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NONUNITAL PRIME RINGS GRADED BY ORDERED GROUPS

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Abstract

Let G be a group with identity element e, and suppose that S is an associative G-graded ring that is not necessarily unital. In the case where G is an ordered group, we show that a graded ideal is prime if and only if it is graded prime. Consequently, in that setting, a graded ring is prime if and only if it is graded prime. For any group G, if S is what we call *ideally symmetrically G-graded*, then we show that there is a bijective correspondence between the G-graded prime ideals of S and the G-prime ideals of S_e. We use this correspondence in the case where G is ordered and S is ideally symmetrically G-graded to show that S is prime if and only if S_e is G-prime. These results generalise classical theorems by Năstăsescu and Van Oystaeyen to a nonunital setting. As applications, we provide a new and more direct proof of a primeness criterion for Leavitt path rings and establish conditions for primeness of symmetrically G-graded subrings of group rings over fully idempotent rings.

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1. Introduction

Throughout this article, S denotes an associative but not necessarily unital ring. Unless explicitly stated, all ideals of S are assumed to be two-sided. Recall that a proper ideal P of S is called *prime* if, for all ideals $A, B \subseteq S$, the inclusion $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. The ring S is called *prime* if the zero ideal $\{0\}$ is prime. This notion generalises that of integral domains to the noncommutative setting. In fact, a commutative ring is prime if and only if it is an integral domain. The class of prime rings includes many familiar constructions, such as simple rings, primitive rings (left or right), and matrix rings over integral domains.

Let G be a group with identity element e. Recall that S is said to be *G-graded* if there are additive subgroups $S_x \subseteq S$, for $x \in G$, such that $S = \bigoplus_{x\in G} S_x$ and $S_x S_y \subseteq S_{xy}$,

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for all $x, y \in G$. If, additionally, $S_xS_y = S_{xy}$ for all $x, y \in G$, then S is said to be *strongly G-graded*. Throughout the rest of this introduction, let S be a G-graded ring.

A fundamental problem, studied over the past five decades, is to determine when a graded ring is prime (see, for example, [3, 4, 9, 12, 13, 16, 17]). In the unital and strongly graded case, Passman provided a complete solution (see Theorem 1.1), relying on the following notions. Let H, N be subgroups of G. Put $S_N := \bigoplus_{x\in N} S_x$. For an ideal I of S_N and $x \in G$, define $I^x := S_{x^{-1}}IS_x$. The ideal I is said to be *H-invariant* if $I^x \subseteq I$ for all $x \in H$.

THEOREM 1.1 (Passman [17, Theorem 1.3]). *Suppose that S is a unital and strongly G-graded ring. Then S is not prime if and only if there exist:*

(i) *subgroups $N \lhd H \subseteq G$ with N finite;*
(ii) *an H-invariant ideal I of S_e such that $I^xI = \{0\}$ for every $x \in G \setminus H$; and*
(iii) *nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.*

In [9, Theorem 1.3], the authors of the present article extended Theorem 1.1 to a class of nonunital rings called *nearly epsilon-strongly G-graded rings* (see [15]). A G-graded ring S belongs to this class if, for each $x \in G$ and each $s \in S_x$, there exist $\epsilon_x \in S_xS_{x^{-1}}$ and $\epsilon'_x \in S_{x^{-1}}S_x$ such that $\epsilon_x s = s = s\epsilon'_x$.

In contrast to the intricate conditions in Theorem 1.1, Năstăsescu and Van Oystaeyen have shown (see Theorem 1.2) that primeness criteria for graded rings become much simpler if we only consider groups G that are *ordered*, meaning that G is equipped with a total order $\leq$ satisfying $a \leq b \Rightarrow xay \leq xby$ for all $a, b, x, y \in G$. Before stating their results, we recall the following notions. An ideal I of S is *graded* if $I = \bigoplus_{x\in G} I_x$, where $I_x := I \cap S_x$. A proper graded ideal P of S is *graded prime* if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for all graded ideals $A, B \subseteq S$. The ring S is *graded prime* if $\{0\}$ is a graded prime ideal. A proper G-invariant ideal $Q \subseteq S_e$ is *G-prime* if $AB \subseteq Q$ implies $A \subseteq Q$ or $B \subseteq Q$ for all G-invariant ideals $A, B \subseteq S_e$. The ring S_e is *G-prime* if $\{0\}$ is a G-prime ideal.

THEOREM 1.2 (Năstăsescu and Van Oystaeyen, [12, Proposition II.1.4] and [13, Proposition 2.11.7]). *Suppose that S is a unital and G-graded ring where G is ordered. The following assertions hold.*

(a) *The ring S is prime if and only if it is graded prime.*
(b) *Suppose that S is strongly G-graded. Then S is prime if and only if S_e is G-prime.*

In this article, we continue the investigation from [9], now focusing on nonunital analogues of Theorem 1.2. We prove that Theorem 1.2(a) holds for *all* nonunital rings (see Theorem 2.5). We also extend Theorem 1.2(b) to rings S that are *ideally symmetrically G-graded*, meaning that for every graded ideal I of S and every $x \in G$, the equalities $S_xS_{x^{-1}}I_x = I_x$ and $I_xS_{x^{-1}}S_x = I_x$ hold (see Theorem 3.13). This class strictly contains all nearly epsilon-strongly G-graded rings, and thus also all unital strongly G-graded rings (see Proposition 5.11). Therefore, the results presented here apply to a broader range of gradings than those considered in [9].

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Here is a detailed outline of this article. In Section 2, we extend some results on prime ideals to a nonunital setting. We introduce *graded m-systems* to establish a primeness criterion for a nonunital S (see Corollary 2.3 and Theorem 2.4). We also prove a nonunital version of Theorem 1.2(a) (see Theorem 2.5). In Section 3, we establish a one-to-one correspondence between G-invariant ideals of S_e and graded ideals of S, in the case where S is ideally symmetrically graded (see Proposition 3.9). We also generalise Theorem 1.2(b) to the class of ideally symmetrically graded rings (see Theorem 3.13). In Section 4, we use our results from Section 3 to give a new short proof (see Theorem 4.4) of a previously established characterisation of prime Leavitt path rings from [9]. In Section 5, we investigate primeness of graded subrings of group rings $R[G]$ for nonunital rings R. In particular, we show that for any nontrivial ordered group G, there are prime G-graded rings which are not nearly epsilon-strongly G-graded (see Proposition 5.11).

2. Graded prime ideals

Throughout this section, S denotes an associative but not necessarily unital ring. Let A be a subset of S. We let $\mathbb{Z}A$ denote the set of all finite sums of integer multiples of elements of A, and we let SA denote the set of all finite sums of products of elements of S with elements of A. The sets AS and SAS are defined analogously. The ideal of S generated by A is denoted by $\langle A\rangle$, and it is defined as the intersection of all ideals of S containing A. Note that $\langle A\rangle = \mathbb{Z}A + AS + SA + SAS$.

For the rest of this section, S is equipped with a G-grading. We let $h(S)$ denote the set $\bigcup_{g\in G} S_g$ of homogeneous elements of S. For a nonzero $s \in h(S)$, we put $\deg(s) := g$ if $s \in S_g$.

PROPOSITION 2.1. *Let A be a subset of $h(S)$. Then $\langle A\rangle$ is a graded ideal of S.*

PROOF. Every element of $\langle A\rangle$ is a finite sum of homogeneous elements of the form a, $-a$, sa, at or sat, for $a \in A \subseteq h(S)$ and $s, t \in h(S)$. Thus, $\langle A\rangle$ is graded. □

We say that a subset T of a graded ring T' is a *graded m-system in T'* if, for all $a, b \in T \cap h(T')$, we have $ab \in T$ or there is some $s \in h(T')$ such that $asb \in T$.

PROPOSITION 2.2. *Let P be a proper graded ideal of S. The following assertions are equivalent.*

(i) *The ideal P is graded prime.*
(ii) *For all $a, b \in h(S)$, if $aSb \subseteq P$ and $ab \in P$, then $a \in P$ or $b \in P$.*
(iii) *The complement $S \setminus P$ is a graded m-system in S.*

PROOF. (i) ⇒ (ii). Suppose that P is graded prime and $a, b \in h(S)$ satisfy $aSb \subseteq P$ and $ab \in P$. By Proposition 2.1, $\langle a\rangle$ and $\langle b\rangle$ are graded ideals of S. Hence,

$$\begin{aligned}\langle a\rangle\langle b\rangle &= (\mathbb{Z}a + aS + Sa + SaS)(\mathbb{Z}b + bS + Sb + SbS)\\ &= \mathbb{Z}ab + \mathbb{Z}abS + \mathbb{Z}aSb + \mathbb{Z}aSbS + \mathbb{Z}aSb + aSbS + aSSb + aSSbS\\ &\quad + \mathbb{Z}Sab + SabS + SaSb + SaSbS + \mathbb{Z}SaSb + SaSbS + SaSSb + SaSSbS \subseteq P.\end{aligned}$$

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Since P is graded prime it follows that $\langle a\rangle \subseteq P$ or $\langle b\rangle \subseteq P$ which, in turn, implies that $a \in P$ or $b \in P$.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. Take $a, b \in h(S) \setminus P$. From (ii), $ab \notin P$ or there is $s \in S$ with $asb \notin P$. In the latter case, there is an $s' \in h(S)$ such that $as'b \notin P$.

(iii) $\Rightarrow$ (i). Suppose that $S \setminus P$ is a graded m-system in S. Let A and B be graded ideals of S with $AB \subseteq P$. Seeking a contradiction, suppose that $A \setminus P \neq \emptyset$ and $B \setminus P \neq \emptyset$. Every element in A and B is a finite sum of homogeneous elements in A and B, respectively, and hence there exist $a \in h(A) \setminus P$ and $b \in h(B) \setminus P$. Since $S \setminus P$ is a graded m-system in S, we have $ab \notin P$ which is a contradiction, or that there is an $s \in h(S)$ with $asb \notin P$. But then we also get a contradiction because $asb \in aSb \subseteq \langle a\rangle\langle b\rangle \subseteq AB \subseteq P$. Therefore, $A \subseteq P$ or $B \subseteq P$. □

If we restrict ourselves to trivial G-gradings on a unital ring S, that is, gradings where $S_g = \{0\}$, for $g \neq e$, then a graded m-system coincides with the classical notion of an m-system (see [8, Definition 10.3]). From Proposition 2.2, we therefore immediately get the following result.

COROLLARY 2.3. *Let P be a proper ideal of S. The following assertions are equivalent.*

(i) *The ideal P is prime.*
(ii) *For all $a, b \in S$, if $aSb \subseteq P$ and $ab \in P$, then $a \in P$ or $b \in P$.*
(iii) *The complement $S \setminus P$ is an m-system in S.*

THEOREM 2.4. *Suppose that G is an ordered group and that S is a G-graded ring. Let P be a proper graded ideal of S. Then P is prime if and only if P is graded prime.*

PROOF. The 'only if' statement is trivial. We now show the 'if' statement. Suppose that P is graded prime. We claim that $S \setminus P$ is an m-system in S. If we assume that the claim holds, then, from Corollary 2.3, it follows that P is prime. To show the claim, take $a, b \in S \setminus P$. Let p (respectively, q) be the sum of the homogeneous components of a (respectively, b) that belong to P. Put $d := a - p$ and $d' := b - q$. Note that all components of d and d' belong to $S \setminus P$. Since G is ordered we can pick the nonzero components r and r' of d and d', respectively, of highest degree. Since P is graded prime, it follows from Proposition 2.2 that $S \setminus P$ is a graded m-system in S. Thus, there exists a nonzero $s \in h(S)$ such that $rsr' \notin P$ or $rr' \notin P$. Seeking a contradiction, suppose that $asb \in P$ and $ab \in P$. From the equality $asb = (d + p)s(d' + q) = dsd' + psd' + dsq + psq$ and the fact that $psd', dsq, psq \in P$, it follows that $dsd' \in P$. Similarly, $ab \in P$ leads to $dd' \in P$. Since P is graded, all components of dsd' and dd' also belong to P. However, the component of dsd' (respectively, dd') of highest degree is rsr' (respectively, rr') which belongs to $S \setminus P$. This is a contradiction. □

Theorem 2.4 immediately implies the following result.

THEOREM 2.5. *Suppose that G is an ordered group and that S is a G-graded ring. Then S is prime if and only if S is graded prime.*

Note that Theorem 2.5 is a partial generalisation of [3, Theorem 3.2].

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3. G-prime ideals

Throughout this section, S denotes an associative G-graded ring that is not necessarily unital. Let J be an ideal of S_e. Recall from Section 1 that J is said to be G-invariant if, for each $x \in G$, the inclusion $S_{x^{-1}}JS_x \subseteq J$ holds. We define the map $(\cdot)_e : S \to S_e$, by $(\sum_{g\in G} s_g)_e := s_e$. By an abuse of notation, the parentheses in $(\cdot)_e$ will often be suppressed.

PROPOSITION 3.1. *Let J be an ideal of S_e. Then J is G-invariant if and only if $\langle J\rangle_e = J$.*

PROOF. Since $J \subseteq \langle J\rangle_e = (J + JS + SJ + SJS)_e = J + JS_e + S_eJ + \sum_{x\in G} S_{x^{-1}}JS_x = J + \sum_{x\in G} S_{x^{-1}}JS_x$, we have $J \subseteq \langle J\rangle_e = J + \sum_{x\in G} S_{x^{-1}}JS_x$, from which it easily follows that J is G-invariant if and only if $\langle J\rangle_e = J$. □

PROPOSITION 3.2. *Let I be a graded ideal of S. Then I_e is a G-invariant ideal of S_e.*

PROOF. Take $x \in G$. Then $S_{x^{-1}}I_eS_x \subseteq (S_{x^{-1}}IS_x) \cap (S_{x^{-1}}S_eS_x) \subseteq I \cap S_e = I_e$. □

DEFINITION 3.3. Let I be a graded ideal of S. We say that I is *identity generated* if $I = \langle I_e\rangle$.

PROPOSITION 3.4. *Let S be a G-graded ring. The map $\langle\cdot\rangle$ is an inclusion-preserving bijection $\{G$-invariant ideals of $S_e\} \to \{$identity generated G-graded ideals of $S\}$ with inverse $(\cdot)_e$.*

PROOF. Clearly the maps $\langle\cdot\rangle$ and $(\cdot)_e$ are inclusion-preserving. By Propositions 2.1 and 3.1, it follows that $\langle\cdot\rangle$ is well defined and $(\cdot)_e \circ \langle\cdot\rangle = \mathrm{id}$. Proposition 3.2 implies that $(\cdot)_e$ is well defined. From the definition of identity generated graded ideals, it follows that $\langle\cdot\rangle \circ (\cdot)_e = \mathrm{id}$. □

Recall that S is said to be *symmetrically G-graded* if $S_xS_{x^{-1}}S_x = S_x$ for every $x \in G$.

DEFINITION 3.5. We say that S is *ideally symmetrically G-graded* if, for every graded ideal I of S and every $x \in G$, the equalities $S_xS_{x^{-1}}I_x = I_x$ and $I_xS_{x^{-1}}S_x = I_x$ hold.

PROPOSITION 3.6. *The following assertions hold.*

(a) *If S is ideally symmetrically G-graded, then S is symmetrically G-graded.*
(b) *If S is nearly epsilon-strongly G-graded, then S is ideally symmetrically G-graded.*

PROOF. (a) Take $I = S$ in Definition 3.5.

(b) Suppose that S is nearly epsilon-strongly G-graded. Take $x \in G$ and a graded ideal I of S. Then $S_xS_{x^{-1}}I_x \subseteq S_eI_x \subseteq I_x$ and $I_xS_{x^{-1}}S_x \subseteq I_xS_e \subseteq I_x$. Now we show the reversed inclusions. Take $s \in I_x$. By the definition of nearly epsilon-strongly G-graded rings (see Section 1), there are $\epsilon_x \in S_xS_{x^{-1}}$ and $\epsilon'_x \in S_{x^{-1}}S_x$ such that $s = \epsilon_x s = s\epsilon'_x$. Thus, $s = \epsilon_x s \in S_xS_{x^{-1}}I_x$ and $s = s\epsilon'_x \in I_xS_{x^{-1}}S_x$. This shows that $S_xS_{x^{-1}}I_x = I_x = I_xS_{x^{-1}}S_x$. □

REMARK 3.7. There exist rings which are symmetrically graded but not ideally symmetrically graded (see Proposition 5.13).

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PROPOSITION 3.8. *Suppose that S is ideally symmetrically G-graded. Let I be a graded ideal of S. Then $I = SI_e = \langle I_e \rangle$ and $I = I_e S = \langle I_e \rangle$.*

PROOF. Note that $I = \sum_{x\in G} I_x = \sum_{x\in G} S_x S_{x^{-1}} I_x \subseteq \sum_{x\in G} S_x I_e \subseteq SI_e \subseteq SI \subseteq I$ since $S_{x^{-1}} I_x \subseteq S_e \cap I = I_e$ for $x \in G$. Thus, $I = SI_e \subseteq I_e + SI_e + I_e S + SI_e S = \langle I_e \rangle \subseteq I$ which establishes the first set of equalities. The second set of equalities are shown analogously. □

PROPOSITION 3.9. *Let S be ideally symmetrically G-graded. Then $\langle \cdot \rangle$ is an inclusion-preserving bijection $\{G\text{-invariant ideals of } S_e\} \to \{G\text{-graded ideals of } S\}$ with inverse $(\cdot)_e$.*

PROOF. This follows from Propositions 3.4 and 3.8. □

PROPOSITION 3.10. *Suppose that S is ideally symmetrically G-graded. Let I be a graded ideal of S such that I_e is a G-prime ideal of S_e. Then I is graded prime.*

PROOF. Let A and B be graded ideals of S with $AB \subseteq I$. Then $A_e B_e \subseteq AB \cap S_e \subseteq I \cap S_e = I_e$. By Proposition 3.2, A_e and B_e are G-invariant. Since I_e is G-prime, we have $A_e \subseteq I_e$ or $B_e \subseteq I_e$. By Propositions 3.8 and 3.9, $A = \langle A_e \rangle \subseteq \langle I_e \rangle = I$ or $B = \langle B_e \rangle \subseteq \langle I_e \rangle = I$. □

PROPOSITION 3.11. *Suppose that S is ideally symmetrically G-graded. Let I be a graded ideal of S which is graded prime. Then I_e is G-prime.*

PROOF. Let J and K be G-invariant ideals of S_e with $JK \subseteq I_e$. From Propositions 2.1, 3.1 and 3.8, it follows that $\langle J \rangle \langle K \rangle = SJKS \subseteq \langle JK \rangle \subseteq \langle I_e \rangle = I$. By graded primeness of I, we have $\langle J \rangle \subseteq I$ or $\langle K \rangle \subseteq I$. Proposition 3.9 now implies that $J = \langle J \rangle_e \subseteq I_e$ or $K = \langle K \rangle_e \subseteq I_e$. Thus I_e is G-prime. □

PROPOSITION 3.12. *Let S be ideally symmetrically G-graded. Then $\langle \cdot \rangle$ is an inclusion-preserving bijection $\{G\text{-prime ideals of } S_e\} \to \{G\text{-graded prime ideals of } S\}$ with inverse $(\cdot)_e$.*

PROOF. This follows from Propositions 3.9, 3.10 and 3.11. □

THEOREM 3.13. *Let G be an ordered group. Suppose that S is ideally symmetrically G-graded. Then S is prime if and only if S_e is G-prime.*

PROOF. This follows from Theorem 2.5 and Proposition 3.12. □

4. Application: prime Leavitt path rings

In [9] a characterisation of prime Leavitt path rings was obtained as an application of some rather involved methods. Here, we give a shorter proof of the same result (see Theorem 4.4) by applying Theorem 3.13. Our new proof also generalises the methods used to prove [2, Theorem 1.4] and [10, Proposition 4.5].

Let $E = (E^0, E^1, s, r)$ be a *directed graph*, where E^0 is the set of vertices, E^1 is the set of edges and $s, r\colon E^1 \to E^0$ are the source and range maps, respectively. For any

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$v \in E^0$, define $s^{-1}(v) := \{e \in E^1 \mid s(e) = v\}$, the set of edges emitted from v. A *path* in E is a sequence $\alpha := f_1 f_2 \cdots f_n$ where $f_i \in E^1$ and $r(f_i) = s(f_{i+1})$, for all i, and such a path is said to have *length* n. Vertices are considered to be paths of length zero. The set of all paths is denoted E^*. Define a preorder on E^0 by writing $u \geq v$ if there is a path from u to v. The graph E is said to satisfy *condition (MT-3)* if, for all $u, v \in E^0$, there is $w \in E^0$ with $u \geq w$ and $v \geq w$.

For the rest of this section, R denotes an associative unital ring. Recall that an *R-ring* S is a ring that is also an R-bimodule such that the following three conditions are satisfied for all $r \in R$ and $s, s' \in S$:

- $(ss')r = s(s'r)$;
- $(rs)s' = r(ss')$;
- $(sr)s' = s(rs')$.

The *Leavitt path ring of E over R*, denoted by $L_R(E)$, is the associative R-ring generated by $\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\}$ subject to the following six families of relations:

(i) $vw = \delta_{v,w} v$ for all $v, w \in E^0$;
(ii) $s(f)f = fr(f) = f$ for every $f \in E^1$;
(iii) $r(f)f^* = f^* s(f) = f^*$ for every $f \in E^1$;
(iv) $f^* f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$;
(v) $\sum_{f \in E^1, s(f)=v} f f^* = v$ for every $v \in E^0$ with $s^{-1}(v)$ nonempty and finite;
(vi) $rv = vr$, $rf = fr$ and $rf^* = f^* r$, for all $r \in R$ and $f \in E^1$.

REMARK 4.1. If $\alpha \in E^*$, then α and α^* are referred to as a *real path* and a *ghost path*, respectively, in $L_R(E)$. By [11, Lemma 3.5] every element in $E^0 \cup E^1 \cup \{f^* \mid f \in E^1\}$ is nonzero in $L_R(E)$, and the set of real (respectively, ghost) paths is linearly independent in the left R-module $L_R(E)$ and in the right R-module $L_R(E)$. From this it follows that all elements of the form $r\alpha\beta^*$ for nonzero $r \in R$ and $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ are nonzero. The element $\alpha\beta^*$ is referred to as a *monomial* in $L_R(E)$.

REMARK 4.2. The ring $L_R(E)$ carries a canonical $\mathbb{Z}$-grading defined by $\deg(f) := 1$ and $\deg(f^*) := -1$ for $f \in E^1$, and $\deg(v) := 0$ for $v \in E^0$ (see [1, Corollary 2.1.5]). This grading is nearly epsilon-strong (see [15]), and thus, by Proposition 3.6(b), $L_R(E)$ is ideally symmetrically $\mathbb{Z}$-graded.

LEMMA 4.3. *Suppose that E is a directed graph and R is a unital ring. Let I be a nonzero $\mathbb{Z}$-invariant ideal of $L_R(E)_0$. Then there exist $v \in E^0$ and a nonzero $r \in R$ such that $rv \in I$.*

PROOF. Put $S := L_R(E)$. Take a nonzero $a \in I$. By [9, Proposition 14.11] there exist $\alpha, \beta \in E^*$, $v \in E^0$ and $r \in R \setminus \{0\}$ with $\alpha^* a \beta = rv$. Since $rv \in S_0$ it follows from the $\mathbb{Z}$-grading of S that $\deg(\alpha) = \deg(\beta) =: n$. Since I is $\mathbb{Z}$-invariant, $rv = \alpha^* a \beta \in S_{-n} I S_n \subseteq I$. □

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THEOREM 4.4. *Suppose that E is a directed graph and R is a unital ring. Then the Leavitt path ring $L_R(E)$ is prime if and only if R is prime and E satisfies condition (MT-3).*

PROOF. Put $S := L_R(E)$. First we show the 'only if' statement. Suppose that R is not prime. Take nonzero ideals I, J of R with $IJ = \{0\}$. Let A and B be the ideals of S consisting of sums of monomials with coefficients in I and J, respectively. By Remark 4.1, A and B are nonzero. Clearly, $AB = \{0\}$, and hence S is not prime. Suppose now that E^0 does not satisfy condition (MT-3). Then there exist $v, w \in E^0$ such that for every $y \in E^0$, $v \not\geq y$ or $w \not\geq y$. Put $C := SvS$ and $D := SwS$. By Remark 4.1, C and D are nonzero ideals of S. Seeking a contradiction, suppose that $CD \neq \{0\}$. Then there exist $\alpha, \beta \in E^*$ with $v\alpha\beta^* w \neq 0$ which gives the contradiction $v \geq u$ and $w \geq u$ with $u := r(\alpha) = r(\beta)$. Therefore, $CD = \{0\}$. Hence, S is not prime.

We now show the 'if' statement. Suppose that R is prime and E satisfies (MT-3). Seeking a contradiction, suppose that S is not prime. By Theorem 3.13 and Remark 4.2, there exist nonzero $\mathbb{Z}$-invariant ideals A and B of S_0 with $AB = \{0\}$. By Lemma 4.3, there are $r, s \in R \setminus \{0\}$ and $v, w \in E^0$ with $rv \in A$ and $sw \in B$. From (MT-3) there are $\alpha, \beta \in E^*$ and $u \in E^0$ with $s(\alpha) = v$, $r(\alpha) = r(\beta) = u$ and $s(\beta) = w$. Consider the nonzero ideals $I := RrR$ and $J := RsR$ of R. By primeness of R and Remark 4.1 it follows that $(IJ)u \neq \{0\}$. Since $\alpha^*\alpha = u = \beta^*\beta$, $\mathbb{Z}$-invariance of A and B imply that $Iu = \alpha^*(Iv)\alpha \subseteq \alpha^* A\alpha \subseteq A$ and $Ju = \beta^*(Jw)\beta \subseteq \beta^* B\beta \subseteq B$. This gives the contradiction sought: $\{0\} = AB \supseteq (Iu)(Ju) = (IJ)u \neq \{0\}$. □

5. Application: prime graded subrings of group rings

The purpose of this section is twofold. Given any nontrivial group G, we show that:

- the class of ideally symmetrically G-graded rings strictly contains the class of epsilon-strongly G-graded rings (see Proposition 5.11);
- the class of symmetrically G-graded rings strictly contains the class of ideally symmetrically G-graded rings (see Proposition 5.13).

To establish this, we consider certain subrings of group rings defined by ideal filters. Throughout this section, we make the following assumptions:

- R is an associative but not necessarily unital ring, and G is a group;
- for each $x \in G$, we are given an additive subgroup I_x of R with $I_e = R$;
- we consider the additive subgroup $S := \bigoplus_{x \in G} I_x x$ of the group ring $R[G]$ of G over R;
- for each $x \in G$, put $S_x := I_x x$, so that the additive group S is G-graded by $(S_x)_{x \in G}$.

We say that $(I_x)_{x \in G}$ is a *G-filter* if, for all $x, y \in G$, the inclusion $I_x I_y \subseteq I_{xy}$ holds.

PROPOSITION 5.1. *The additive G-graded group S is a G-graded subring of $R[G]$ if and only if $(I_x)_{x \in G}$ is a G-filter of ideals of R.*

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PROOF. Suppose that S is a G-graded subring of $R[G]$. Take $x, y \in G$. Since I_x is an additive subgroup of R and the inclusions $S_eS_x \subseteq S_x$ and $S_xS_e \subseteq S_x$ hold, it follows that I_x is an ideal of R. The inclusion $S_xS_y \subseteq S_{xy}$ implies that $I_xI_y \subseteq I_{xy}$. This establishes the 'only if' part of the statement. The 'if' part of the statement follows analogously. □

PROPOSITION 5.2. *Suppose that S is a G-graded subring of $R[G]$. Then S is symmetrically G-graded if and only if R is idempotent and, for each $x \in G$, the equality $I_xI_{x^{-1}}I_x = I_x$ holds.*

PROOF. The ring S is symmetrically G-graded if and only if the equality $S_xS_{x^{-1}}S_x = S_x$ holds for every $x \in G$. This is, in turn, equivalent to $I_xI_{x^{-1}}I_x = I_x$ for every $x \in G$. Putting $x = e$ in this equality yields $RRR = R$ so that $R = RRR \subseteq RR \subseteq R$. Hence, $R = RR$. □

Recall that a ring R is called *fully idempotent* if every ideal of R is idempotent. In that case, in particular, R itself has to be idempotent. From [5, Theorem 1.2] we recall the following result.

PROPOSITION 5.3. *A ring is fully idempotent if and only if, for all ideals I and J of the ring, the equality $IJ = I \cap J$ holds.*

PROPOSITION 5.4. *Let S be a G-graded subring of $R[G]$ where R is fully idempotent. Then S is symmetrically G-graded if and only if, for every $x \in G$, the equality $I_x = I_{x^{-1}}$ holds.*

PROOF. By Propositions 5.2 and 5.3, S is symmetrically G-graded $\Leftrightarrow$ $I_x \cap I_{x^{-1}} = I_x$ for all $x \in G$ $\Leftrightarrow$ $I_x \subseteq I_{x^{-1}}$ for all $x \in G$ $\Leftrightarrow$ $I_x = I_{x^{-1}}$ for all $x \in G$. □

PROPOSITION 5.5. *Let S be a G-graded subring of $R[G]$ where R is fully idempotent. Then S is symmetrically G-graded if and only if S is ideally symmetrically G-graded.*

PROOF. The 'if' statement follows from Proposition 3.6(a). Turning to the 'only if' statement, suppose that S is symmetrically G-graded. Let J be a graded ideal of S. Take $x \in G$. Since $RJ \subseteq J$ and $JR \subseteq J$, it follows that there is an ideal K_x of R such that $J_x = K_xx$. From $J \subseteq S$ we have $K_x \subseteq I_x$. By Propositions 5.3 and 5.4, it follows that $S_xS_{x^{-1}}J_x = I_xI_{x^{-1}}K_xx = I_xI_xK_xx = (I_x \cap I_x \cap K_x)x = K_xx = J_x$. Analogously, one can show that $J_xS_{x^{-1}}S_x = J_x$. □

THEOREM 5.6. *Suppose that S is a symmetrically G-graded subring of $R[G]$ where the group G is ordered and the ring R is fully idempotent. Then S is prime if and only if R is prime.*

PROOF. This follows from Theorem 3.13 and Proposition 5.5, since ideals of R are G-invariant. □

We now proceed to show that there exist symmetrically graded subrings of $R[G]$ that are not nearly epsilon-strongly graded (see Proposition 5.11). For this, we recall some definitions from [14, 18]. Let T be an associative (not necessarily unital) ring and M a left (respectively, right) T-module. Then M is *s-unital* if $m \in Tm$ (respectively,

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$m \in mT$) for all $m \in M$. The ring T is s-unital if it is s-unital as both a left and right T-module. From [6, Corollary 1.5], we recall the following result.

PROPOSITION 5.7. *Let R be a fully idempotent ring. The centre of R is von Neumann regular. In particular, if R is commutative, then R is von Neumann regular and hence s-unital.*

Thus, to find a non-s-unital fully idempotent R, we must consider a noncommutative R. We also aim to consider rings S that are not nearly epsilon-strongly G-graded.

PROPOSITION 5.8. *Suppose that S is a G-graded subring of $R[G]$. Then S is nearly epsilon-strongly G-graded if and only if, for every $x \in G$, the ideal I_x is s-unital both as a left $I_xI_{x^{-1}}$-module and as a right $I_{x^{-1}}I_x$-module.*

PROOF. The claim follows from the chain of equivalences: S is nearly epsilon-strongly G-graded $\Leftrightarrow$ for all $x \in G$, S_x is s-unital both as a left $S_xS_{x^{-1}}$-module and as a right $S_{x^{-1}}S_x$ module $\Leftrightarrow$ for all $x \in G$, I_x is s-unital both as a left $I_xI_{x^{-1}}$-module and as a right $I_{x^{-1}}I_x$ module. □

COROLLARY 5.9. *Let S be a symmetrically G-graded subring of $R[G]$ where R is a fully idempotent unital domain. Suppose that there is $x \in G$ with $0 \subsetneq I_x \subsetneq R$. Then S is not nearly epsilon-strongly G-graded.*

PROOF. Seeking a contradiction, suppose that S is nearly epsilon-strongly G-graded. Take a nonzero $a \in I_x$. From Propositions 5.3 and 5.4, it follows that $I_xI_{x^{-1}} = I_xI_x = I_x$. By Proposition 5.8, there is $b \in I_xI_{x^{-1}} = I_x$ with $ba = a$. But R is unital and hence $1_Ra = a$, so that $(b - 1_R)a = ba - 1_Ra = a - a = 0$. Since $a \neq 0$ and R is a domain, this implies that $b = 1_R$ which contradicts $I_x \subsetneq R$. □

EXAMPLE 5.10. We follow [7, Example 12.2] closely. Let F_0 be a field of characteristic zero. Let A be the first Weyl algebra over F_0 with generators X and Y satisfying the relation $XY - YX = 1$. Let δ be the differentiation operator on $F := F_0[Y]$. Then A is the differential polynomial ring $F[X; \delta]$. Put $L := AX$. Then L is a simple domain without identity. Put $R := L \oplus F_0$. Then R is a fully idempotent domain with exactly three ideals: $\{0\}$, L and R.

PROPOSITION 5.11. *Let R and L be defined as in Example 5.10. For each $x \in G$, we take $I_x \in \{R, L\}$ such that the set $G_R := \{x \in G \mid I_x = R\}$ is a subgroup of G. The following assertions hold.*

(a) *The ring S is symmetrically G-graded.*
(b) *The ring S is strongly G-graded $\Leftrightarrow$ S is nearly epsilon-strongly G-graded $\Leftrightarrow$ $G_R = G$.*
(c) *If G is an ordered group, then S is prime.*

PROOF. (a) Since G_R is a subgroup of G, Propositions 5.1 and 5.4 imply that S is a symmetrically G-graded subring of $R[G]$.

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(b) Since R, and hence also S, are unital, the implication S is strongly G-graded $\Rightarrow$ S is nearly epsilon-strongly G-graded holds. From Corollary 5.9, the implication S is nearly epsilon-strongly G-graded $\Rightarrow$ $G_R = G$ holds. The implication $G_R = G \Rightarrow S$ is strongly G-graded is trivial.

(c) This follows from Theorem 5.6, since R is a domain and hence prime. □

We conclude this article by providing an example of the claim made in Remark 3.7 (see Proposition 5.13). To this end, we consider the following example.

EXAMPLE 5.12. Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers, that is, the algebraic closure of $\mathbb{Q}$. Let O denote the ring of algebraic integers in $\overline{\mathbb{Q}}$. Let J be the ideal of O generated by $\{2^{1/2^n} \mid n \geq 1\}$. Since $2^{1/2^n} = (2^{1/2^{n+1}})^2$ for $n \geq 1$ it follows that J is an idempotent ideal. We claim that $1 \notin J$. Assume, for a moment, that the claim holds. Then J is a proper ideal of O. Put $K := O\sqrt{2}$. Then K is an ideal of the ring J. Since $1 \notin J$ it follows that $\sqrt{2} \notin J\sqrt{2}$ and hence that $K = O\sqrt{2} \supsetneq JO\sqrt{2} = JK$. Note that $K \subsetneq J$ since, supposing that $J = K$, using idempotency of J would give $J = J^2 = JK \subsetneq J$. Summarising, we have $JK \subsetneq K \subsetneq J$.

We now show the claim. Seeking a contradiction, suppose that $1 \in J$. Then there exist $m \geq 1$, $a_1, \ldots, a_m \in O$ and integers $n_1, \ldots, n_m \geq 1$ with $1 = \sum_{i=1}^{m} a_i 2^{1/2^{n_i}}$. Let $N := \max\{n_1, \ldots, n_m\}$. For each i, we have $2^{1/2^{n_i}} = (2^{1/2^N})^{2^{N-n_i}}$, so each summand $a_i 2^{1/2^{n_i}}$ lies in the principal ideal $(2^{1/2^N})$. Therefore, $1 \in (2^{1/2^N})$ and so $1 = 2^{1/2^N} \sum_{i=1}^{m} a_i (2^{1/2^N})^{2^{N-n_i}-1}$. Since O is a ring and $2^{1/2^N}$ is an algebraic integer, the sum belongs to O. Hence, $2^{1/2^N}$ is a unit in O, which is a contradiction. Indeed, if $2^{1/2^N}$ were a unit in O, then $2 = (2^{1/2^N})^{2^N}$ would also be a unit in O, which is impossible since $1/2 \notin O$.

PROPOSITION 5.13. *Let J and K be defined as in Example 5.12 and let $I_x := J$ for every $x \in G$. Then S is symmetrically G-graded. Moreover, if G is nontrivial, then S is not ideally symmetrically G-graded.*

PROOF. By Proposition 5.2, S is symmetrically G-graded, since J is idempotent. Now suppose that $G \neq \{e\}$. Put $M := \sum_{x \in G} Kx$ and note that M is a graded ideal of S. Fix $x \in G \setminus \{e\}$. Then $S_x S_{x^{-1}} M_x = J^2 Kx = JKx \subsetneq Kx = M_x$. Thus, S is not ideally symmetrically G-graded. □

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