109.42 Euler's constant via an integral

Euler's constant is

$$\gamma = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right).$$

That the limit exists can be established in a number of ways (e.g. [1, 2, 3, 4, 5, 6]). In [7] it is done by observing that

$$\int_0^1 \frac{t}{k(k+t)} dt = \frac{1}{k} - \log\left(\frac{k+1}{k}\right), \qquad (k \geqslant 1). \tag{1}$$

Then, since

$$0 < \int_0^1 \frac{t}{k(k+t)} dt < \int_0^1 \frac{t}{k(k+0)} dt = \frac{1}{k^2},$$

so, summing over k, we get

$$0 < \gamma_n < \sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Therefore $\gamma = \lim_{n\to\infty} \gamma_n$ exists.

Here we obtain a few well-known facts about γ by looking at the above integrands $f_k(t) = \dfrac{t}{k(k+t)}$ more carefully, as one might do in an introductory calculus course.

Wherever our analysis is independent of k, we shall suppress the subscript: $f = f_k$.

(i) The integrands $f(t) = \dfrac{t}{k(k+t)}$ have $f'(t) = \dfrac{1}{(k+t)^2} > 0$, so f is increasing. Therefore

$$0 < \gamma_n = \sum_{k=1}^{n} \int_0^1 \frac{t}{k(k+1)} dt < \sum_{k=1}^{n} \int_0^1 \frac{1}{k(k+1)} dt$$

$$= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1},$$

whence

$$0 < \gamma \leqslant 1.$$

And in the same way,

$$\gamma - \gamma_n = \sum_{k=n+1}^{\infty} \int_0^1 \frac{t}{k(k+t)} dt < \sum_{k=n+1}^{n} \int_0^1 \frac{1}{k(k+1)} dt$$

$$= \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=n+1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{n+1}.$$

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Therefore, since also $\int_0^1 \frac{t}{k(k+t)}dt > 0$, we have

$$0 < \gamma - \gamma_n < \frac{1}{n+1}.$$

(ii) Further, $f''(t) = \frac{-2}{(k+t)^3} < 0$, and so f is concave. Therefore the Trapezium Rule is an underestimate of the integral over $[0, 1]$ and the Midpoint Rule is an overestimate:

$$\tfrac{1}{2}\big(f(0) + f(1)\big) < \int_0^1 f(t)\,dt < f\big(\tfrac{1}{2}\big).$$

That is,

$$\frac{1}{2}\left(0 + \frac{1}{k(k+1)}\right) < \int_0^1 \frac{t}{k(k+1)}dt < \frac{\frac{1}{2}}{k(k+\frac{1}{2})}.$$

Summing over k:

$$\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k(k+1)} < \sum_{k=1}^{n}\int_0^1 \frac{t}{k(k+t)}dt < \sum_{k=1}^{n}\frac{\frac{1}{2}}{k(k+\frac{1}{2})}dt = \sum_{k=1}^{n}\frac{1}{k(2k+1)}.$$

Employing partial fractions:

$$\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) < \gamma_n < \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{2k+1}\right).$$

Letting $n \to \infty$, this gives

$$\frac{1}{2} \leqslant \gamma \leqslant 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \cdots$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots\right).$$

So, knowing that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, we get

$$\frac{1}{2} \leqslant \gamma \leqslant 2(1 - \log 2) \approx 0.6137.$$

(iii) The Trapezium Rule also gives

$$\gamma - \gamma_n = \sum_{k=n+1}^{\infty}\int_0^1 \frac{t}{k(k+t)}dt > \frac{1}{2}\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2(n+1)}.$$

And so we obtain the left-hand side of the much-cited estimates (see [8]) :

$$\frac{1}{2(n+1)} < \gamma - \gamma_n < \frac{1}{2n}. \tag{2}$$

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(iv) It is reasonable to think that the Midpoint Rule employed in a similar way might yield the right-hand side of (2). Here we get

$$\gamma - \gamma_n = \sum_{k=n+1}^{\infty} \int_0^1 \frac{t}{k(k+t)} dt < \sum_{k=n+1}^{\infty} \frac{1}{k(2k+1)}.$$

Taking large numerical values for n does suggest that

$$\sum_{k=n+1}^{\infty} \frac{1}{k(2k+1)} < \frac{1}{2n},$$

but ultimately, this series is not so manageable. However, we can use another stalwart of calculus – the integral test – to obtain the right-hand side of (2). Observe that $\gamma = \sum_{k=1}^{\infty} F(k)$, where

$$F(x) = \int_0^1 \frac{t}{x(x+t)} dt \qquad (x \geqslant 1).$$

Here, $F > 0$ and $F'(x) = \dfrac{-1}{x^2(x+1)} < 0$, so F satisfies the hypotheses of the Integral Test. Its associated upper error bound gives

$$\gamma - \gamma_n < \int_n^{\infty} F(x)\, dx.$$

Evaluating the integral we get

$$\gamma - \gamma_n < (n+1) \log\left(\frac{n+1}{n}\right) - 1.$$

Now $1/x$ is convex on $(0, \infty)$ and so (here the Trapezium Rule is an overestimate):

$$\log\left(\frac{n+1}{n}\right) = \int_n^{n+1} \frac{1}{x} dx < \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) = \frac{2n+1}{2n(n+1)}.$$

Therefore

$$\gamma - \gamma_n < \frac{2n+1}{2n} - 1 = \frac{1}{2n},$$

which is the right-hand side of (2).

Remarks

(a) The associated lower error bound for the Integral Test namely,

$$\int_{n+1}^{\infty} F(x)\, dx < \gamma - \gamma_n,$$

is not quite as good as the left-hand side of (2).

(b) In [9], (1) is iterated to give expressions for $\gamma - \gamma_n$ which yield substantial improvements of (2).

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(v) Write $f_k(t) = \dfrac{t}{k(k+t)} = \dfrac{t}{k^2(1+\frac{t}{k})} = \dfrac{t}{k^2} - \dfrac{t^2}{k^3} + \dfrac{t^3}{k^4} - \dfrac{t^4}{k^5} + \dots$.

Then

$$\int_0^1 f_k(t)\,dt = \frac{1}{2}\frac{1}{k^2} - \frac{1}{3}\frac{1}{k^3} + \frac{1}{4}\frac{1}{k^4} - \frac{1}{5}\frac{1}{k^5} + \dots,$$

and therefore

$$\begin{aligned}\gamma = \sum_{k=1}^{\infty}\int_0^1 f_k(t)\,dt = {} & \frac{1}{2}\frac{1}{1^2} - \frac{1}{3}\frac{1}{1^3} + \frac{1}{4}\frac{1}{1^4} - \frac{1}{5}\frac{1}{1^5} + \dots \\ & + \frac{1}{2}\frac{1}{2^2} - \frac{1}{3}\frac{1}{2^3} + \frac{1}{4}\frac{1}{2^4} - \frac{1}{5}\frac{1}{2^5} + \dots \\ & + \frac{1}{2}\frac{1}{3^2} - \frac{1}{3}\frac{1}{3^3} + \frac{1}{4}\frac{1}{3^4} - \frac{1}{5}\frac{1}{3^5} + \dots \\ & + \frac{1}{2}\frac{1}{4^2} - \frac{1}{3}\frac{1}{4^3} + \frac{1}{4}\frac{1}{4^4} - \frac{1}{5}\frac{1}{4^5} + \dots \\ & \vdots \ \downarrow\end{aligned}$$

Summing the right-hand side down the columns instead of along the rows, this reads

$$\gamma = \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - \frac{1}{3}\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots\right) + \frac{1}{4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) - \dots$$

That is,

$$\gamma = \sum_{j=2}^{\infty} \frac{(-1)^j}{j}\zeta(j),$$

where ζ is the famous Riemann-Zeta function $\zeta(j) = \sum_{k=1}^{\infty} \frac{1}{k^j}$, which is defined for $j > 1$. (This is usually verified using the integral test). It is in this form that Euler himself first discovered γ (see [2]).

(vi) Observe also that

$$\int_{-1/2}^{1/2} f_k(t) = \frac{1}{k} - \log\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right).$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n}\int_{-1/2}^{1/2} f_k(t) &= \sum_{k=1}^{n}\left(\frac{1}{k} - \log\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right)\right) \\ &= \sum_{k=1}^{n}\frac{1}{k} - \log\left(n+\frac{1}{2}\right) + \log\left(\frac{1}{2}\right)\end{aligned}$$

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$$= \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) - \log 2 + \log\left(\frac{n+1}{n+\frac{1}{2}}\right).$$

Having $n \to \infty$, we get

$$\sum_{k=1}^{\infty} \int_{-1/2}^{1/2} f_k(t)\,dt = \gamma - \log 2.$$

Now, writing $f_k(t)$ as an infinite series exactly as above, then integrating from $-\frac{1}{2}$ to $\frac{1}{2}$, then again summing down the columns instead of along the rows, one obtains

$$\sum_{j=2}^{\infty} \frac{\zeta(2j+1)}{4^j(2j+1)} = \log 2 - \gamma.$$

This was also obtained by Euler. An excellent account of series involving γ and ζ can be found in [10].

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