\hat{v}_2$ is up to conjugacy $g \cdot B$ where g is a diagonal matrix $\mathrm{Diag}(g_1, \ldots, g_p)$, with $g_i \in U(m)$ a diagonal element of order p and B as defined in the proof of Lemma 3·1.

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The fact that $(g \cdot B)^p = 1$ implies that $g_p \cdot g_{p-1} \cdots g_1 = 1$, hence we can find $a_i \in U(m)$, for $i = 1, \dots, p$, such that $a_1 \cdot g_1 \cdot a_p^{-1} = a_2 \cdot g_2 \cdot a_1^{-1} = \cdots = a_p \cdot g_p \cdot a_{p-1} = 1$, that is, $a \cdot g \cdot B \cdot a^{-1} = B$ where a is the diagonal matrix $\mathrm{Diag}(a_1, \dots, a_p)$. Therefore $\hat{v}_2 = B$, up to conjugacy, and

$$\begin{aligned} C_{\pi^{-1}(C_{PU(n)}(v_1))}(\hat{v}_2) &= \{(y_1, \dots, y_p; w) \in U(m) \wr \mathbb{Z}/p \mid y_i = y_{i+1}\} \\ &\cong \Delta U(m) \times \mathbb{Z}/p, \end{aligned}$$

where $\Delta U(m)$ is a diagonal copy of $U(m)$ in $\prod_{i=1}^{p} U(m)$.

Now, l_1 in (6) is surjective ($l_2(B) = 1$) with kernel $C_{U(n)}(\pi^{-1}(V)) = \Delta U(m)$. Hence diagram (6) looks like

$$\begin{array}{ccccc} \Delta U(m) & = & \Delta U(m) & & \\ \downarrow & & \downarrow & & \\ \Delta U(m) \times \mathbb{Z}/p & \rightarrowtail & \pi^{-1}(C_{PU(n)}(V)) & \xrightarrow{l_2} & \mathbb{Z}/p \\ \downarrow l_1 & & \downarrow l & & \| \\ \mathbb{Z}/p & \rightarrowtail & \mathbb{Z}/p \oplus \mathbb{Z}/p & \twoheadrightarrow & \mathbb{Z}/p \end{array}$$

Now, $A \notin \Delta U(m) \times \mathbb{Z}/p$, but $A \in \pi^{-1}(C_{PU(n)}(V))$, thus l_2 is surjective and A generates a section, that is, the sequence

$$\Delta U(m) \times \mathbb{Z}/p \rightarrowtail \pi^{-1}\big(C_{PU(n)}(V)\big) \xrightarrow{l_2} \mathbb{Z}/p$$

is exact. But this sequence is not split, as the generators of the sections of l_1 and l_2 (B and A respectively) do not commute ($AB = BA\zeta$). Nevertheless, these generators does commute when considering their classes in $PU(n)$, and therefore the exact sequence

$$PU(m) \longrightarrow C_{PU(n)}(V) \longrightarrow V$$

is split. Since the splitting is induced by the inclusion $V \subset Z(C_{PU(n)}(V))$, $PU(m)$ commutes with V and $C_{PU(n)}(V) = PU(m) \times V$, that is the group Υ_4.

COROLLARY 3·4. *The subgroup $W_2 \subset PU(n)$ generated by the classes represented by the matrices A and B (defined in the proof of Lemma 3·1) is, up to conjugacy, the only non-toral elementary abelian p-subgroup of rank two in $PU(n)$. In particular this can only occur if $n = mp$, and $C_{PU(n)}(W_2) = W_2 \times PU(m)$.*

4. *Proof of Theorem 1·3 and Corollary 1·4*

The proof is done by induction on n. Assume the result holds for $r < n$, now we show that $PU(n)$ meets the conditions of Theorem 2·8.

According to Lemmas 3·1 and 3·3, the centralizer in $PU(n)$ of any rank one or two elementary abelian p-subgroup is a central quotient of a finite number of unitary groups of rank less that n. By the induction hypothesis those centralizers are totally N-determined, hence condition (1) in Theorem 2·8 holds.

By Corollary 3·4 there is at most one conjugacy class of non-toral elementary abelian p-subgroups of rank two in $PU(n)$. This class occurs when $n = mp$ in which case is represented by the group W_2 generated by the elements $[A]$ and $[B]$ of $PU(n)$, and $C_{PU(n)}(W_2) = W_2 \times PU(m)$. Hence condition (2) in Theorem 2·8 holds.

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To check that condition (3) in Theorem 2·8 is also satisfied, we need to calculate $\mathrm{Aut}_{\mathscr{A}_p(PU(n))}(W_2)$.

LEMMA 4·1. *Let W_2 be the subgroup of $PU(n)$ ($n = pm$) generated by the classes of the matrices A and B. Then $\mathrm{Aut}_{\mathscr{A}_p(PU(n))}(W_2) \cong \mathrm{SL}_2(p)$, where the action on W_2 is the natural one.*

Proof. Let σ be an element of $\mathrm{Aut}_{\mathscr{A}_p(PU(n))}(W_2) \cong N_{PU(n)}(W_2)/C_{PU(n)}(W_2)$, then:

(i) σ is represented by a $\mathbb{F}_p$-matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, where $\sigma([A]) = [A]^a[B]^b$ and $\sigma([B]) = [A]^c[B]^d$;

(ii) σ is an inner automorphism of $PU(n)$, thus it lifts to $U(n)$. The relation $AB = \zeta BA$, where $\zeta = e^{\frac{2\pi i}{p}}$, gives rise to the relation $ad - bc = 1$.

Therefore, we have a natural embedding of $W_{PU(n)}(W_2)$ into $\mathrm{SL}_2(p)$.

Finally, the lift of W_2 to $U(n)$ is conjugated to a diagonal inclusion of the group Γ_p^U, as defined in [**21**]. Thus $\mathrm{SL}_2(p) = W_{U(p)}(\Gamma_p^U) \subset W_{PU(n)}(W_2) \subset \mathrm{SL}_2(p)$, which finishes the proof.

Therefore $\mathrm{Aut}_{\mathscr{A}_p(PU(n))}(W_2)$ acts transitively on the set of lines of W_2. Moreover, according to a Theorem of Dickson [**10**], $\mathrm{SL}_2(p)$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, which are of order p, and

$$\mathrm{Aut}(PU(m)^\wedge_p) = \begin{cases} \hat{\mathbb{Z}}_p^*, & m > 2 \\ \hat{\mathbb{Z}}_p^*/_{\pm}, & m = 2 \end{cases}$$

has no element of order p. Hence any group homomorphism from $\mathrm{Aut}_{\mathscr{A}_p(PU(n))}(W_2)$ to $\mathrm{Aut}(PU(m)^\wedge_p)$ is trivial, and condition (3) in Theorem 2·8 is satisfied.

Finally, as $SU(n)^\wedge_p$ is a N-determined p-compact group [**17**], then $PU(n)^\wedge_p$ is also N-determined.

Acknowledgements. The authors acknowledge the support of the Centre de Recerca Matemàtica and Universidad de Málaga, that have made this paper possible.

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