DISTORTION RISK MEASURES, AMBIGUITY AVERSION AND OPTIMAL EFFORT

BY

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ABSTRACT

We consider the class of concave distortion risk measures to study how choice is influenced by the decision-maker's attitude to risk and provide comparative statics results. We also assume ambiguity about the probability distribution of the risk and consider a framework à la Klibanoff, Marinacci and Mukerji (2005; A smooth model of decision making under ambiguity. *Econometrica*, **73**, 1849–1892) to study the value of information that resolves ambiguity. We show that this value increases with greater ambiguity, with greater ambiguity aversion, and in some cases with greater risk aversion. Finally, we examine whether a more risk-averse and a more ambiguity-averse individual will invest in more effort to shift his initial risk distribution to a better target distribution.

KEYWORDS

Ambiguity, ambiguity aversion, comparative statics results, distortion risk measures, risk aversion.

1. INTRODUCTION

This paper is concerned with the ambiguity about the probability distribution that is used to evaluate risk measures. Almost all models used in the theory of risk measures assume that the distributions of risks are perfectly known. We rather assume here that there exists an uncertainty about a parameter of the risk distribution and that this uncertainty may modify decisions of the individuals if they are averse to ambiguity. This paper aims to examine the effect of ambiguity-aversion on the valuation of risk reduction. It will also give an example where ambiguity-aversion and risk-aversion have different consequences on effort to reduce risk.

A risk measure is defined as a mapping from the set of random variables representing the risks to the real numbers. It may be interpreted as the amount of money that should be added as a buffer to a risk so that it becomes acceptable to an internal or external risk controller. It may be used, e.g. for evaluating capital requirements in order to avoid insolvency, for determining provisions and

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for calculating insurance premiums. In this paper, the risk X may be thought as an insurance company's risk related to a particular policy, a particular line-of-business or to the entire insurance portfolio over a specified time horizon. A negative outcome for X means that a gain has occurred. This means that a decision-maker will always try to act to minimize the risk measure associated to X.

We consider the class of distortion risk measures introduced by Wang (1996). This class of risk measures used the concept of distortion function as proposed in Yaari's dual theory of choice (see Yaari, 1987). A distortion function is defined as a non-decreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$. The distortion risk measure associated with distortion function g is defined by

$$\mathbb{H}_g[X] = -\int_{-\infty}^{0} \left(1 - g(\bar{F}_X(x))\right) dx + \int_{0}^{\infty} g(\bar{F}_X(x))\, dx, \tag{1.1}$$

where X is a (random) risk with survival function $\bar{F}_X$. Under the assumption that $g(\bar{F}_X(\cdot))$ is right-continuous, $\mathbb{H}_g[X]$ may also be interpreted as a distorted expectation of X evaluated with a distorted survival function $g(\bar{F}_X(\cdot))$.

Yaari's dual theory and distortion risk measures have been considered for two decades by several authors to study insurance decisions, such as pricing, reserving, designing optimal (re)insurance contracts, setting capital, optimizing risk exchanges (see among others: Denneberg, 1990; Doherty and Eeckhoudt, 1995; Wang and Young, 1998; Wang, 2000; Tsanakas and Desli, 2003; Tsanakas and Christofides, 2006).

Ellsberg (1961) observed that individuals prefer a lottery with a known probability distribution to a lottery with an unknown distribution. This preference violates the expected utility theory, but can be explained by ambiguity-aversion of the individuals. Among the several approaches to model this aversion that have been provided in the economic literature (see Etner *et al.*, 2012), the approach developed by Klibanoff *et al.* (2005) is very useful to study the effect of ambiguity about the risk distribution because it permits to separate the effect of ambiguity-aversion from that of risk-aversion. In their model, the ambiguity-aversion is captured by introducing a concave transform of the expected utility. This approach has been recently used to examine the effect of ambiguity-aversion on insurance demand (see, e.g. Snow, 2011; Alary *et al.*, 2012) and it is shown that the demand for self-insurance rises with ambiguity-aversion whereas the demand for self-protection is less clear.

In this paper, we assume that the decision-maker is uncertain about a parameter of the risk distribution in order to examine the effect of ambiguity-aversion on the valuation of risk reduction. Our results are established using a similar recursive model of ambiguity preferences as developed by Klibanoff *et al.* (2005), but with second-order probability distribution for the unknown parameter.

Although the problem of parameter and model uncertainty in insurance has been raised for several years (see, e.g. Cairns, 2000 for a general point of view

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and e.g. Landsman and Tsanakas, 2012 for an example), the impact of parameter uncertainty on risk measures has only been recently studied. Bignozzi and Tsanakas (2013) have introduced the notion of residual estimation risk when the risk measures are positive homogeneous, translation invariant and law invariant. The residual estimation risk is then defined as the risk measure of the difference between the risk itself and its risk measure when the parameter is replaced by a random estimator. This estimation risk reflects the amount of capital that needs to be subtracted from the difference between the risk and its (random) risk measure such that it becomes acceptable. It is expected that, as the size of the sample that is used to construct the estimator increases, the residual risk decreases. Our approach differs from Bignozzi and Tsanakas (2013) since we are interested in the measure of aversion to parameter uncertainty rather than only its measure. Moreover, we want to understand how this aversion modifies decisions. Finally, note that we don't need to know the distribution of the difference between the risk and its (random) risk measure.

The remainder of the paper is organized as follows. In Section 2, we explain how the notion of risk aversion can be translated into the class of distortion risk measures in order to permit comparative statics results analysis of greater risk aversion. We present the framework à la Klibanoff *et al.* (2005) to take into account ambiguity about the probability distribution in Section 3. We also introduce characterizations for ambiguity measures in order to study the value of information that resolves ambiguity. We show that this value increases with greater ambiguity, with greater ambiguity-aversion, and in some cases with greater-risk aversion. In Section 4, we consider the case of a decision-maker that is able to invest in effort to reduce risk by shifting the initial risk distribution to a better target distribution. We examine whether a more risk-averse and a more ambiguity-averse individual will invest in more effort to shift his initial risk distribution. Section 5 contains brief conclusions and proofs are postponed to Section 6.

2. Comparative statics results for the class of distortion risk measures

In Yaari's dual theory of choice, a decision-maker is expected to act in order to maximize the distorted expectation of his wealth. Here, we say that the decision-maker bases his preferences on the distorted expectations hypothesis if he acts in order to minimize the distorted expectation of his risk: if his distortion function is denoted by g, then he prefers the risk X over the risk Y when $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$. Note that a wealth, W, may be viewed as a negative risk, and that $\mathbb{H}_g[-W] = -\mathbb{H}_{g^*}[W]$, where $g^*(u) = 1 - g(1-u)$ is the dual distortion operator of g. Therefore, the decision-maker's preferences are equivalent if he uses the distortion function g^* for calculating the distorted expectation of his wealth.

Several definitions of risk aversion have been proposed for non-expected utility theories of choice under uncertainty, see, e.g. Montesano (1990). A first

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definition is linked with the risk premium, i.e. the difference between the expected value of the prospect under consideration and its certainty equivalent: a decision-maker is said to be risk-averse if he always prefers a certain prospect to a risky prospect with the same expectation. A second defines risk aversion as aversion to a mean preserving spread in the distribution of prospects. These definitions are equivalent to the concavity of the von Neumann–Morgenstern utility function for the expected utility theory. But they differ in our case. The former refers to the condition $g^*(p) \leq p$, or equivalently, to $g(p) \geq p$ (a risk-averse decision-maker systematically overestimates the tail probabilities related to the levels of his risk), while the latter is stronger and refers to the condition that g^* is convex, or equivalently, that g is concave (see, e.g. Röell, 1987; Yaari, 1987) and Section 2.6.1.4 of Denuit *et al.* (2005)).

We rather consider the second definition and assume that the distortion function of an individual always belongs to the class of concave distortion functions. We shall say that a decision-maker with distortion function h is more risk-averse than a decision-maker with distortion function g, if there exists a concave distortion function f such that $h = f \circ g$ (see, e.g. Ross, 1981 for equivalent definitions when twice differentiability may be assumed).

Note that, if g is a concave distortion function, then $\mathbb{H}_g$ is also a coherent risk measure (i.e. it satisfies translativity, subadditivity, positive homogeneity and monotonicity, see Artzner *et al.* 1999). Moreover, the set where g' fails to exist is countable, and therefore its associated distortion risk measure can be written as

$$\mathbb{H}_g[X] = \int_0^1 g'(1-u)\, Va\, R[X;u]\, du, \tag{2.2}$$

where $Va\, R[X;u] = \inf\{x \in \mathbb{R} : F_X(x) \geq u\}$ is the quantile function of X and is known as the Value-at-Risk. The distortion risk measure is written as a spectral risk measure (see Acerbi, 2002).

Example 1. The class of Normal distortion function has been introduced by Wang (2000) and is characterized by the functions $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$ with $\gamma > 0$, where Φ is the standard normal cumulative distribution function. These functions are concave and satisfy $g_{\gamma_1+\gamma_2} = g_{\gamma_1} \circ g_{\gamma_2} = g_{\gamma_2} \circ g_{\gamma_1}$ for $\gamma_1, \gamma_2 > 0$. Hence, a decision-maker with distortion function $g_{\tilde{\gamma}}$ is more risk-averse than another decision-maker with distortion function g_γ if $\tilde{\gamma} > \gamma$.

Consider an agent who has to choose between two risks X and Y. Would a more risk-averse agent make the same choice?

Example 1 (continued). Let us consider an agent with a Normal distortion function g_γ and assume that his choice has to be made between two Lognormal distributions, $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{LN}(\mu_Y, \sigma_Y^2)$. The quantile function of X is given by

$$Va\, R[X;\alpha] = \exp\left(\mu_X + \sigma_X \Phi^{-1}(\alpha)\right),$$

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and its risk measure by

$$\begin{aligned}
&\mathbb{H}_{g_\gamma}[X] \\
&= \int_0^1 VaR[X;u]\, g'_\gamma(1-u)du = \int_0^1 VaR[X;u]\, \exp\left(\gamma\Phi^{-1}(u) - \gamma^2/2\right) du \\
&= \int_0^1 \exp\left(\mu_X - \gamma^2/2 + (\gamma+\sigma_X)\,\Phi^{-1}(u)\right) du = \exp\left(\mu_X + \gamma\sigma_X + \sigma_X^2/2\right) \\
&= \mathbb{E}[X]\exp(\gamma\sigma_X).
\end{aligned}$$

Assume moreover that the agent prefers X over Y, i.e. $H_{g_\gamma}[X] \leq H_{g_\gamma}[Y]$. This condition is equivalent to

$$\mu_X - \mu_Y + \sigma_X^2/2 - \sigma_Y^2/2 \leq \gamma(\sigma_Y - \sigma_X).$$

It follows that a more risk-averse agent with a Normal distortion function $g_{\tilde{\gamma}}$ such that $\tilde{\gamma} > \gamma$ will always choose X rather than Y if and only if $(\sigma_Y - \sigma_X) > 0$.

We now give a general comparative statics result. It may be discussed and compared to the results given by Hammond (1974), Diamond and Stiglitz (1974) and Jewitt (1989) for the expected utility framework.

Theorem 1. *Consider a decision-maker with concave distortion function g who prefers X over Y. Then, the following properties are equivalent:*

i) A more risk-averse decision-maker prefers X over Y;

ii) for all $\alpha \in (0,1)$,

$$\int_\alpha^1 g'(1-u)\,(VaR[X;u] - VaR[Y;u])\,du \leq 0. \tag{2.3}$$

Let us compare this last condition to the condition obtained for Example 1.

Example 1 (continued). Condition ii) of Theorem 1 holds if and only if $(\sigma_Y - \sigma_X) > 0$. Indeed, let

$$\begin{aligned}
d(\alpha) &= \int_\alpha^1 g'(1-u)\,(VaR[X;u] - VaR[Y;u])\,du \\
&= \exp\left(\tilde{\mu}_X^{(\gamma)}\right)\Phi\left(\tilde{\sigma}_X^{(\gamma)} - \Phi^{-1}(\alpha)\right) - \exp\left(\tilde{\mu}_X^{(\gamma)}\right)\Phi\left(\tilde{\sigma}_Y^{(\gamma)} - \Phi^{-1}(\alpha)\right).
\end{aligned}$$

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where $\tilde{\mu}_X^{(\gamma)} = \mu_X + \gamma\sigma_X + \sigma_X^2/2$ and $\tilde{\sigma}_X^{(\gamma)} = \gamma + \sigma_X$ *(resp. for* Y*)*. We have $d(0) = H_{g_\gamma}[X] - H_{g_\gamma}[Y] \leq 0$, $d(1) = 0$ and

$$\begin{aligned} d'(\alpha) &= -\exp\left(\tilde{\mu}_X^{(\gamma)} + \Phi^{-1}(\alpha)\,\tilde{\sigma}_X^{(\gamma)} - \left(\tilde{\sigma}_X^{(\gamma)}\right)^2/2\right) \\ &\quad + \exp\left(\tilde{\mu}_Y^{(\gamma)} + \Phi^{-1}(\alpha)\,\tilde{\sigma}_Y^{(\gamma)} - \left(\tilde{\sigma}_Y^{(\gamma)}\right)^2/2\right) \\ &= -\exp\left(\mu_X - \gamma^2/2 + \Phi^{-1}(\alpha)\,(\gamma + \sigma_X)\right) \\ &\quad + \exp\left(\mu_Y - \gamma^2/2 + \Phi^{-1}(\alpha)\,(\gamma + \sigma_Y)\right). \end{aligned}$$

Note that $d'(\alpha) < 0$ (resp. $>$) if and only if $\mu_X - \mu_Y > \Phi^{-1}(\alpha)(\sigma_Y - \sigma_X)$ (resp. $<$). Therefore, $d(\alpha) \leq 0$ for all $\alpha \in [0, 1]$ if and only if $(\sigma_Y - \sigma_X) > 0$. This condition does not depend on γ and is actually more general than needed for a more risk-averse decision-maker with a Normal distortion function.

We derive from Theorem 1 several sufficient conditions for a risk-averse decision-maker to have the same preferences as a less risk-averse decision-maker. Let

$$Va\,R^{(2)}[X;\alpha] = \int_0^{1-\alpha} Va\,R[X;1-u]du = \int_\alpha^1 Va\,R[X;u]\,du, \qquad \alpha \in [0,1]. \tag{2.4}$$

Note that $Va\,R^{(2)}[X;\alpha]$ is related to the Tail-Value-at-Risk in the following way: $Va\,R^{(2)}[X;\alpha] = (1-\alpha)\,TVa\,R[X;\alpha]$.

Corollary 2. *Consider a decision-maker with distortion function* g *who prefers risk* X *over risk* Y*. Assume that one of the following condition holds:*

i) $Va\,R[X;\alpha] \leq Va\,R[Y;\alpha]$ *for all* $\alpha \in (0,1)$*;*

ii) there exists $\alpha_0 \in (0,1)$ *such that* $Va\,R[X;\alpha] \geq Va\,R[Y;\alpha]$ *for* $0 < \alpha \leq \alpha_0$ *and* $Va\,R[X;\alpha] \leq Va\,R[Y;\alpha]$ *for* $\alpha_0 \leq \alpha < 1$*;*

iii) g *is twice differentiable and* $Va\,R^{(2)}[X;\alpha] \leq Va\,R^{(2)}[Y;\alpha]$ *for all* $\alpha \in (0,1)$*;*

iv) g *is twice differentiable and there exists* $\alpha_0 \in (0,1)$ *such that* $Va\,R^{(2)}[X;\alpha_0] = Va\,R^{(2)}[Y;\alpha_0]$*, for* $0 < \alpha \leq \alpha_0$*,* $Va\,R^{(2)}[X;\alpha] - Va\,R^{(2)}[Y;\alpha]$ *is non-increasing and, for* $\alpha_0 \leq \alpha < 1$*,* $Va\,R^{(2)}[X;\alpha] \leq Va\,R^{(2)}[Y;\alpha]$*.*

Then, a more risk-averse decision-maker also prefers X *over* Y*.*

These sufficient conditions are strongly linked with well-known stochastic order relations:

- The risk X is said to be smaller than the risk Y in stochastic dominance (denoted by $X \preceq_{ST} Y$) if $\bar{F}_X(d) \leq \bar{F}_Y(d)$ for all $d \in \mathbb{R}$, or equivalently if $\mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)]$ for all non-decreasing functions u (such that the expectations exist). This is also equivalent to Condition i) of Corollary 2 (see, e.g. Section 3.3 in Denuit *et al.*, 2005).

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- The risk X is said to precede the risk Y in the stop-loss order (denoted by $X \preceq_{SL} Y$) if $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \in \mathbb{R}$. It is well-known that this condition is equivalent to one of the following conditions (a) $\mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)]$ for all non-decreasing and concave functions u (such that the expectations exist), (b) $TVaR[X;\alpha] \leq TVaR[Y;\alpha]$ for all $\alpha \in (0,1)$, (c) $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$ for any concave distortion function g (see, e.g. Section 3.4 in Denuit *et al.* (2005)). This is also equivalent to Condition iii) of Corollary 2 since $VaR^{(2)}[X;\alpha] = (1-\alpha)TVaR[X;\alpha]$. Moreover, it is well-known that, if $X \preceq_{ST} Y$, then $X \preceq_{SL} Y$ (Condition i) implies Condition iii) if g is twice differentiable). Finally, note that, if $\mathbb{E}[X] \leq \mathbb{E}[Y]$, then Condition ii) of Corollary 2 also implies Condition iii) (see, e.g. the proof for Condition ii) and take $g(x) = x$).

Condition iv) of Corollary 2 looks like Condition (2.13) in Theorem 2 of Jewitt (1989) where $VaR[X;\cdot]$ is replaced by $\bar{F}_X(\cdot)$. This condition is necessary and sufficient in the expected utility framework for an agent to make the same choice between two risks as a less risk-averse agent.

We now study the willingness to pay for reducing risk. Let X and Y be two risks such that X is preferred over Y, i.e. $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$. The willingness to pay for reducing risk from Y to X, $\pi = \pi(Y, X; g) \geq 0$, is the amount of money an agent would be willing to pay in order to face risk X rather than Y, and is defined by

$$\pi(Y, X; g) = \mathbb{H}_g[Y] - \mathbb{H}_g[X]. \tag{2.5}$$

This a well-known concept in the economics literature, see, e.g. Eeckhoudt *et al.* (1997) for a discussion and a comparison with the risk premium in the expected utility framework.

We are interested in a condition for a more risk-averse decision-maker to be willing to pay more for an improvement from Y to X. Let us first consider the case of the Lognormal distributed risks of Example 1.

Example 1 (continued). The willingness to pay, $\pi(Y, X; g_\gamma)$, increases with γ (i.e. with risk aversion in the class of Normal distortion function) if and only if

$$\frac{\partial \pi\left(Y, X; g_\gamma\right)}{\partial \gamma} = \sigma_Y \mathbb{H}_{g_\gamma}[Y] - \sigma_X \mathbb{H}_{g_\gamma}[X] > 0. \tag{2.6}$$

If we assume moreover that, any more risk-averse decision-maker with distortion function $g_{\tilde{\gamma}}$ ($\tilde{\gamma} > \gamma$) will always prefer X over Y, which is equivalent to $\sigma_Y > \sigma_X$, we see that Condition (2.6) necessarily holds.

The following corollary gives a necessary and sufficient condition in the general case.

Corollary 3. *Assume a decision-maker with distortion function g prefers X over Y, i.e. $\pi(Y, X; g) \geq 0$, and that any more risk-averse decision-maker also prefers X over Y. Then, the following properties are equivalent:*

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i) A more risk-averse decision-maker with distortion function h has willingness to pay $\pi(Y, X; h)$ *that is larger than* $\pi(Y, X; g)$*;*

ii) for all $\alpha \in (0, 1)$*,*

$$\int_\alpha^1 g'(1-u)\,(Va\,R[X;u] - Va\,R[Y;u])\,du \le g(1-\alpha) \times \int_0^1 g'(1-u)\,(Va\,R[X;u] - Va\,R[Y;u])\,du. \tag{2.7}$$

Let $(g_\alpha)_{\alpha\in(0,1)}$ be the class of distortion functions defined by

$$g_\alpha(u) = \begin{cases} g(u)/g(1-\alpha), & u \le 1-\alpha, \\ 1, & u > 1-\alpha. \end{cases}$$

The distortion function g_α is the distortion function of a more risk-averse decision-maker than the decision-maker with distortion function g. Condition (2.7) is then equivalent to the condition that any decision-maker with distortion function g_α has a larger willingness to pay than the decision-maker with distortion function g

$$\pi(Y, X; g_\alpha) \ge \pi(Y, X; g), \quad \alpha \in (0, 1).$$

We may also note that a sufficient condition for the more risk-averse decision-maker to have a larger willingness to pay is given by Condition ii) of Corollary 2: there exists $\alpha_0 \in (0, 1)$ such that $Va\,R[X;\alpha] \ge Va\,R[Y;\alpha]$ for $0 < \alpha \le \alpha_0$ and $Va\,R[X;\alpha] \le Va\,R[Y;\alpha]$ for $\alpha_0 \le \alpha < 1$.

Example 1 (continued). Condition (2.7) is equivalent to

$$\mathbb{H}_g[Y]\left[\Phi\left(\sigma_Y + \gamma - \Phi^{-1}(\alpha)\right) - \Phi\left(\gamma - \Phi^{-1}(\alpha)\right)\right] > \mathbb{H}_g[X]\left[\Phi\left(\sigma_X + \gamma - \Phi^{-1}(\alpha)\right) - \Phi\left(\gamma - \Phi^{-1}(\alpha)\right)\right],$$

for all $\alpha \in (0, 1)$. Since a more risk-averse decision-maker with distortion function h also prefers X over Y if and only if $\sigma_Y > \sigma_X$, we see that Condition (2.7) always holds for this particular case.

3. Distortion risk measures and ambiguity aversion

We assume that the risk X is a random variable whose probability distribution belongs to a certain family of distributions $\{F_\theta, \theta \in \mathcal{T}\}$ (θ is a parameter for this family), called the parametric model, so that $F_X = F_{\theta_0}$ for some $\theta_0 \in \mathcal{T}$. The value θ_0 is unknown and is referred to as the true value of the parameter. We denote by X_θ a risk whose probability distribution function is F_θ. The risk measure of X, $\mathbb{H}_g[X]$, is then equal to $\mathbb{H}_g[X_{\theta_0}]$. We assume that $|\mathbb{H}_g[X_\theta]| < \infty$ for all $\theta \in \mathcal{T}$.

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The subjective beliefs about the value of θ_0 are captured by a random variable Θ with density probability function π. A decision-maker is said to be ambiguity-neutral if he uses as a risk measure the average value of the distortion risk measures with respect to the distribution of Θ:

$$\int_{\mathcal{T}} \mathbb{H}_g[X_\theta]\pi\,(\theta)\,d\theta = \mathbb{E}[\eta\,(\Theta)], \tag{3.8}$$

where $\eta(\theta) = \mathbb{H}_g[X_\theta]$ (under the condition that $\int_{\mathcal{T}} |\mathbb{H}_g[X_\theta]|\pi(\theta)d\theta < \infty$, which will be assumed from now).

We denote by X_Θ the random variable such that $(X_\Theta|\Theta = \theta) \sim F_\theta$ and by $X^{(\Theta)}$ the random variable whose quantile function is defined by

$$Va\,R[X^{(\Theta)};\alpha] = \int_{\mathcal{T}} Va\,R[X_\theta;\alpha]\pi\,(\theta)\,d\theta, \quad 0 \le \alpha \le 1. \tag{3.9}$$

X_Θ has a mixture distribution: its probability distribution function is equal to the average of the probability distribution functions of X_θ while, for $X^{(\Theta)}$, the average is made over the quantile functions. Under the condition of Fubini's theorem, the distortion risk measure of $X^{(\Theta)}$ equals to the average value of the distortion risk measures of X_θ with respect to the distribution of Θ:

$$\begin{aligned}
\mathbb{H}_g[X^{(\Theta)}] &= \int_0^1 g'(1-\alpha)\left[\int_{\mathcal{T}} Va\,R[X_\theta;\alpha]\pi\,(\theta)\,d\theta\right]d\alpha \\
&= \int_{\mathcal{T}}\left[\int_0^1 g'(1-\alpha)\,Va\,R[X_\theta;\alpha]d\alpha\right]\pi\,(\theta)\,d\theta \\
&= \int_{\mathcal{T}} \mathbb{H}_g[X_\theta]\pi\,(\theta)\,d\theta = \mathbb{E}[\mathbb{H}_g[X_\Theta|\Theta]].
\end{aligned}$$

It follows that an ambiguity-neutral decision maker implicitly assumes that, if there was no uncertainty, the distribution of its risk would be given by the distribution of $X^{(\Theta)}$.

In the case where the probability distribution functions, F_θ, are probability distribution functions of comonotonic risks, $X^{(\Theta)}$ is an average of these risks over θ (see also the discussion in Wang and Young (1998)). For example, suppose that $\mathcal{T} = \{\theta_{-1}, \theta_0, \theta_1\}$ and that $\mathbb{P}(\Theta = \theta_{-1}) = p$ and $\mathbb{P}(\Theta = \theta_1) = q$ with $0 < p < 1$, $0 < q < 1$, $p + q < 1$. Assume moreover that there exists a non-decreasing function h_{-1} (resp. h_1) such that $X_{\theta_{-1}} = h_{-1}(X)$ (resp. $X_{\theta_1} = h_1(X)$). Then,

$$\begin{aligned}
Va\,R[X^{(\Theta)};\alpha] &= p\,Va\,R[X_{\theta_{-1}};\alpha] + (1-p-q)\,Va\,R[X_{\theta_0};\alpha] + q\,Va\,R[X_{\theta_1};\alpha] \\
&= (ph_{-1} + (1-p-q) + ph_1)(Va\,R[X;\alpha]) \\
&= Va\,R[(ph_{-1} + (1-p-q) + ph_1)(X);\alpha],
\end{aligned}$$

and therefore $X^{(\Theta)}$ has the same distribution as $ph_{-1}(X)+(1-p-q)\,X+qh_1(X)$.

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The recursive model of ambiguity preferences with second-order probability distribution developed by Klibanoff *et al.* (2005) is now adapted to the class of distortion risk measures to investigate the value of information that resolves ambiguity or resolves risk. The main element of this model is a non-decreasing and convex transformation function φ that captures the ambiguity preference. The decision-maker's risk measure is defined by

$$\tilde{\mathbb{H}}_{g,\varphi}[X,\Theta] = \varphi^{-1}\left(\mathbb{E}[\varphi\left(\mathbb{H}_g[X_\Theta|\Theta]\right)]\right) = \varphi^{-1}\left(\int_{\mathcal{T}} \varphi\left(\mathbb{H}_g[X_\theta]\right)\pi\left(\theta\right)d\theta\right). \tag{3.10}$$

By Jensen's inequality, we derive that

$$\varphi^{-1}\left(\mathbb{E}[\varphi\left(\mathbb{H}_g[X_\Theta|\Theta]\right)]\right) \geq \mathbb{E}[\mathbb{H}_g[X_\Theta|\Theta]] = \mathbb{H}_g[X^{(\Theta)}].$$

Therefore, $\tilde{\mathbb{H}}_{g,\varphi}[X,\Theta]$ is always larger than $\mathbb{H}_g[X^{(\Theta)}]$, the risk measure of an ambiguity-neutral decision maker. The convexity of φ expresses ambiguity aversion. Note that the idea proposed by Klibanoff *et al.* to introduce such a function to represent the decision maker's uncertainty about the risk distribution has also been used for several years by actuaries to price insurance policies and is known as the "mean value principle" (see, e.g. Laeven and Goovaerts, 2011 for a detailed comparison of the main premium calculation principles). It could have been possible to consider other actuarial pricing principles that satisfy the compulsory risk-loading property to take into account uncertainty, but in this paper we restrict to the mean value principle as suggested by Klibanoff *et al.* for the expected utility framework because we can derive interesting conclusions about the effect of ambiguity-aversion on the valuation of risk reduction.

Example 1 (continued). Recall that $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$ and $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$. Suppose that the mean of X is ambiguous in the sense that the decision-maker has an imperfect knowledge of it. Assume that $X_\theta \sim \mathcal{LN}(\log(\theta) - \sigma_X^2/2, \sigma_X^2)$ with $\theta \in T = (0,\infty)$. Note that $\mathbb{H}_{g_\gamma}[X_\theta] = \theta\exp(\gamma\sigma_X)$. The ambiguity takes the form of probability distributions for Θ and we may assume, for example, that $\Theta \sim Gamma(\theta_m\beta, \beta)$, with $\theta_m > 0$ and $\beta > 0$. Assume moreover that $\varphi(x) = \exp(\alpha x)$ with $\alpha \geq 0$, then for $\beta > \alpha\exp(\gamma\sigma_X)$

$$\begin{aligned}\tilde{\mathbb{H}}_{g_\gamma,\varphi}[X,\Theta] &= -\frac{\beta}{\alpha}\ln\left(1 - \frac{\alpha}{\beta}\exp\left(\gamma\sigma_X\right)\right)\theta_m \\ &= -\rho^{-1}(\alpha,\beta,\gamma)\ln\left(1-\rho(\alpha,\beta,\gamma)\right)\mathbb{H}_{g_\gamma}[X^{(\Theta)}],\end{aligned}$$

where $\rho(\alpha,\beta,\gamma) = \alpha\beta^{-1}\exp(\gamma\sigma_X)$. The ambiguity-averse decision-maker's risk measure, $\tilde{\mathbb{H}}_{g_\gamma,\varphi}[X,\Theta]$, decreases with β, but increases with α. As α tends to zero or β tends to infinity, the risk measure tends to the risk measure of the ambiguity-neutral decision maker, i.e. $\mathbb{H}_{g_\gamma}[X^{(\Theta)}]$.

We see from Example 1 that the existence of $\mathbb{E}[\varphi(\mathbb{H}_{g_\gamma}[X_\Theta|\Theta])$ is not necessarily satisfied (if $\alpha \geq \beta\exp(-\gamma\sigma_X)$). The choice of a function φ should be made

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to ensure the existence of $\tilde{\mathbb{H}}_{g_\gamma,\varphi}[X, \Theta]$ for large families of distributions π on $\mathcal{T}$. This issue is not new and has been raised by several authors in the case of the mean value principle (see e.g. references in Laeven and Goovaerts, 2011).

Let Θ_1 and Θ_2 be two random variables that capture the subjective beliefs about the value θ_0 and that satisfy the mean-preserving condition

$$\mathbb{E}[\mathbb{H}_g[X_{\Theta_1}|\Theta_1]] = \mathbb{E}[\mathbb{H}_g[X_{\Theta_2}|\Theta_2]]. \tag{3.11}$$

This condition ensures that the behavior of an ambiguity-neutral decision maker will be unaffected into the choice setting if Θ_1 was replaced by Θ_2. We say that the information used for constructing the prior distribution of Θ_2 "reduces" ambiguity with respect to the information used for constructing the prior distribution of Θ_1 if

$$\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_1] \geq \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_2].$$

Information that reduces ambiguity has a positive value for decision makers who are ambiguity averse. A sufficient condition is given by the condition that $\mathbb{H}_g[X_{\Theta_1}|\Theta_1]$ dominates $\mathbb{H}_g[X_{\Theta_2}|\Theta_2]$ in the stop-loss order, i.e. $\mathbb{H}_g[X_{\Theta_1}|\Theta_1] \succeq_{SL} \mathbb{H}_g[X_{\Theta_2}|\Theta_2]$.

Example 2. A location-scale family is a family of probability distributions parametrized by a location parameter and a (non-negative) scale parameter: if X is any random variable whose probability distribution belongs to such a family, then $Y = a + bX$, $b > 0$, also belongs to this family. Let μ be the mean of X and σ its standard deviation, then

$$VaR[X_{\mu,\sigma}; \alpha] = \mu + \sigma q_\alpha, \quad 0 \leq \alpha \leq 1.$$

where q_α is the quantile function of the centered and standardized distribution and

$$\mathbb{H}_g[X_{\mu,\sigma}] = \mu + \sigma \int_0^1 g'(1-\alpha) q_\alpha d\alpha.$$

The ambiguity may then concern μ or σ. In this case, $\mathbb{H}_g[X_{\mu_1,\sigma}|\mu_1] \succeq_{SL} \mathbb{H}_g[X_{\mu_2,\sigma}|\mu_2]$ is equivalent to $\mu_1 \succeq_{SL} \mu_2$ when μ_1 and μ_2 are random variables and σ is fixed, or $\mathbb{H}_g[X_{\mu,\sigma_1}|\sigma_1] \succeq_{SL} \mathbb{H}_g[X_{\mu,\sigma_2}|\sigma_2]$ is equivalent to $\sigma_1 \succeq_{SL} \sigma_2$ if $\int_0^1 g'(1-\alpha) q_\alpha d\alpha > 0$, σ_1 and σ_2 are random variables and μ is fixed.

Example 3. Let X be a random variable whose values are larger than 1 and let $Y_\tau = X^\tau$ for a positive constant τ. Then, we have

$$VaR[Y_\tau; \alpha] = (VaR[X; \alpha])^\tau, \quad 0 \leq \alpha \leq 1.$$

The ambiguity may concern τ. In this case, $\mathbb{H}_g[Y_{\tau_1}|\tau_1] \succeq_{SL} \mathbb{H}_g[Y_{\tau_2}|\tau_2]$ is equivalent to $\tau_1 \succeq_{SL} \tau_2$, because $\mathbb{H}_g[Y_\tau]$ is an increasing and convex function of τ.

For a large number of parametric families of distributions, it is possible to order the distribution by using a distortion risk measure. It is the case, for example, for the families of distributions of Examples 2 and 3.

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Definition 1. *We say that* $\{F_\theta, \theta \in \mathcal{T}\}$ *is "invariant" with respect to* $\mathbb{H}_g$*, if, for all* $\theta_1, \theta_2 \in \mathcal{T} \subset \mathbb{R}$*, such that* $\theta_1 \geq \theta_2$*, then* $\mathbb{H}_g[X_{\theta_1}] \geq \mathbb{H}_g[X_{\theta_2}]$*. If* Θ *satisfies the mean preserving condition* $\mathbb{H}_g[X_{\theta_0}] = \mathbb{E}[\mathbb{H}_g[X_\Theta|\Theta]]$*, then* $\theta^*_{\Theta,g,\varphi}$*, defined implicitly by*

$$\mathbb{H}_g[X_{\theta^*_{\Theta,g,\varphi}}] = \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta], \tag{3.12}$$

is larger than θ_0 *and is called the cautious parameter associated to* Θ*,* g *and* φ*.*

The mean preserving condition ensures that the risk measure for X of a decision maker who knows the value of θ_0 and the risk measure of an ambiguity-neutral decision maker who models his subjective beliefs by Θ are equal. An ambiguity-averse decision maker will therefore have a larger risk measure for X, and $\theta^*_{\Theta,g,\varphi}$ enables him to take into account the imperfect knowledge of θ_0 without using the distribution of Θ.

Note that if, for all $\alpha \in (0, 1)$ and $\theta_1, \theta_2 \in \mathcal{T}$, such that $\theta_1 \geq \theta_2$, we have

$$\int_\alpha^1 g'(1-u)\left(Va\,R[X_{\theta_2}; u] - Va\,R[X_{\theta_1}; u]\right) du \leq 0,$$

we deduce by Theorem 1 that, for any more risk-averse decision-maker with distortion function h, $\{F_\theta, \theta \in \mathcal{T}\}$ is also invariant with respect to $\mathbb{H}_h$.

Example 1 (continued). If $\Theta \sim Gamma(\theta_0\beta, \beta)$, with $\beta > 0$, $\varphi_\alpha(x) = \exp(\alpha x)$ with $0 < \alpha < \beta \exp(-\gamma\sigma_X)$, then $\{F_\theta, \theta \in T\}$ is invariant with respect to H_{g_γ} and, for $X = X_{\theta_0}$,

$$\theta^*_{\Theta,g_\gamma,\varphi_\alpha} = -\rho^{-1}(\alpha, \beta, \gamma) \ln(1 - \rho(\alpha, \beta, \gamma))\,\theta_0, \tag{3.13}$$

where $\rho(\alpha, \beta, \gamma) = \alpha\beta^{-1} \exp(\gamma\sigma_X)$t.

We are now concerned with the willingness to pay for information that reduces ambiguity or resolves ambiguity.

Let Θ_1 and Θ_2 be two random variables that satisfy the mean-preserving condition (3.11) and such that $\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_1] \geq \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_2]$. The willingness to pay for information that reduces ambiguity from Θ_1 to Θ_2, $\omega = \omega(\Theta_1, \Theta_2)$, is defined implicitly by

$$\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_1] = \tilde{\mathbb{H}}_{g,\varphi}[X + \omega, \Theta_2]. \tag{3.14}$$

It is the amount of money that is necessary to use to pass from the a priori knowledge Θ_1 to that of Θ_2.

The value of information that resolves ambiguity, $\omega(\Theta, \theta_0)$, is the value of information that eliminates ambiguity from the a priori knowledge Θ to the true value θ_0 and is therefore defined by

$$\omega(\Theta, \theta_0) = \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta] - \mathbb{H}_g[X]. \tag{3.15}$$

Note that the mean preserving condition $\mathbb{E}[\mathbb{H}_g[X_\Theta|\Theta]] = \mathbb{H}_g[X_{\theta_0}]$ (which is implicitly assumed here) has an equivalent form in the work by Klibanoff *et al.*

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(2005) for the expected utility theory framework to disentangle the effect of ambiguity-aversion from that of risk-aversion. For the location-scale families, it is equivalent to the condition that the decision-maker has an unbiased view of the parameter θ_0 (μ_0 or σ_0), but this condition could be more difficult to interpret for other pairs (family of distributions; distortion function) like the pair introduced in Example 3. However it is not difficult to construct a distribution π that satisfies the mean-preserving condition from a sample data from F_{θ_0} as explained in the following remark.

Remark. Parameter θ_0 is typically estimated from an i.i.d random sample $\{X_1, \ldots, X_n\}$ where $X_i \sim F_{\theta_0}$. Assume that $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$ is an unbiased estimator of θ_0 such that

$$\hat{\theta}_n = \theta_0 + \frac{\sigma(\theta_0)}{\sqrt{n}} Z,$$

where Z is a centered random variable with variance equal to 1 and $\sigma(\cdot)$ is a known function. Let us assume that $\eta(\cdot) = H_g[X_{(\cdot)}]$ is three times differentiable at θ_0 with $\eta'(\theta_0) \neq 0$ and let

$$\tilde{\theta}_n = \hat{\theta}_n - \frac{\eta''(\hat{\theta}_n)}{\eta'(\hat{\theta}_n)} \frac{\sigma^2(\hat{\theta}_n)}{n}.$$

By the delta method, we get

$$\mathbb{H}_g[X_{\tilde{\theta}_n} | \tilde{\theta}_n] = \mathbb{H}_g[X_{\theta_0}] + \eta'(\theta_0) \frac{\sigma(\theta_0)}{\sqrt{n}} Z + o_P\left(n^{-1}\right),$$

as n tends to infinity. Therefore, the bias of $H_g[X_{\tilde{\theta}_n} | \tilde{\theta}_n]$ is asymptocally negligeable with repect to its variance. To construct a distribution for $\tilde{\theta}_n$, one may, for example, resample the sample data $\{X_1, \ldots, X_n\}$ (by using, e.g. bootstrap, jackknife,...) and performs inference on to get several values $\tilde{\theta}_{i,n}$, $i = 1, \ldots, N$. Then, use a Kernel estimator of the density to obtain a candidate for π that satisfies asymptotically the mean preserving condition.

We immediately derive the following proposition.

Proposition 4. *The willingness to pay for information that resolves ambiguity increases with the stop-loss order:*

$$\mathbb{H}_g[X_{\Theta_1} | \Theta_1] \succeq_{SL} \mathbb{H}_g[X_{\Theta_2} | \Theta_2] \Longrightarrow \omega(\Theta_1, \theta_0) \geq \omega(\Theta_2, \theta_0).$$

As for the risk-aversion, we define the notion of a greater ambiguity-aversion: a decision-maker with ambiguity function ψ is said to be more ambiguity-averse than a decision-maker with ambiguity function φ, if there exists an increasing and convex function χ such that $\psi = \chi \circ \varphi$.

Theorem 5. *For ambiguity-averse decision makers, the willingness to pay for information that resolves ambiguity increases with greater ambiguity aversion.*

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This result is coherent with Theorem 3 in Snow (2010) which studies the value of information in an expected utility model of ambiguity with second-order probabilities. The next theorem gives a sufficient condition on the distortion functions of two decision-makers with the same ambiguity function such that the willingness to pay for information that resolves ambiguity is larger than for the more risk-averse decision-maker.

Let us consider a decision-maker with distortion function g and ambiguity function φ, and a more risk-averse decision-maker with distortion function h and ambiguity function φ. Let Θ be a random variable that captures the subjective beliefs about the value θ_0 such that

$$\mathbb{E}[\mathbb{H}_g[X_\Theta|\Theta]] = \mathbb{H}_g[X_{\theta_0}] \text{ and } \mathbb{E}[\mathbb{H}_h[X_\Theta|\Theta]] = \mathbb{H}_h[X_{\theta_0}].$$

Theorem 6. *Assume that $\{F_\theta, \theta \in \mathcal{T}\}$ is invariant with respect to $\mathbb{H}_g$ and $\mathbb{H}_h$. The willingness to pay for information that resolves ambiguity is larger for the decision-maker with distortion function h than for the decision-maker with distortion function g if $\theta^*_{\Theta,h,\varphi} \geq \theta^*_{\Theta,g,\varphi}$ and, for all $\alpha \in (0,1)$,*

$$\int_\alpha^1 g'(1-u)\left(VaR[X_{\theta_0};u] - VaR[X_{\theta^*_{\Theta,g,\varphi}};u]\right)du \leq g(1-\alpha)\int_0^1 g'(1-u)\left(VaR[X_{\theta_0};u] - VaR[X_{\theta^*_{\Theta,g,\varphi}};u]\right)du. \quad (3.16)$$

Example 1 (continued). Assume that $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$ and $X_\theta \sim \mathcal{LN}(\log(\theta) - \sigma_X^2/2, \sigma_X^2)$ with $\theta \in T = (0,\infty)$. If $X = X_{\theta_0}$, $\Theta \sim Gamma(\theta_0\beta, \beta)$, with $\beta > 0$, and $\varphi_\alpha(x) = \exp(\alpha x)$ with $0 < \alpha < \beta\exp(-\gamma\sigma_X)$, then we have $\theta^*_{\Theta,g_{\tilde\gamma},\varphi_\alpha} \geq \theta^*_{\Theta,g_\gamma,\varphi_\alpha}$ for $\tilde\gamma \geq \gamma$ by Equation (3.13), and Condition (3.16) is equivalent to

$$\left(1+\rho^{-1}(\alpha,\beta,\gamma)\ln(1-\rho(\alpha,\beta,\gamma))\right)\exp(\gamma\sigma_X) \times \left[\Phi\left(\sigma_X+\gamma-\Phi^{-1}(\alpha)\right) - \Phi\left(\gamma-\Phi^{-1}(\alpha)\right)\right] \leq 0,$$

where $\rho(\alpha,\beta,\gamma) = \alpha\beta^{-1}\exp(\gamma\sigma_X)$. This condition holds since $0 < \rho(\alpha,\beta,\gamma) < 1$. We therefore deduce that the willingness to pay for information that resolves ambiguity is larger for the more risk-averse decision-maker.

4. Ambiguity aversion and optimal effort

We consider a decision-maker with distortion function g who faces a risk Z. Assume that this decision-maker could make an effort $e \in [0,1]$ to shift his risk distribution toward a better target distribution of a risk Y such that Y is preferred over Z, i.e. $\mathbb{H}_g[Y] \leq \mathbb{H}_g[Z]$. But this effort has a monetary cost characterized by a function c with $c(0) = 0$, $c' > 0$ and $c'' > 0$. After the investment in effort

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e, the decision maker's final risk is denoted by $X^{(e)}$ whose quantile function is assumed to be given by

$$VaR[X^{(e)};\alpha] = e\,VaR[Y;\alpha] + (1-e)\,VaR[Z;\alpha], \quad 0 < \alpha < 1.$$

The objective function of the decision-maker is then given by

$$\min_{e\in[0,1]} \left(\mathbb{H}_g[X^{(e)}] + c(e)\right), \tag{4.17}$$

with

$$\mathbb{H}_g[X^{(e)}] = e\mathbb{H}_g[Y] + (1-e)\mathbb{H}_g[Z].$$

Remark. An equivalent optimization problem is that of an insurance company that wants to purchase an excess-of-loss cover with attachment point a and limit h for a risk Z_{nr} (not reinsured) such that the risk that is passed to the reinsurer is given by $Z = \min((Z_{nr}-a)_+, h)$. The risk measure of Z is then given by the price of the reinsurance contract:

$$\mathbb{H}_g[Z] = \int_0^\infty g\left(\bar{F}_Z(u)\right) du = \int_a^{a+h} g\left(\bar{F}_{Z_{nr}}(u)\right) du,$$

where the distortion function g is chosen by the resinsurer. The ceding company may limit the exposure to losses and modify the risk such that its new distribution is given by a random variable Y_{nr}. The new price of reinsurance is then given by

$$\mathbb{H}_g[Y] = \int_0^\infty g\left(\bar{F}_Y(u)\right) du = \int_a^{a+h} g\left(\bar{F}_{Y_{nr}}(u)\right) du,$$

where $Y = \min((Y_{nr}-a)_+, h)$. But the effort to limit the exposure has a monetary cost, and the insurance company has to make a trade-off between the reinsurance cost and the limit of his exposure.

We assume that the optimal effort e_g^* belongs to the interval $(0, 1)$ or equivalently that $\mathbb{H}_g[Y] + c'(0) < \mathbb{H}_g[Z] < \mathbb{H}_g[Y] + c'(1)$.

Theorem 7. *Assume that for all* $\alpha \in (0, 1)$,

$$\int_\alpha^1 g'(1-u)\,(VaR[Y;u] - VaR[Z;u])\,du \le g(1-\alpha) \times \int_0^1 g'(1-u)\,(VaR[Y;u] - VaR[Z;u])\,du. \tag{4.18}$$

Then, a more risk-averse decision-maker will invest more in effort.

Note that, if Condition (4.18) holds, then, by Corollary 3, a more risk-averse decision-maker with distortion function h has a willingness to pay $\pi(Z, Y; h)$ that is larger than $\pi(Z, Y; g)$. This explains why he will invest more in effort than the decision-maker with distortion function g.

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We now impose ambiguity on the target distribution of Y as in Huang (2012) for the framework of the expected utility. Let $\underline{Y}$ and $\overline{Y}$ be two risks such that $\mathbb{H}_g[\underline{Y}] \leq \mathbb{H}_g[\overline{Y}] \leq \mathbb{H}_g[Z]$. We assume that Y is a random variable whose probability distribution belongs to the family of distributions $\{F_\theta, \theta \in [0,1]\}$ characterized by the random variables Y_θ with probability distribution functions F_θ and quantile functions

$$Va\,R[Y_\theta;\alpha] = (1-\theta)\ Va\,R[\underline{Y};\alpha] + \theta\, Va\,R[\overline{Y};\alpha], \quad 0 \leq \alpha \leq 1.$$

For a given $\theta \in [0,1]$, the decision-maker's final risk after the investment in effort e is denoted by $X_\theta^{(e)}$ and has a quantile function which is given by

$$\begin{aligned} Va\,R[X_\theta^{(e)};\alpha] = e\left[(1-\theta)\ Va\,R[\underline{Y};\alpha] + \theta\, Va\,R[\overline{Y};\alpha\right] \\ +(1-e)\,Va\,R[Z;\alpha], \quad 0 < \alpha < 1. \end{aligned}$$

The objective function of the decision maker is then given by

$$\min_{e\in[0,1]} \tilde{\mathbb{H}}_{g,\varphi}[X^{(e)} + c(e), \Theta] = \min_{e\in[0,1]} \varphi^{-1}\left(\mathbb{E}\left[\varphi\left(\mathbb{H}_g[X_\Theta^{(e)}|\Theta] + c(e)\right)\right]\right). \quad (4.19)$$

Example 4. Assume that $\varphi_\alpha(x) = \exp(\alpha x)$, then

$$\begin{aligned} \tilde{\mathbb{H}}_{g,\varphi_\alpha}[X^{(e)} + c(e), \Theta] = \frac{1}{\alpha} \ln \mathbb{E}\left[\exp\left(\alpha e\left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}]\right)\right)\Theta\right] \\ +e\mathbb{H}_g[\overline{Y}] + (1-e)\mathbb{H}_g[Z] + c(e). \end{aligned}$$

The first-order condition of the objective function is given by

$$\frac{\partial \tilde{\mathbb{H}}_{g,\varphi_\alpha}}{\partial e} = \frac{1}{e}\pi_{Es}\left(e\left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}]\right)\Theta;\alpha\right) + \left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[Z]\right) + c'(e) = 0,$$

where

$$\begin{aligned} &\pi_{Es}\left(e\left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}]\right)\Theta;\alpha\right) \\ &= \frac{\mathbb{E}\left[e\left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}]\Theta\right)\exp\left(\alpha e\left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}]\right)\Theta\right)\right]}{\mathbb{E}\left[\exp\left(\alpha e\left(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}]\right)\Theta\right)\right]} \end{aligned}$$

is the Esscher premium of $e(\mathbb{H}_g[\overline{Y}] - \mathbb{H}_g[\underline{Y}])\Theta$ with index α. It is well-known that the Esscher premium increases with α. It follows that, if there exists an optimal effort for some α, then a more ambiguity-averse decision-maker (with a larger α) will make less effort to reduce its risk.

For the general case, we assume that the optimal effort e_φ^* belongs to the interval $(0,1)$.

Theorem 8. *A more ambiguity-averse decision-maker will invest less in effort.*

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This result shows that risk-aversion and ambiguity-aversion may have different impact on the effort to reduce risk. It is clear that a more risk-averse decision-maker will invest more in effort if its willingness to pay to reduce risk is larger. At a first glance, it could be counterintuitive that a more ambiguity-averse decision-maker will invest less in effort, but it is not the case because the uncertainty concerns the target distribution. If the decision-maker has an imperfect knowledge of this distribution, a more ambiguity-averse decision-maker will be more reluctant to enter in a procedure to reduce risk.

5. Conclusion

We consider concave distortion risk measure minimizers choosing two risks and examine how choice is influenced by the decision-maker's attitude to risk. We provide a necessary and sufficient condition such that, if a decision-maker prefers a risk over another, then a more risk-averse decision-maker will make the same choice. We derive a condition for a more risk-averse decision-maker to be willing to pay more for an improvement in risk measures.

We then assume that the decision-maker is uncertain about a parameter of the risk distribution and we use this last condition to study the effect of ambiguity-aversion on the valuation of risk reduction. We show that the willingness to pay for information that resolves ambiguity is always larger for a more ambiguity-averse decision-maker, but is not necessarily larger for a more risk-averse decision maker. Moreover, we prove that, if a decision-maker is able to make an effort to reduce his risk, then a more risk-averse decision-maker will invest more in effort if its willingness to pay to reduce risk is larger than the less risk-averse decision-maker. However a more ambiguity-averse decision-maker will always invest less in effort when the ambiguous beliefs concern the distribution of the less risky distribution. This result may be, for example, of practical interest for loss prevention and safety incentive programs if there exists ambiguity about the target distributions of risks.

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APPENDIX

Proof of Theorem 1

Suppose i) holds. $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$ is equivalent to

$$\int_0^1 g'(1-u)\,(VaR[X;u] - VaR[Y;u])\,du \leq 0.$$

Since g is concave and $g(0) = 0$, we have, for $u_0 \in (0,1)$, $g(u_0) > 0$. Let us define the concave distortion function f_0 by

$$f_0(u) = \begin{cases} u/g(u_0), & u \leq g(u_0), \\ 1, & u > g(u_0), \end{cases}$$

Then, we have

$$h(u) = \begin{cases} g(u)/g(u_0), & u \leq u_0, \\ 1, & u > u_0, \end{cases}$$

and $\mathbb{H}_h[X] \leq \mathbb{H}_h[Y]$ is equivalent to $\int_{1-u_0}^1 g'(1-u)(VaR[X;u] - VaR[Y;u])\,du \leq 0$. Therefore, for all $\alpha \in (0,1)$, $\int_\alpha^1 g'(1-u)(VaR[X;u] - VaR[Y;u])du \leq 0$.

Suppose ii) holds. Let $h = f \circ g$ (where f is a concave distortion function) be the distortion function of the more risk-averse agent. Let $\mathcal{E}_g$ (resp. $\mathcal{E}_f$) be the set where g' (resp. f') fails to exist. For $u \in \mathcal{E}_g^c \cap \mathcal{E}_f^c$, we have

$$h'(u) = f'(g(u))g'(u),$$

and therefore

$$\int_0^1 h'(1-u)\,(VaR[X;u] - VaR[Y;u])\,du$$
$$= \int_0^1 f'(g(1-u))g'(1-u)\,(VaR[X;u] - VaR[Y;u])\,du.$$

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We now use the same type of arguments as in the proof of Theorem 5.2.1 in Dhaene *et al.* (2006). We first prove the implication for concave piecewise linear distortion functions f. Any such distortion function can be written in the form

$$f(x) = \sum_{i=1}^{n} a_i(\beta_i - \beta_{i+1}) \min(x/a_i, 1),$$

where $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = 1$. Further, β_i is the derivative of f in the interval (a_{i-1}, a_i) and is non-negative, and $\beta_{n+1} = 0$. Because of the concavity of f , we have that β_i is a decreasing function of i. And since f is a distortion function $f(1) = \sum_{i=1}^{n} a_i(\beta_i - \beta_{i+1}) = 1$. Since

$$f'(x) = \sum_{i=1}^{n} (\beta_i - \beta_{i+1}) I_{\{0<x<a_i\}},$$

we deduce that

$$\begin{aligned}
&\int_0^1 h'(1-u)\,(Va\,R[X;u] - Va\,R[Y;u])\,du \\
&\quad = \sum_{i=1}^{n} (\beta_i - \beta_{i+1}) \int_0^1 g'(1-u)\,(Va\,R[X;u] - Va\,R[Y;u])\, I_{\{0<g(1-u)<a_i\}} du \\
&\quad = \sum_{i=1}^{n} (\beta_i - \beta_{i+1}) \int_{1-g^{-1}(a_i)}^1 g'(1-u)\,(Va\,R[X;u] - Va\,R[Y;u])\,du \le 0
\end{aligned}$$

and then $\mathbb{H}_h[X] \le \mathbb{H}_h[Y]$. The result is derived for general concave distortion functions f by using the monotone convergence theorem as in Dhaene *et al.* (2006).

Proof of Corollary 2

By Theorem 1, it is sufficient to prove that for all $\alpha \in (0, 1)$,

$$d(\alpha) = \int_\alpha^1 g'(1-u)\,(Va\,R[X;u] - Va\,R[Y;u])\,du \le 0.$$

Note that

$$d'(\alpha) = -g'(1-\alpha)\,(Va\,R[X;\alpha] - Va\,R[Y;\alpha])\,, \qquad \alpha \in (0,1), \tag{5.20}$$

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and that by integrating by parts (when g is twice differentiable)

$$\begin{aligned}
d(\alpha) &= \int_0^{1-\alpha} g'(u)\,(VaR[X;1-u]-VaR[Y;1-u])\,du \\
&= \left[g'(u)\left(\int_0^u (VaR[X;1-v]-VaR[Y;1-v])\,dv\right)\right]_0^{1-\alpha} \\
&\quad -\int_0^{1-\alpha} g''(u)\left(\int_0^u (VaR[X;1-v]-VaR[Y;1-v])\,dv\right)du \\
&= g'(1-\alpha)\left(\int_0^{1-\alpha} (VaR[X;1-v]-VaR[Y;1-v])\,dv\right) \\
&\quad -\int_0^{1-\alpha} g''(u)\left(\int_0^u (VaR[X;1-v]-VaR[Y;1-v])\,dv\right)du \\
&= g'(1-\alpha)\left(VaR^{(2)}[X;\alpha]-VaR^{(2)}[Y;\alpha]\right) \\
&\quad -\int_0^{1-\alpha} g''(u)\left(VaR^{(2)}[X;1-u]-VaR^{(2)}[Y;1-u]\right)du \\
&= g'(1-\alpha)\left(VaR^{(2)}[X;\alpha]-VaR^{(2)}[Y;\alpha]\right) \\
&\quad -\int_\alpha^1 g''(1-u)\left(VaR^{(2)}[X;u]-VaR^{(2)}[Y;u]\right)du. \qquad (5.21)
\end{aligned}$$

i) Obvious.

ii) By Equation (5.20), d is non-increasing for $0 < \alpha \leq \alpha_0$ and non-decreasing for $\alpha_0 \leq \alpha < 1$. Since $d(0) \leq 0$ and $d(1) = 0$, $d(\alpha) \leq 0$ for all $\alpha \in (0,1)$.

iii) By Equation (5.21) and the concavity of g, $d(\alpha) \leq 0$ for all $\alpha \in (0,1)$.

iv) By Equation (5.20), d is non-increasing for $0 < \alpha \leq \alpha_0$ and by Equation (5.21) and the concavity of g, $d(\alpha) \leq 0$ for all $\alpha_0 \leq \alpha < 1$. The result follows.

Proof of Corollary 3

First recall that

$$\pi(Y, X; g) = \mathbb{H}_g[Y] - \mathbb{H}_g[X] = \int_0^1 g'(1-u)\,(VaR[Y;u]-VaR[X;u])\,du.$$

Let $X_\pi = X + \pi(Y, X; g)$. We have $\mathbb{H}_g[X_\pi] = \mathbb{H}_g[Y]$. By Theorem 1, a more risk-averse decision-maker with distortion function h prefers X_π to Y, i.e.

$$\mathbb{H}_h[X_\pi] \leq \mathbb{H}_h[Y] \Leftrightarrow \mathbb{H}_h[X] + \pi(Y, X; g) \leq \mathbb{H}_h[Y] \Leftrightarrow \pi(Y, X; g) \leq \pi(Y, X; h),$$

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if and only if, for all $\alpha \in (0, 1)$,

$$\begin{aligned}
&\int_\alpha^1 g'(1-u)\,(VaR[X_\pi; u] - VaR[Y; u])\,du \le 0 \\
&\Leftrightarrow \int_\alpha^1 g'(1-u)\,(VaR[X; u] - VaR[Y; u])\,du \\
&\quad \le -\pi\,(Y, X; g) \int_\alpha^1 g'(1-u)du \\
&\Leftrightarrow \int_\alpha^1 g'(1-u)\,(VaR[X; u] - VaR[Y; u])\,du \le g\,(1-\alpha) \\
&\quad \times \int_0^1 g'(1-u)\,(VaR[X; u] - VaR[Y; u])\,du.
\end{aligned}$$

Proof of Theorem 5

We have

$$\begin{aligned}
\psi\left(\mathbb{H}_g[X_{\theta_0}] + \omega\,(\Theta, \theta_0; \psi)\right) &= \mathbb{E}[\psi\left(\mathbb{H}_g[X_\Theta]\right)] \\
&= \mathbb{E}[\chi\left(\varphi\left(\mathbb{H}_g[X_\Theta]\right)\right)] \\
&\ge \chi\left(\mathbb{E}[\varphi\left(\mathbb{H}_g[X_\Theta]\right)]\right) \\
&\ge \chi\left(\varphi\left(\mathbb{H}_g[X_{\theta_0}] + \omega\,(\Theta, \theta_0; \varphi)\right)\right) \\
&= \psi\left(\mathbb{H}_g[X_{\theta_0}] + \omega\,(\Theta, \theta_0; \varphi)\right)
\end{aligned}$$

and then, since ψ is an increasing function, we deduce that

$$\omega\,(\Theta, \theta_0; \psi) \ge \omega\,(\Theta, \theta_0; \varphi)\,.$$

Proof of Theorem 6

Since $\{F_\theta, \theta \in \mathcal{T}\}$ is invariant with respect to $\mathbb{H}_g$, the willingness to pay for information that resolves ambiguity of the decision-maker with distortion function g is

$$\begin{aligned}
\omega\,(\Theta, \theta_0; g) &= \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta] - \mathbb{H}_g[X_{\theta_0}] = \mathbb{H}_g[X_{\theta^*_{\Theta,g,\varphi}}] - \mathbb{H}_g[X_{\theta_0}] \\
&= \pi\,(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; g),
\end{aligned}$$

and, since $\{F_\theta, \theta \in \mathcal{T}\}$ is invariant with respect to $\mathbb{H}_h$, the willingness to pay for information that resolves ambiguity of the decision-maker with distortion

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function h is

$$\begin{aligned}\omega(\Theta, \theta_0; h) &= \tilde{\mathbb{H}}_{h,\varphi}[X, \Theta] - \mathbb{H}_h[X_{\theta_0}] = \mathbb{H}_h[X_{\theta^*_{\Theta,h,\varphi}}] - \mathbb{H}_h[X_{\theta_0}] \\ &= \pi(X_{\theta^*_{\Theta,h,\varphi}}, X_{\theta_0}; h).\end{aligned}$$

Therefore,

$$\begin{aligned}&\omega(\Theta, \theta_0; h) - \omega(\Theta, \theta_0; g) \\ &\quad= \pi(X_{\theta^*_{\Theta,h,\varphi}}, X_{\theta_0}; h) - \pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; g) \\ &\quad= \left[\pi(X_{\theta^*_{\Theta,h,\varphi}}, X_{\theta_0}; h) - \pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; h)\right] \\ &\qquad+ \left[\pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; h) - \pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; g)\right] \\ &\quad= \left(\mathbb{H}_h[X_{\theta^*_{\Theta,h,\varphi}}] - \mathbb{H}_h[X_{\theta^*_{\Theta,g,\varphi}}]\right) + \left[\pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; h) - \pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; g)\right].\end{aligned}$$

First, we have

$$\mathbb{H}_h[X_{\theta^*_{\Theta,h,\varphi}}] - \mathbb{H}_h[X_{\theta^*_{\Theta,g,\varphi}}] \geq 0 \Leftrightarrow \theta^*_{\Theta,h,\varphi} \geq \theta^*_{\Theta,g,\varphi}.$$

Then, by Corollary 3

$$\pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; h) - \pi(X_{\theta^*_{\Theta,g,\varphi}}, X_{\theta_0}; g) \geq 0,$$

if for, all $\alpha \in (0, 1)$,

$$\begin{aligned}&\int_\alpha^1 g'(1-u)\left(VaR[X_{\theta_0}; u] - VaR[X_{\theta^*_{\Theta,g,\varphi}}; u]\right) du \\ &\qquad\leq g(1-\alpha)\int_0^1 g'(1-u)\left(VaR[X_{\theta_0}; u] - VaR[X_{\theta^*_{\Theta,g,\varphi}}; u]\right) du,\end{aligned}$$

and the result follows.

Proof of Theorem 7

The objective function of the decision-maker with distortion function g is given by

$$\min_{e\in[0,1]} F_g(e) = \min_{e\in[0,1]} \left(\mathbb{H}_g[X^{(e)}] + c(e)\right).$$

The first-order condition of the objective function is

$$\left.\frac{\partial F_g(e)}{\partial e}\right|_{e=e^*_g} = \mathbb{H}_g[Y] - \mathbb{H}_g[Z] + c'(e^*_g) = 0,$$

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and the second-order condition is

$$\left.\frac{\partial^2 F(e)}{\partial e^2}\right|_{e=e_g^*} = c''(e_g^*) > 0.$$

Let us now consider a more risk-averse decision-maker with distortion function $h = f \circ g$ where f is a concave distortion function. His objective function is given by

$$\min_{e\in[0,1]} F_h(e) = \min_{e\in[0,1]} \mathbb{H}_h[X^{(e)}] + c(e)$$

whose first-order condition is given by

$$\left.\frac{\partial F_h(e)}{\partial e}\right|_{e=e_h^*} = \mathbb{H}_h[Y] - \mathbb{H}_h[Z] + c'(e_h^*) = 0.$$

But note that

$$\left.\frac{\partial F_h(e)}{\partial e}\right|_{e=e_g^*} = (\mathbb{H}_h[Y] - \mathbb{H}_h[Z]) - \left(\mathbb{H}_g[Y] - \mathbb{H}_g[Z]\right) < 0$$

by Corollary 3. Therefore, $e_h^* > e_g^*$ and the result follows.

Proof of Theorem 8

The objective function of the decision-maker is given by

$$\min_{e\in[0,1]} \varphi^{-1}\left(\mathbb{E}\left[\varphi\left(\mathbb{H}_g[X_\Theta^{(e)}|\Theta] + c(e)\right)\right]\right)$$

which is equivalent to

$$\min_{e\in[0,1]} G_\varphi(e) = \min_{e\in[0,1]} \mathbb{E}\left[\varphi\left(\mathbb{H}_g[X_\Theta^{(e)}|\Theta] + c(e)\right)\right].$$

The first-order condition of the objective function is

$$\left.\frac{\partial G_\varphi(e)}{\partial e}\right|_{e=e_\varphi^*} = \mathbb{E}\left[\varphi'\left(\mathbb{H}_g[X_\Theta^{(e_\varphi^*)}|\Theta] + c(e_\varphi^*)\right) \times \left(\left.\frac{\partial \mathbb{H}_g[X_\Theta^{(e)}|\Theta]}{\partial e}\right|_{e=e_\varphi^*} + c'(e_\varphi^*)\right)\right] = 0. \tag{5.22}$$

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The second-order condition is given by

$$\left.\frac{\partial^2 G_\varphi(e)}{\partial e^2}\right|_{e=e_\varphi^*} = \mathbb{E}\left[\varphi''\left(\mathbb{H}_g[X_\Theta^{(e_\varphi^*)}|\Theta]+c(e_\varphi^*)\right)\left(\frac{\partial \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}|\Theta]}{\partial e}+c'(e_\varphi^*)\right)^2\right] + c''(e_\varphi^*)\mathbb{E}\left[\varphi'\left(\mathbb{H}_g[X_\Theta^{(e_\varphi^*)}|\Theta]+c(e_\varphi^*)\right)\right]$$

which is positive since $\varphi' > 0$, $\varphi'' > 0$ and $c'' > 0$.

Note that

$$\frac{\partial \mathbb{H}_g[X_\theta^{(e)}]}{\partial e} = (1-\theta)\left(\mathbb{H}_g[\underline{Y}]-\mathbb{H}_g[Z]\right)+\theta\left(\mathbb{H}_g[\overline{Y}]-\mathbb{H}_g[Z]\right)$$

is non-positive, increases with θ and does not depend on e. Since $\varphi' > 0$ and $c' > 0$, we deduce from the first-order condition on e_φ^* (5.22) that there exists $\theta_b \in (0,1)$ such that, for $0 < \theta \leq \theta_b$, $\partial \mathbb{H}_g[X_\theta^{(e)}]/\partial e|_{e=e_\varphi^*} + c'(e_\varphi^*) \leq 0$ and, for $\theta_b < \theta < 1$, $\partial \mathbb{H}_g[X_\theta^{(e)}]/\partial e|_{e=e_\varphi^*} + c'(e_\varphi^*) \geq 0$.

Let us now consider a more ambiguity-averse decision maker with ambiguity function $\psi = \chi \circ \varphi$ where χ is an increasing and convex function. His objective function is given by

$$\min_{e\in[0,1]} G_\psi(e) = \min_{e\in[0,1]}\left(\mathbb{E}\left[\psi\left(\mathbb{H}_g[X_\Theta^{(e)}|\Theta]+c(e)\right)\right]\right),$$

whose first-order condition is given by

$$\mathbb{E}\left[\psi'\left(\mathbb{H}_g[X_\Theta^{(e_\psi^*)}|\Theta]+c(e_\psi^*)\right)\left(\left.\frac{\partial \mathbb{H}_g[X_\Theta^{(e)}|\Theta]}{\partial e}\right|_{e=e_\psi^*}+c'(e_\psi^*)\right)\right]=0,$$

or equivalently by

$$\mathbb{E}\left[\chi'\left(\varphi\left(\mathbb{H}_g[X_\Theta^{(e_\psi^*)}|\Theta]+c(e_\psi^*)\right)\right)\varphi'\left(\mathbb{H}_g[X_\Theta^{(e_\psi^*)}|\Theta]+c(e_\psi^*)\right) \times\left(\left.\frac{\partial \mathbb{H}_g[X_\Theta^{(e)}|\Theta]}{\partial e}\right|_{e=e_\psi^*}+c'(e_\psi^*)\right)\right]=0.$$

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Since $\chi' > 0$, $\varphi' > 0$, $\chi'' > 0$ and $\mathbb{H}_g[X_\theta^{(e_\varphi^*)}]$ increases with θ, it follows from the first-order condition on e_φ^* (5.22) and the existence of θ_b that

$$\mathbb{E}\left[\chi'\left(\varphi\left(\mathbb{H}_g[X_\Theta^{(e_\varphi^*)}|\Theta] + c(e_\varphi^*)\right)\right)\varphi'\left(\mathbb{H}_g[X_\Theta^{(e_\varphi^*)}|\Theta] + c(e_\varphi^*)\right)\right.$$
$$\left.\times\left(\left.\frac{\partial \mathbb{H}_g[X_\Theta^{(e)}|\Theta]}{\partial e}\right|_{e=e_\varphi^*} + c'(e_\varphi^*)\right)\right] > 0.$$

Therefore, $e_\psi^* < e_\varphi^*$ and the result follows.

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