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Some identities for terminating q-series

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1. *Introduction*

We present the following sequence of polynomial identities:

$$\left.\begin{aligned}
1 &= 1,\\
1+q &= \frac{(1-q^2)}{(1-q)},\\
1+q\frac{(1-q^2)}{(1-q)}+q^4 &= -q^2+\frac{(1-q^3)(1-q^4)}{(1-q)(1-q^2)}-q^3,\\
1+q\frac{(1-q^3)}{(1-q)}+q^4\frac{(1-q^3)}{(1-q)}+q^9 &\\
&= -q^2\frac{(1-q^6)}{(1-q)}+\frac{(1-q^4)(1-q^5)(1-q^6)}{(1-q)(1-q^2)(1-q^3)}-q^3\frac{(1-q^6)}{(1-q)},\\
1+q\frac{(1-q^4)}{(1-q)}+q^4\frac{(1-q^3)(1-q^4)}{(1-q)(1-q^2)} &\\
+q^9\frac{(1-q^4)}{(1-q)}+q^{16} &= q^9-q^2\frac{(1-q^7)(1-q^8)}{(1-q)(1-q^2)}+\frac{(1-q^5)(1-q^6)(1-q^7)(1-q^8)}{(1-q)(1-q^2)(1-q^3)(1-q^4)}\\
&\quad -q^3\frac{(1-q^7)(1-q^8)}{(1-q)(1-q^2)}+q^{11},\\
&\vdots\\
\sum_{m=0}^{N} q^{m^2}\begin{bmatrix}N\\m\end{bmatrix}_q &= \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{5}{2}m^2+\frac{1}{2}m}\begin{bmatrix}2N\\N+2m\end{bmatrix}_q.\\
&\vdots
\end{aligned}\right\} \quad (1\cdot1)$$

$\begin{bmatrix}N\\m\end{bmatrix}_q$ is the Gaussian polynomial defined to be zero for $m<0$ or $m>N$, one for $m=0$ or N and

$$\begin{bmatrix}N\\m\end{bmatrix}_q = \prod_{i=1}^{m}\frac{(1-q^{N-m+i})}{(1-q^i)} \quad \text{for} \quad 0<m<N.$$

(Note that the right side of (1·1) is zero for all but finitely many values of m.) It is a monic polynomial of degree $m(N-m)$ with positive integral coefficients, a fact readily verified from its recursive definition:

$$\begin{bmatrix}N\\m\end{bmatrix}_q = \begin{bmatrix}N-1\\m-1\end{bmatrix}_q + q^m\begin{bmatrix}N-1\\m\end{bmatrix}_q.$$

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If we restrict q to $|q| < 1$ and let N approach infinity in (1·1), both sides converge, and we obtain:

$$1+\sum_{m=1}\frac{q^{m^2}}{(1-q)(1-q^2)\dots(1-q^m)}=\prod_{i=1}^{\infty}\frac{1}{(1-q^i)}\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{5}{2}m^2+\frac{1}{2}m}.$$

The summation on the right can be summed to an infinite product by Jacobi's Triple Product Identity:

$$\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}am^2+bm}=\prod_{i=0}^{\infty}(1-q^{\frac{1}{2}a-b+ai})(1-q^{\frac{1}{2}a+b+ai})(1-q^{ai}).$$

We thus have, as a corollary to (1·1):

$$1+\sum_{m=1}^{\infty}\frac{q^{m^2}}{(1-q)(1-q^2)\dots(1-q^m)}=\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+1})(1-q^{5i+4})}, \tag{1·2}$$

an identity of analytic and combinatorial significance known as the First Rogers-Ramanujan Identity. Some of the history of (1·2) is given by Hardy on p. 91 of *Ramanujan* (10).

There is a Second Rogers-Ramanujan Identity which is also the limit of a sequence of polynomial identities:

$$\sum_{n=0}^{N}q^{n^2+n}\begin{bmatrix}N\\n\end{bmatrix};q]=\frac{1}{(1-q^{N+1})}\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{5}{2}m^2+\frac{3}{2}m}\begin{bmatrix}2N+2\\N+2m+2\end{bmatrix};q]. \tag{1·3}$$

In the limit $N\to\infty$, this becomes:

$$1+\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+2})(1-q^{5i+3})}. \tag{1·4}$$

(1·1) and (1·3) are here proved by making certain modifications in a standard proof of the Rogers-Ramanujan Identities. The technique is sufficiently general to yield identities for terminating forms of many classical q-series. The following examples will also be proved:

$$\sum_{n=0}^{N}q^{nN}\begin{bmatrix}N\\n\end{bmatrix};q]=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n}\begin{bmatrix}2N\\N+2n\end{bmatrix};q], \tag{1·5}$$

$$1+\sum_{n=1}^{N}q^{2n^2}\begin{bmatrix}N\\n\end{bmatrix};q^2]\prod_{i=1}^{n}\frac{(1+q^{2N-2n+2i-1})}{(1+q^{2i-1})}=\sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2+n}\begin{bmatrix}2N\\N+2n\end{bmatrix};q^2] \tag{1·6}$$

$$\sum_{n=0}^{N}q^{2n^2+2n}\begin{bmatrix}N\\n\end{bmatrix};q^2]\prod_{i=1}^{n+1}\frac{(1+q^{2N-2n+2i+1})}{(1+q^{2i-1})}=\frac{(1+q^{2N+3})}{(1-q^{2N+2})}\sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2+3n}\begin{bmatrix}2N+2\\N+2n+2\end{bmatrix};q^2] \tag{1·7}$$

$$1+\sum_{n=1}^{N}q^{2n^2}\begin{bmatrix}N\\n\end{bmatrix};q^2]\prod_{i=1}^{2n}\frac{(1+q^{2N-2n+i})}{(1+q^i)}\prod_{j=1}^{n}\frac{(1-q^{2N+2j})}{(1-q^{4N-2n+2j})}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{7}{2}n^2+\frac{1}{2}n}\begin{bmatrix}2N\\N+2n\end{bmatrix};q^2]\prod_{i=1}^{\infty}\frac{(1-q^{2N+i})(1-q^{2N+i})}{(1-q^{2N-n+i})(1-q^{2N+n+i})} \tag{1·8}$$

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$$1+\sum_{n=1}^{N} q^{2n^2+2n}\begin{bmatrix}N\\n\end{bmatrix};q^2]\prod_{i=1}^{2n}\frac{(1+q^{2N-2n+i})}{(1+q^i)}\prod_{j=1}^{n}\frac{(1-q^{2N+2j})}{(1-q^{4N-2n+2j})}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{7}{2}n^2+\frac{3}{2}n}\begin{bmatrix}2N+2\\N+2n+2\end{bmatrix};q^2]\prod_{i=1}^{\infty}\frac{(1-q^{2N+i})(1-q^{2N+2+i})}{(1-q^{2N-n+i})(1-q^{2N+n+1+i})}, \quad (1\cdot9)$$

$$\sum_{n=0}^{N} q^{2n^2+2n}\begin{bmatrix}N\\n\end{bmatrix};q^2]\prod_{i=1}^{2n+1}\frac{(1+q^{2N-2n+1+i})}{(1+q^i)}\prod_{j=1}^{n+1}\frac{(1-q^{2N+2j})}{(1-q^{4N-2n+2+2j})}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{7}{2}n^2+\frac{5}{2}n}\begin{bmatrix}2N+2\\N+2n+2\end{bmatrix};q^2]\prod_{i=1}^{\infty}\frac{(1-q^{2N+2+i})(1-q^{2N+2+i})}{(1-q^{2N-n+1+i})(1-q^{2N+n+2+i})}. \quad (1\cdot10)$$

All of these become well-known q-series identities in the limit as N approaches infinity.

It should be pointed out that (1·1) and (1·3) are not the only polynomial identities which imply the Rogers-Ramanujan Identities. Andrews described a pair of less attractive polynomial identities in (1) (see also (5), p. 50). Two identities similar but not obviously equivalent to (1·1) and (1·3) are given by:

$$\sum_{n=0}^{N} q^{n^2}\begin{bmatrix}N\\n\end{bmatrix};q]=\frac{(1-q)(1-q^2)\dots(1-q^N)}{(1-q)(1-q^2)\dots(1-q^{2N})}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5}{2}n^2+\frac{1}{2}n}\begin{bmatrix}2N\\N+n\end{bmatrix};q], \quad (1\cdot11)$$

$$\sum_{n=0}^{N} q^{n^2+n}\begin{bmatrix}N\\n\end{bmatrix};q]=\frac{(1-q)(1-q^2)\dots(1-q^N)}{(1-q)(1-q^2)\dots(1-q^{2N+1})}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5}{2}n^2+\frac{3}{2}n}\begin{bmatrix}2N+1\\N+n+1\end{bmatrix};q]. \quad (1\cdot12)$$

These were proved in a much more general setting by Watson (21) (see equation (A 6) in the Appendix to this paper) and first stated in this form by Andrews (2).

2. *The Technique*

There is a general technique for obtaining identities similar to the Rogers-Ramanujan identities. It was discovered by Rogers (14), (16), developed by Dyson (9), refined by Bailey (6), (7) and exhaustively used by Slater (18), (19), whose list of 130 such identities is testimony to the power of the method. The essential idea is almost a triviality.

THEOREM (Bailey). *If* $B_n=\sum_{r=0}^{n}A_r U_{n-r}V_{n+r}$

and $C_n=\sum_{r=n}^{\infty}D_r U_{r-n}V_{r+n}$,

then $\sum_{n=0}^{\infty}A_n C_n=\sum_{n=0}^{\infty}B_n D_n$,

provided that all series converge.

Proof. $\sum_{n=0}^{\infty}A_n C_n=\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}A_n D_r U_{r-n}V_{r+n}$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}D_r A_n U_{r-n}V_{r+n}$$

$$=\sum_{r=0}^{\infty}D_r B_r.$$ (We assume that change in the order of summation is permitted.)

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The difficult part of this method is deciding what expressions for A_r, D_r, U_r and V_r will yield closed forms for B_n and C_n. Of course, the applicability of this theorem is not limited to q-series. Bailey points out in (7) that his transformations of ordinary hypergeometric series are special cases of this theorem. Before giving an example of the use of this theorem to find q-series identities, we will need some notational shorthand.

The rising 'q-factorial' is denoted by

$$(a)_n = (a;q)_n \equiv \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

Note that, if n is a positive integer, then

$$(a)_n = (a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}),$$

and

$$(a)_\infty = (a;q)_\infty = (1-a)(1-aq)(1-aq^2)\dots .$$

The generalized basic hypergeometric function is denoted by

$${}_{m+1}\phi_m\left[\begin{matrix}a_1,\dots,a_{m+1};\\ b_1,\dots,b_m\end{matrix}\; x\right] = {}_{m+1}\phi_m\left[\begin{matrix}a_1,\dots,a_{m+1};\\ b_1,\dots,b_m\end{matrix}\; q,x\right]$$
$$\equiv \sum_{n=0}^{\infty} \frac{(a_1)_n\dots(a_{m+1})_n x^n}{(q)_n (b_1)_n\dots(b_m)_n},$$

In finding closed form expressions for B_n and C_n, we shall need several classical summations and transformations for basic hypergeometric series. These are given with references in the appendix to this paper, and are referred to by the designation (A-).

The Rogers-Ramanujan identities can be proved by using the Theorem with the following choices for A_n, D_n, U_n, V_n:

$$A_n = \frac{(1-aq^{2n})(a)_n}{(1-a)(q)_n}(-1)^n a^n q^{\frac{3}{2}n^2-\frac{1}{2}n},$$

$$D_n = a^n q^{n^2},$$

$$U_n = \frac{1}{(q)_n},$$

$$V_n = \frac{1}{(aq)_n}.$$

Then we get for B_n and C_n:

$$B_n = \sum_{k=0}^{n} \frac{(1-aq^{2k})(a)_k}{(1-a)(q)_k}(-1)^k a^k q^{\frac{3}{2}k^2-\frac{1}{2}k} \frac{1}{(q)_{n-k}(aq)_{n+k}}$$
$$= \frac{1}{(q)_n (aq)_n} \sum_{k=0}^{n} \frac{(1-aq^{2k})(a)_k a^k q^{nk+k^2}(q^{-n})_k}{(1-a)(q)_k (aq^{n+1})_k},$$

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since we have the identity

$$\frac{(q)_n}{(q)_{n-k}} = (q^{n-k+1})_k = (-1)^k q^{nk-\frac{1}{2}k^2+\frac{1}{2}k}(q^{-n})_k.$$

So we can say that

$$B_n = \frac{1}{(q)_n (aq)_n} \lim_{b\to\infty} {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, b, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aqb^{-1}, aqb^{-1}, aq^{1+n} \end{matrix}; ab^{-2}q^{1+n}\right]$$

$$= \frac{1}{(q)_n}, \quad \text{using} \quad \text{(A 5)}.$$

Similarly,

$$C_n = \sum_{k=n}^{\infty} \frac{a^k q^{k^2}}{(q)_{k-n}(aq)_{k+n}}$$

$$= \frac{a^n q^{n^2}}{(aq)_{2n}} \sum_{j=0}^{\infty} \frac{a^j q^{j^2+2jn}}{(q)_j (aq^{2n+1})_j}$$

$$= \frac{a^n q^{n^2}}{(aq)_{2n}} \lim_{b\to\infty} {}_2\phi_1\left[\begin{matrix} b, b \\ aq^{2n+1} \end{matrix}; ab^{-2}q^{2n+1}\right]$$

$$= \frac{a^n q^{n^2}}{(aq)_\infty}, \quad \text{using} \quad \text{(A 3)}.$$

By the Theorem we have

$$\sum_{n=0}^{\infty} B_n D_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n} = \sum_{n=0}^{\infty} A_n C_n = \frac{1}{(aq)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{\frac{5}{2}n^2-\frac{1}{2}n}(1-aq^{2n})(a)_n}{(q)_n (1-a)}. \quad (2{\cdot}1)$$

The first Rogers-Ramanujan identity (1·2) is the case $a = 1$, while the second (1·4) is the case $a = q$.

In all of the work of Bailey and Slater on q-series, they use only one choice for D_n, U_n and V_n, namely:

$$D_n = (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n$$

($D_n = a^n q^{n^2}$ is the limit $\rho_1, \rho_2 \to \infty$),

$$U_n = \frac{1}{(q)_n},$$

and

$$V_n = \frac{1}{(aq)_n},$$

and only experiment with different values for A_n. In this paper, we make a different choice for D_n, U_n, V_n:

$$D_n = r^n,$$

$$U_n = \frac{(\beta)_n}{(q)_n},$$

and

$$V_n = \frac{(a\beta)_n}{(aq)_n}.$$

By taking the same values for A_n which were taken by Bailey and Slater, identities very similar to theirs can be obtained. While we have lost the parameters ρ_1 and ρ_2,

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we have picked up two new parameters which will have a very different effect on the identities. In particular, when β is set equal to $a^{-1}q^{-N}$, our infinite series will terminate at or before the N-th term.

3. *Proof of Finite Rogers-Ramanujan*

In the Theorem, we make the following choices for A_n, D_n, U_n, V_n:

$$A_n = \frac{(1-aq^{2n})(a)_n\gamma^n(\gamma^{-1})_n}{(1-a)(q)_n(a\gamma q)_n},$$

$$D_n = r^n,$$

$$U_n = \frac{(\beta)_n}{(q)_n},$$

and

$$V_n = \frac{(a\beta)_n}{(aq)_n}.$$

Then, we have

$$\begin{aligned}
B_n &= \sum_{k=0}^{n} \frac{(1-aq^{2k})(a)_k\gamma^k(\gamma^{-1})_k(\beta)_{n-k}(a\beta)_{n+k}}{(1-a)(q)_k(a\gamma q)_k(q)_{n-k}(aq)_{n+k}} \\
&= \frac{(a\beta)_n(\beta)_n}{(aq)_n(q)_n}\sum_{k=0}^{n}\frac{(a)_k(1-aq^{2k})(\gamma^{-1})_k(a\beta q^n)_k(q^{-n})_k}{(q)_k(1-a)(a\gamma q)_k(\beta^{-1}q^{1-n})_k(aq^{n+1})_k}\left(\frac{\gamma q}{\beta}\right)^k \\
&= \frac{(a\beta)_n(\beta)_n}{(aq)_n(q)_n}\,{}_6\phi_5\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, \gamma^{-1}, a\beta q^n, q^{-n} \\ \sqrt{a}, -\sqrt{a}, a\gamma q, \beta^{-1}q^{1-n}, aq^{n+1}\end{matrix}; \frac{\gamma q}{\beta}\right] \\
&= \frac{(a\beta)_n(\beta)_n(\gamma\beta^{-1}q^{1-n})_n}{(q)_n(a\gamma q)_n(\beta^{-1}q^{1-n})_n}, \quad \text{using} \quad \text{(A 5)},
\end{aligned}$$

so that

$$B_n = \frac{(a\beta)_n(\beta\gamma^{-1})_n\gamma^n}{(q)_n(a\gamma q)_n},$$

and

$$\begin{aligned}
C_n &= \sum_{k=n}^{\infty} r^k\frac{(\beta)_{k-n}(a\beta)_{k+n}}{(q)_{k-n}(aq)_{k+n}} = r^n\frac{(a\beta)_{2n}}{(aq)_{2n}}\sum_{j=0}^{\infty}\frac{(\beta)_j(a\beta q^{2n})_j r^j}{(q)_j(aq^{2n+1})_j} \\
&= r^n\frac{(a\beta)_{2n}}{(aq)_{2n}}\,{}_2\phi_1\left[\begin{matrix}\beta, a\beta q^{2n} \\ aq^{2n+1}\end{matrix}; r\right] = r^n\frac{(a\beta)_\infty(\beta r)_\infty}{(aq)_\infty(r)_\infty}\,{}_2\phi_1\left[\begin{matrix}q\beta^{-1}, r \\ \beta r\end{matrix}; a\beta q^{2n}\right], \quad \text{using} \quad \text{(A 2)}.
\end{aligned}$$

Thus, we deduce from the theorem,

$$\begin{aligned}
\sum_{n=0}^{\infty} B_nD_n &= \sum_{n=0}^{\infty} r^n\gamma^n\frac{(a\beta)_n(\beta\gamma^{-1})_n}{(q)_n(a\gamma q)_n} = \sum_{n=0}^{\infty} A_nC_n \\
&= \frac{(a\beta)_\infty(\beta r)_\infty}{(aq)_\infty(r)_\infty}\sum_{n=0}^{\infty}\frac{r^n\gamma^n(1-aq^{2n})(a)_n(\gamma^{-1})_n}{(1-a)(q)_n(a\gamma q)_n}\times{}_2\phi_1\left[\begin{matrix}q\beta^{-1}, r \\ \beta r\end{matrix}; a\beta q^{2n}\right]. \qquad (3\cdot1)
\end{aligned}$$

The argument of the basic hypergeometric series of the right side of (3·1) depends on n, and thus this series can only be summed if it reduces to $a_1\phi_0$. That is to say, it

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can only be expressed as a product for all n if either $r = 0$ or $r = q\beta^{-2}$. Setting $r = 0$ reduces (3·1) to the q-binomial theorem (A 1). If $r = q\beta^{-2}$, we use (A 1) to get:

$$\sum_{n=0}^{\infty} q^n\gamma^n\beta^{-2n}\frac{(a\beta)_n\,(\beta\gamma^{-1})_n}{(q)_n\,(a\gamma q)_n} = \frac{(a\beta^{-1}q)_\infty\,(q\beta^{-1})_\infty}{(aq)_\infty\,(q\beta^{-2})_\infty}\sum_{n=0}^{\infty}\frac{q^n\gamma^n\beta^{-2n}(1-aq^{2n})\,(a)_n\,(\gamma^{-1})_n\,(a\beta)_{2n}}{(1-a)\,(q)_n\,(a\gamma q)_n\,(a\beta^{-1}q)_{2n}}. \tag{3·2}$$

As promised in § 2, we set $\beta = a^{-1}q^{-N}$ so that both series will terminate. (Note that $(q^{-N})_n = 0$ for $n > N$.) Then we have

$$\sum_{n=0}^{\infty} a^{2n}\gamma^n q^{n(1+2N)}\frac{(q^{-N})_n\,(a^{-1}\gamma^{-1}q^{-N})_n}{(q)_n\,(a\gamma q)_n}$$
$$= \frac{(a^2q^{N+1})_\infty\,(aq^{N+1})_\infty}{(aq)_\infty\,(a^2q^{2N+1})_\infty}\sum_{n=0}^{\infty}\frac{a^{2n}\gamma^n q^{n(1+2N)}(1-aq^{2n})}{(1-a)\,(q)_n\,(a\gamma q)_n\,(a^2q^{N+1})_{2n}}(a)_n\,(\gamma^{-1})_n\,(q^{-N})_{2n}. \tag{3·3}$$

We now use the following simplifications:

$$(a^{-1})_n = (-1)^n a^{-n} q^{\frac{1}{2}n^2-\frac{1}{2}n}(aq^{1-n})_n,$$

$$\frac{(q^{N-n+1})_n}{(q)_n} = \begin{bmatrix} N \\ n \end{bmatrix}; q] \quad \text{(Gaussian Polynomial)},$$

to rewrite (3·3) as:

$$\sum_{n=0}^{\infty} a^n q^{n^2}\begin{bmatrix} N \\ n \end{bmatrix}; q]\frac{(a\gamma q)_N}{(a\gamma q)_n\,(a\gamma q)_{N-n}} = \frac{(a^2q)_{2N}\,(q)_N}{(aq)_N}\sum_{n=0}^{\infty}\frac{a^{2n}\gamma^n q^{2n^2}(1-aq^{2n})\,(a)_n\,(\gamma^{-1})_n}{(1-a)\,(q)_n\,(a\gamma q)_n\,(q)_{N-2n}(a^2q)_{N+2n}}. \tag{3·4}$$

Note that by the definition of $(a)_n$ in § 2, the right side is zero if $2n > N$.

To get the finite form of the Rogers-Ramanujan Identities, we set $\gamma = 0$:

$$\sum_{n=0}^{\infty} a^n q^{n^2}\begin{bmatrix} N \\ n \end{bmatrix}; q] = \frac{(a^2q)_{2N}\,(q)_N}{(aq)_N}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{\frac{5}{2}n^2-\frac{1}{2}n}(1-aq^{2n})\,(a)_n}{(1-a)\,(q)_n\,(q)_{N-2n}(a^2q)_{N+2n}}. \tag{3·5}$$

Equation (1·1) results when $a = 1$, and equation (1·3) is the case $a = q$.

Other choices of γ lead to other interesting polynomial identities. In the limit as γ approaches ∞, we get:

$$\sum_{n=0}^{\infty} a^n q^{nN}\begin{bmatrix} N \\ n \end{bmatrix}; q] = \frac{(a^2q)_{2N}\,(q)_N}{(aq)_N}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{\frac{3}{2}n^2-\frac{1}{2}n}(1-aq^{2n})\,(a)_n}{(1-a)\,(q)_n(q)_{N-2n}(a^2q)_{N+2n}}. \tag{3·6}$$

For $a = 1$, this becomes equation (1·5). As N approaches ∞, the left side of (1·5) approaches 1 and (1·5) becomes Euler's Pentagonal Number Theorem.

If, in (3·4), we replace q by q^2 and then set $\gamma = -q^{-1}$, we get that:

$$\sum_{n=0}^{\infty} a^n q^{2n^2}\begin{bmatrix} N \\ n \end{bmatrix}; q^2]\frac{(-aq;q^2)_N}{(-aq;q^2)_n\,(-aq;q^2)_{N-n}}$$
$$= \frac{(a^2q^2;q^2)_{2N}\,(q^2;q^2)_N}{(aq^2;q^2)_N}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{4n^2-n}(1-aq^{4n})}{(1-a)\,(q^2;q^2)_n\,(-aq;q^2)_n}$$
$$\times\frac{(a;q^2)_n\,(-q;q^2)_n}{(q^2;q^2)_{N-2n}\,(a^2q^2;q^2)_{N+2n}}. \tag{3·7}$$

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This yields equations (1·6) and (1·7) for the respective values $a = 1, a = q^2$. In the limit as $N \to \infty$, (1·6) and (1·7) become the analytic counterpart of the Göllnitz-Gordon identities for the modulus eight (see (8), § III).

4. *Finite Form of Rogers-Selberg Identities*

While it is possible to obtain finite forms for all of Slater's q-series identities, the task of suitably specializing parameters and reducing series to their most compact form becomes increasingly difficult. We shall content ourselves with only one more example, the finite form of the following identities known as the Rogers-Selberg identities for the modulus seven:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n (-q)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7}{2}n^2+\frac{1}{2}n} = \frac{(q^3;q^7)_\infty (q^4;q^7)_\infty (q^7;q^7)_\infty}{(q^2;q^2)_\infty}, \quad (4\cdot1)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n (-q)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7}{2}n^2+\frac{3}{2}n} = \frac{(q^2;q^7)_\infty (q^5;q^7)_\infty (q^7;q^7)_\infty}{(q^2;q^2)_\infty}, \quad (4\cdot2)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n (-q)_{2n+1}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7}{2}n^2+\frac{5}{2}n} = \frac{(q;q^7)_\infty (q^6;q^7)_\infty (q^7;q^7)_\infty}{(q^2;q^2)_\infty}. \quad (4\cdot3)$$

These were first proved by Rogers((14) and (16)) and then rediscovered by Selberg (17) The finite forms are, respectively, equations (1·8), (1·9) and (1·10).

In the theorem of § 2, we make the following choices for A_n, D_n, U_n, V_n:

$$A_n = \frac{(1-aq^{2n})(a)_n \gamma^n (\gamma^{-1})_n}{(1-a)(q)_n (a\gamma q)_n},$$

$$D_n = r^n,$$

$$U_n = \frac{(\beta^2;q^2)_n}{(q^2;q^2)_n},$$

and

$$V_n = \frac{(a^2\beta^2;q^2)_n}{(a^2q^2;q^2)_n}.$$

Then, we have

$$B_n = \sum_{k=0}^{n} \frac{(1-aq^{2k})(a)_k \gamma^k (\gamma^{-1})_k (\beta^2;q^2)_{n-k} (a^2\beta^2;q^2)_{n+k}}{(1-a)(q)_k (a\gamma q)_k (q^2;q^2)_{n-k} (a^2q^2;q^2)_{n+k}}$$

$$= \frac{(a^2\beta^2;q^2)_n (\beta^2;q^2)_n}{(a^2q^2;q^2)_n (q^2;q^2)_n} \sum_{k=0}^{n} \frac{(a)_k (1-aq^{2k})(\gamma^{-1})_k}{(q)_k (1-a)(a\gamma q)_k} \frac{(a^2\beta^2q^{2n};q^2)_k (q^{-2n};q^2)_k}{(\beta^{-2}q^{2-2n};q^2)_k (a^2q^{2+2n};q^2)_k} \left(\frac{\gamma q^2}{\beta^2}\right)^k$$

$$= \frac{(a^2\beta^2;q^2)_n (\beta^2;q^2)_n}{(a^2q^2;q^2)_n (q^2;q^2)_n} {}_8\phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \gamma^{-1}, -q^{-n}, a\beta q^n, -a\beta q^n, q^{-n} \\ \sqrt{a}, -\sqrt{a}, a\gamma q, -aq^{n+1}, \beta^{-1}q^{1-n}, -\beta^{-1}q^{1-n}, aq^{n+1} \end{matrix}; \frac{\gamma q^2}{\beta^2}\right]$$

$$= \frac{(a^2\beta^2;q^2)_n (\beta^2;q^2)_n (aq)_n (-a^{-1}\beta^{-2}q^{1-2n})_n}{(a^2q^2;q^2)_n (q^2;q^2)_n (\beta^{-1}q^{1-n})_n (-\beta^{-1}q^{1-n})_n} {}_4\phi_3 \left[\begin{matrix} -a\gamma q^{n+1}, a\beta q^n, -a\beta q^n, q^{-n} \\ -a\beta^2 q^n, a\gamma q, -aq^{n+1} \end{matrix}; q\right]$$

using (A 6). (4·4)

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So

$$B_n = \frac{(a^2\beta^2;q^2)_n\,(aq)_n\,(-a\beta^2)_{2n}}{(a^2q^2;q^2)_n\,(q^2;q^2)_n\,(-a\beta^2)_n}\,(-1)^n a^{-n} q^{-\frac{1}{2}n^2-\frac{1}{2}n}\;{}_4\phi_3\left[\begin{matrix}-a\gamma q^{n+1}, a\beta q^n, -a\beta q^n, q^{-n}\\ -a\beta^2 q^n, a\gamma q, -aq^{n+1}\end{matrix};q\right],$$

and (4·5)

$$C_n = \sum_{k=n}^{\infty} r^k \frac{(\beta^2;q^2)_{k-n}\,(a^2\beta^2;q^2)_{k+n}}{(q^2;q^2)_{k-n}\,(a^2q^2;q^2)_{k+n}}$$

$$= r^n \frac{(a^2\beta^2;q^2)_{2n}}{(a^2q^2;q^2)_{2n}}\,{}_2\phi_1\left[\begin{matrix}\beta^2, a^2\beta^2q^{4n}\\ a^2q^{4n+2}\end{matrix};q^2, r\right]$$

$$= r^n \frac{(a^2\beta^2;q^2)_\infty\,(\beta^2r;q^2)_\infty}{(a^2q^2;q^2)_\infty\,(r;q^2)_\infty}\,{}_2\phi_1\left[\begin{matrix}q^2\beta^{-2}, r\\ \beta^2r\end{matrix};q^2, a^2\beta^2q^{4n}\right], \quad \text{using (A 2).} \tag{4·6}$$

Before stating the series identity, we observe that our expressions will be greatly simplified if we set $\gamma = \beta^2q^{-1}$ and use (A 4) to sum the basic hypergeometric series appearing in the expression for B_n and set $r = q^2\beta^{-4}$ and use (A 3) to sum C_n. This yields:

$$B_n = (-1)^n a^{-n} q^{-\frac{1}{2}n^2-\frac{1}{2}n} \frac{(a^2\beta^2;q^2)_n\,(aq)_n\,(-a\beta^2)_{2n}}{(a^2q^2;q^2)_n\,(q^2;q^2)_n\,(-a\beta^2)_n} \frac{(\beta^2q^{-2n};q^2)_n}{(a\beta^2)_n(-a^{-1}q^{-2n})_n}$$

$$= \beta^{2n}q^{-n} \frac{(a^2\beta^2;q^2)_n\,(\beta^{-2};q^2)_n\,(-a\beta^2)_{2n}}{(q^2;q^2)_n\,(a^2\beta^4;q^2)_n\,(-aq)_{2n}},$$

and

$$C_n = q^{2n}\beta^{-4n} \frac{(a^2\beta^{-2}q^2;q^2)_\infty\,(\beta^{-2}q^2;q^2)_\infty\,(a^2\beta^2;q^2)_{2n}}{(a^2q^2;q^2)_\infty\,(\beta^{-4}q^2;q^2)_\infty\,(a^2\beta^{-2}q^2;q^2)_{2n}}.$$

Thus,

$$\sum_{n=0}^{\infty} B_nD_n = \sum_{n=0}^{\infty} q^n\beta^{-2n} \frac{(a^2\beta^2;q^2)_n\,(\beta^{-2};q^2)_n\,(-a\beta^2)_{2n}}{(q^2;q^2)_n\,(a^2\beta^4;q^2)_n\,(-aq)_{2n}} = \sum_{n=0}^{\infty} A_nC_n$$

$$= \frac{(a^2\beta^{-2}q^2;q^2)_\infty(\beta^{-2}q^2;q^2)_\infty}{(a^2q^2;q^2)_\infty\,(\beta^{-4}q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-aq^{2n})\,(a)_n}{(1-a)\,(q)_n} \frac{q^n\beta^{-2n}(q\beta^{-2})_n\,(a^2\beta^2;q^2)_{2n}}{(a\beta^2)_n\,(a^2\beta^{-2}q^2;q^2)_{2n}}. \tag{4·7}$$

Again, both series will terminate if we set $\beta = a^{-1}q^{-N}$. With this substitution and after some simplification, we get:

$$\sum_{n=0}^{\infty} a^{2n}q^{2n^2}\begin{bmatrix}N\\ n\end{bmatrix};q^2\Big] \frac{(-aq)_{2N}}{(-aq)_{2n}\,(-aq)_{2N-2n}} \frac{(a^2q^2;q^2)_{N+n}\,(a^2q^2;q^2)_{2N-n}}{(a^2q^2;q^2)_N\,(a^2q^2;q^2)_{2N}}$$

$$= \frac{(q^2;q^2)_N}{(a^2q^2;q^2)_N} \sum_{n=0}^{\infty} \frac{(-1)^n a^{3n}q^{\frac{1}{2}n^2-\frac{1}{2}n}(1-aq^{2n})\,(a)_n}{(1-a)\,(q)_n}$$

$$\times \frac{(a^4q^2;q^2)_{2N}}{(q^2;q^2)_{N-2n}\,(a^4q^2;q^2)_{N+2n}} \frac{(a^2q)_{2N+n}\,(aq)_{2N-n}}{(a^2q)_{2N}\,(aq)_{2N}} \tag{4·8}$$

Equations (1·8) and (1·10) are the cases $a = 1$ and $a = q$, respectively.

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We use the same values for D_n, U_n and V_n but change the value for A_n in order to get (1·9). If

$$A_n = \frac{(1-a^2q^{4n})\,(a)_n\,\gamma^n q^{-n}(\gamma^{-1})_n}{(1-a^2)\,(q)_n\,(a\gamma q)_n}, \quad \text{then}$$

$$B_n = \sum_{k=0}^{\infty} \frac{(1-a^2q^{4k})\,(a)_k\,\gamma^k q^{-k}(\gamma^{-1})_k\,(\beta^2;q^2)_{n-k}\,(a^2\beta^2;q^2)_{n+k}}{(1-a^2)\,(q)_n\,(a\gamma q)_n\,(q^2;q^2)_{n-k}\,(a^2q^2;q^2)_{n+k}}$$

$$= \frac{(a^2\beta^2;q^2)_n\,(\beta^2;q^2)_n}{(a^2q^2;q^2)_n\,(q^2;q^2)_n} \sum_{k=0}^{n} \frac{(a)_k\,(1-a^2q^{4k})\,(\gamma^{-1})_k}{(q)_k\,(1-a^2)\,(a\gamma q)_k} \frac{(a^2\beta^2q^{2n};q^2)_k\,(q^{-2n};q^2)_k}{(\beta^{-2}q^{2-2n};q^2)_k\,(a^2q^{2+2n};q^2)_k} \left(\frac{\gamma q}{\beta^2}\right)^k$$

$$= \frac{(a^2\beta^2;q^2)_n\,(\beta^2;q^2)_n}{(a^2q^2;q^2)_n\,(q^2;q^2)_n}$$

$$\times {}_{10}\phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \gamma^{-1}, -q^{-n}, iq\sqrt{a}, -iq\sqrt{a}, a\beta q^n, -a\beta q^n, q^{-n} \\ \sqrt{a}, -\sqrt{a}, a\gamma q, -aq^{n+1}, i\sqrt{a}, -i\sqrt{a}, \beta^{-1}q^{1-n}, -\beta^{-1}q^{1-n}, aq^{n+1} \end{matrix}; \gamma q\beta^{-2}\right]$$

So

$$B_n = \frac{(a^2\beta^2;q^2)_n\,(\beta^2;q^2)_n\,(aq)_n\,(-a^{-1}\beta^{-2}q^{1-2n})_n}{(a^2q^2;q^2)_n\,(q^2;q^2)_n\,(\beta^{-1}q^{1-n})_n(-\beta^{-1}q^{1-n})_n}$$

$$\times \sum_{j=0}^{\infty} \frac{(-a\gamma q^{n+1})_j\,(iq\sqrt{a})_j\,(-iq\sqrt{a})_j\,(a\beta q^n)_j\,(-a\beta q^n)_j\,(q^{-n})_j}{(q)_j\,(a\gamma q)_j\,(-aq^{n+1})_j\,(i\sqrt{a})_j\,(-i\sqrt{a})_j\,(-a\beta^2q^n)_j}$$

$$\times \sum_{k=0}^{\infty} \frac{(q^{-1})_k\,(a^2\beta^2q^{2n+2j};q^2)_k\,(q^{-n+j})_k\,q^k}{(q)_k\,(-aq^{2j};q^2)_k\,(-a\beta^2q^{n+j})_k} \qquad \text{using} \quad (A\ 7)$$

Hence

$$B_n = \frac{(a^2\beta^2;q^2)_n\,(aq)_n\,(-a\beta^2)_{2n}}{(a^2q^2;q^2)_n\,(q^2;q^2)_n\,(-a\beta^2)_n}\,(-1)^n a^{-n}q^{-\frac{1}{2}n^2-\frac{1}{2}n}$$

$$\times \sum_{j=0}^{\infty} \frac{(-a\gamma q^{n+1})_j\,(a\beta q^n)_j\,(-a\beta q^n)_j\,(q^{-n})_j\,(1+aq^{2j})}{(q)_j\,(a\gamma q)_j\,(-aq^{n+1})_j\,(-a\beta^2q^n)_j\,(1+a)}$$

$$\times \frac{(1+a\beta^2q^{2n})\,(1+aq^{n+j})\,q^{-n+j}}{(1+aq^{2j})\,(1+a\beta^2q^{n+j})}$$

$$= \frac{(a^2\beta^2;q^2)_n\,(-a\beta^2q)_{2n}}{(-a)_n\,(q^2;q)_n\,(-a\beta^2q)_n}\,(-1)^n a^{-n}q^{-\frac{1}{2}n^2-\frac{3}{2}n}\,{}_4\phi_3\left[\begin{matrix} -a\gamma q^{n+1}, a\beta q^n, -a\beta q^n, q^{-n} \\ a\gamma q, -aq^n, -a\beta^2q^{n+1} \end{matrix}; q\right].$$

If we set $\gamma = \beta^2$, then the basic hypergeometric series can be summed by (A 4), and we get

$$B_n = (-1)^n a^{-n}q^{-\frac{1}{2}n^2-\frac{3}{2}n} \frac{(a^2\beta^2;q^2)_n\,(-a\beta^2q)_{2n}}{(-a)_n\,(q^2;q^2)_n\,(-a\beta^2q)_n} \frac{(\beta^2q^{2-2n};q^2)_n}{(a\beta^2q)_n\,(-a^{-1}q^{1-2n})_n}$$

$$= q^{-n}\beta^{2n} \frac{(a^2\beta^2;q^2)_n\,(\beta^{-2};q^2)_n\,(-a\beta^2q)_{2n}}{(q^2;q^2)_n\,(a^2\beta^4q^2;q^2)_n\,(-a)_{2n}}.$$

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Thus, with $\gamma = \beta^2$ and $r = q^2\beta^{-4}$, we have from the theorem

$$\sum_{n=0}^{\infty} B_n D_n = \sum_{n=0}^{\infty} \beta^{-2n} q^n \frac{(a^2\beta^2;q^2)_n\,(\beta^{-2};q^2)_n\,(-a\beta^2q)_{2n}}{(q^2;q^2)_n\,(a^2\beta^4q^2;q^2)_n\,(-a)_{2n}} = \sum_{n=0}^{\infty} A_n C_n$$

$$= \frac{(a^2\beta^{-2}q^2;q^2)_\infty\,(q^2\beta^{-2};q^2)_\infty}{(a^2q^2;q^2)_\infty\,(q^2\beta^{-4};q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n\beta^{-2n}(1-aq^{4n})}{(1-a^2)} \frac{(a)_n\,(\beta^{-2})_n\,(a^2\beta^2;q^2)_{2n}}{(q)_n\,(a\beta^2q)_n\,(a^2\beta^{-2}q^2;q^2)_{2n}}, \tag{4·9}$$

Both series will terminate if we set $\beta = a^{-1}q^{-N}$. With this substitution and after some simplification, we get finally the result

$$\sum_{n=0}^{\infty} a^{2n}q^{2n^2}\begin{bmatrix}N\\n\end{bmatrix};q^2\Big] \frac{(-a)_{2N}}{(-a^2)_{2n}\,(-a)_{2N-2n}} \frac{(a^2;q^2)_{N+n}\,(a^2;q^2)_{2N-n}}{(a^2;q^2)_N\,(a^2;q^2)_{2N}}$$

$$= \frac{(q^2;q^2)_N}{(a^2q^2;q^2)_N} \sum_{n=0}^{\infty} \frac{(-1)^n a^{3n}q^{\frac{7}{2}n^2-\frac{3}{2}n}(1-a^2q^{4n})\,(a)_n}{(1-a^2)\,(q)_n}$$

$$\times \frac{(a^4q^2;q^2)_{2N}}{(q^2;q^2)_{N-2n}\,(a^4q^2;q^2)_{N+2n}} \frac{(a^2)_{2N+n}\,(a)_{2N-n}}{(a^2)_{2N}\,(a)_{2N}}. \tag{4·10}$$

Equation (1·9) is the case $a = q$.

Appendix: Identities for Basic Hypergeometric Series

Our first identity is

(A 1) $$ {}_1\phi_0[^a;x] = \frac{(ax)_\infty}{(x)_\infty}. $$

This is the *q-binomial theorem* (for $a = q^{-\lambda}$ it reduces to the binomial theorem: $\Sigma_{n=0}^{\infty}\binom{\lambda}{n}(-x)^n = (1-x)^\lambda$ in the limit as $q \to 1^-$). It was discovered a number of times during the nineteenth century. R. Askey has found the earliest known reference in an 1811 book by Rothe (22). Proofs can be found in (5), Theorem 2·1, and (3), Theorem 2·4. The first uses difference equations; the second is combinatorial.

The second identity is

(A 2) $$ {}_2\phi_1[^{a,b}_{c};x] = \frac{(b)_\infty\,(ax)_\infty}{(c)_\infty\,(x)_\infty}\,{}_2\phi_1[^{c/b,x}_{ax};b]. $$

This is *Heine's fundamental transformation*, which can be iterated to obtain additional transformations (after interchanging the numerator parameters of the right side). It was first stated and proved by Heine in 1847 (11). A simple proof requiring only (A 1) can be found in (5), Corollary 2·3.

Further we have

(A 3) $$ {}_2\phi_1[^{a,b}_{c};c/ab] = \frac{(c/a)_\infty\,(c/b)_\infty}{(c)_\infty\,(c/ab)_\infty}. $$

This is also due to Heine. It is proved by first applying (A 2) to the basic hypergeometric series, and then using (A 1) to sum the resulting series.

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Also

$$\text{(A 4)} \qquad {}_3\phi_2\left[\begin{matrix}a, b, q^{-N}\\ c, abc^{-1}q^{1-N}\end{matrix}; q\right] = \frac{(c/a)_N (c/b)_N}{(c)_N (c/ab)_N}.$$

This is the q-analogue of Saalschutz's identity. This was first proved by Jackson in 1910 (**12**). As Watson has pointed out, (**21**), it can be proved by comparing the coefficients of x^N in the third iterate of the transformation (A 2).

The fifth identity which we use is

$$\text{(A 5)} \qquad {}_6\phi_5\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-N}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{N+1}\end{matrix}; \frac{aq^{N+1}}{bc}\right] = \frac{(aq)_N (aq/bc)_N}{(aq/b)_N (aq/c)_N}.$$

Usually referred to as a limiting form of Jackson's theorem, this was first proved but incorrectly stated by Rogers in 1895 (**15**), p. 29. It was rediscovered and correctly stated by Jackson (**13**), equation (5). A third proof can be found in (**3**), Theorem 3·2. It should be noted that the parameters $q\sqrt{a}$, $\sqrt{a}$, $-q\sqrt{a}$, $-\sqrt{a}$ serve only to introduce the factor

$$\frac{(q\sqrt{a})_n (-q\sqrt{a})_n}{(\sqrt{a})_n (-\sqrt{a})_n} = \frac{(aq^2; a^2)_n}{(a; q^2)_n} = \frac{(1-aq^{2n})}{(1-a)}.$$

The sixth result is

$$\text{(A 6)} \qquad {}_8\phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{N+1}\end{matrix}; \frac{a^2q^{2+N}}{bcde}\right]$$
$$= \frac{(aq)_N (aq/de)_N}{(aq/d)_N (aq/e)_N} {}_4\phi_3\left[\begin{matrix}aq/bc, d, e, q^{-N}\\ deq^{-N}/a, aq/b, aq/c\end{matrix}; q\right].$$

Known as the *q-analogue of Whipple's transformation*, this was discovered and proved by Watson in 1929 (**21**). Watson's proof is by induction on N and only requires (A 4). (A 5) is the special case of this identity where $bc = aq$. Alternatively, Slater (**20**), p. 100, deduced this identity from (A 5) using the method of § 2 of this paper. In the limit as b, c, d, e approach infinity and for $a = 1$ or q, respectively, this identity becomes the finite form of the Rogers–Ramanujan identities stated by Andrews (equations (1·11) and (1·12) of this paper).

Finally, we have

$$\text{(A 7)} \qquad {}_{10}\phi_9\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, g, q^{-N}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq^{N+1}\end{matrix}; \frac{a^3q^{3+N}}{bcdefg}\right]$$
$$= \frac{(aq)_N (aq/fg)_N}{(aq/f)_N (aq/g)_N} \sum_{j=0}^{\infty} \frac{(aq/bc)_j (d)_j (e)_j (f)_j (g)_j}{(q)_j (aq/b)_j (aq/c)_j (aq/d)_j (aq/e)_j} \frac{(q^{-N})_j a^j q^{2j}}{(fgq^{-N}/a)_j d^j e^j}$$
$$\times {}_4\phi_3\left[\begin{matrix}aq/de, fq^j, gq^j, q^{j-N}\\ aq^{j+1}d^{-1}, aq^{j+1}e^{-1}, fgq^{j-N}a^{-1}\end{matrix}; q\right].$$

This is a special case of a very general theorem proved by Andrews in 1975 ((**4**), Theorem 4) for 'well-poised' series with an arbitrary number of parameters. His proof is by induction on the number of parameters, taking (A 6) as given. The induction argument requires only (A 4).

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