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Bernstein-wave instability in a collisional plasma with a quasistatic electric field

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Plasma-wave instability caused entirely by an external large-scale electric field $\mathbf{E}_0$ is considered for the pure electron Bernstein mode. It is assumed for the angle α between the wave vector $\mathbf{k}$ and the normal $\mathbf{n}$ to the direction of the external magnetic field $\mathbf{B}_0$ that the relation $\alpha \approx \tan\alpha \ll 1$ is fulfilled. The quasistatic time variation of the field $\mathbf{E}_0$, which is parallel to $\mathbf{B}_0$, provides the possibility of realizing the direct initiation (by the large-scale field $\mathbf{E}_0$) mechanism for the development of instability. An expression for the growth rate is obtained for the case when the external electric field $\mathbf{E}_0$ together with Landau damping and pair Coulomb collisions are taken into account in a modified dispersion relation (MDR). This expression is correct in the framework of linear theory for the frequency ranges typical of electron cyclotron waves and Bernstein modes. The influence of the electric field in the MDR is taken into account through the method of Brinca and Dysthe. A numerical analysis of obtained instability is performed for specific values of the plasma parameters.

1. Introduction

In our previous papers (Kryshtal and Kucherenko 1995, 1996) we have briefly explained the main reasons for the enhanced interest in the behaviour of a collisional plasma in the presence of an external electric field $\mathbf{E}_0$ in the solar atmosphere. The results obtained earlier for solid state (Pines and Schrieffer 1961) and magnetospheric (Misra *et al.* 1979; Brinca 1980; Brinca and Dysthe 1983) plasmas have been used as the starting point for similar investigations in solar active regions. The conditions for the growth of instability of Bernstein modes in a solar plasma have been studied in a number of papers, and are now well known. There have been, for example, investigations of the fine structure of solar radio emission (Zheleznyakov and Zlotnik 1974, 1975) and (closely related to the problem of Bernstein modes) fundamental studies of the possibilities of realization of the mechanism of electron cyclotron maser emission (ECME) (Melrose and Dulk 1982; Melrose 1991). In these papers the plasma was investigated in the coronal region of the current circuit of a flare loop, where the influence of collisions can be neglected and where beams of energetic electrons are the main source of instability. As a rule, the loss-cone distribution was considered for the electron velocity distribution function. At the same time, data from observational programmes over the last decade has provided researchers with evidence of the considerable contribution of

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photospheric processes to the energy balance of a flare (Benka 1994; Alikajeva *et al.* 1995*a*, *b*).

For these reasons, any possible source of waves in the region under investigation – at the footpoints of the loop structures, if we are studying the flare plasma – is of great interest. Plasma-wave instability can be one of these sources. This instability can manifest itself in different ways (plasma-wave generation, decay and coalescence, turbulence, anomalous resistance, etc.), but it is clear that a beam of energetic electrons cannot be its main cause. First, values of the plasma density in the investigated area that are too high prevent the growth of the instability, and, secondly, the large-scale electric field $\mathbf{E}_0$ in one of the loop's legs is directed in such a way that it removes electrons from the beam (Gopasyuk and Gopasyuk 1995). In such a case this large-scale field becomes the main reason for instability, while Coulomb collisions together with Landau damping tend to neutralize this process. In the absence of the beams, the equilibrium distribution function of the electrons of the background plasma can be approximated as a shifted Maxwellian, and the instability arises as a result of the direct initiation mechanism and develops in accordance with the stationary scenario.

This process was described in detail in Kryshtal and Kucherenko (1995, 1996), where we studied the ion-acoustic and Langmuir instabilities. Here we study in the same way the instability of the second harmonic of pure electron Bernstein modes. Willes and Robinson (1996) have shown, for the ECME mechanism, the way in which ion-acoustic waves together with Langmuir waves and Bernstein modes take part in the process of decay and coalescence. As a result of these processes, electromagnetic radiation can be produced, and can escape from the generation area. If we assume that it was this emission that was observed in the frequency range $\omega \approx 17$–20 GHz by the *Yohkoh* programme near the footpoints of some flare loops (Benka 1994) then it can be considered as a valuable source of information about the structure of large-scale electric and magnetic fields in the lower levels of the flare atmosphere.

The modes studied in the present paper are approximate solutions of the dispersion relation (DR) for longitudinal dielectric permeability

$$\varepsilon_l(\omega, \mathbf{k}) = \frac{k_i k_j}{k^2} \varepsilon_{ij}(\omega, \mathbf{k}) = 0, \qquad (i, j = x, y, z), \tag{1}$$

where k_i are the components of the perturbation wave vector and $\varepsilon_{ij}(\omega, \mathbf{k})$ are the Cartesian components of the dielectric permeability tensor of a hot magneto-active plasma. Expressions for these components have been obtained in Alexandrov *et al.* (1989) for the case where the particle velocity distribution is of purely Maxwellian type and Coulomb collisions are described by the BGK integral (Bhatnagar *et al.* 1954). That is why it is better to call the waves studied here 'quasi-Bernstein weakly oblique' modes. They turn into ordinary Bernstein modes with dispersion law:

$$\omega^2 = m^2\Omega^2(1 + \alpha_m), \qquad m = 1, 2, 3, \ldots, \tag{2}$$

with

$$\alpha_m = 2\omega_*^2 z_\beta^{-1} \tilde{A}_m(z_\beta), \tag{3}$$

only as $\nu_{\alpha\beta} \to 0$ and $k_z \to 0$. Here $\nu_{\alpha\beta}$ is the mutual collision frequency of particles of species α with particles of species β (in plasma $\alpha, \beta = i$ (ions), e (electrons) or

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n (neutrals)), and the wave-vector amplitude $k = |\mathbf{k}|$ can be expressed in the ordinary form

$$k^2 = k_\perp^2 + k_z^2. \tag{4}$$

In (2) and (3) the following notation has been used:

$$\tilde{A}_m(z_\beta) = I_m(z_\beta)\exp(-z_\beta), \qquad z_\beta = \frac{k_\perp^2 V_{T_\beta}^2}{\Omega_\beta^2}, \qquad \omega_* = \frac{\omega_{p_\beta}}{\Omega_\beta},$$

$$V_{T_\beta}^2 = \frac{k_B T_\beta}{m_\beta}, \qquad \Omega_\beta = \frac{q_\beta B_0}{m_\beta c}, \qquad \omega_{P_\beta}^2 = \frac{4\pi n_\beta e^2}{m_\beta},$$

and, in turn, m_β, q_β, T_β and n_β are the mass, charge, kinetic temperature and number density of particles of species β respectively. For the pure electron Bernstein modes, $\beta = e$ and $q_\beta = -e$, where $|e|$ is the electronic charge. $I_m(z_\beta)$ are the modified mth-order Bessel functions.

If consideration of collisions and a small (but nonzero) component k_z of the wave vector leads to only small perturbations in the DR, i.e. if this description of the process in the framework of linear theory is legitimate, then the dispersion law can be considered as unchanged. In this situation the small perturbations mentioned above contribute their corrections only to the imaginary part of the frequency, i.e. to the growth rate (Krall and Trivelpiece 1973). In this case the following conditions have to be satisfied:

$$\frac{\nu_{\alpha\beta}}{\Omega_\beta} \ll 1 \qquad (\alpha, \beta = e, i, n), \tag{5}$$

$$\frac{k_z}{k_\perp} \equiv k_* \left(\approx \frac{k_z}{|k|} \right) \ll 1. \tag{6}$$

In the present paper a Cartesian coordinate system is used, with the Z axis taken to be in the direction of the vectors $\mathbf{E}_0$ and $\mathbf{B}_0$. Taking into account the specific location of the studied plasma layers in the solar atmosphere and the fact that $\mathbf{B}_0$ is the (external) magnetic field of the loop structure, this Z axis can, in most cases, be considered as normal to the photospheric surface. In the case of a longitudinal electric field $\mathbf{E}_0$ (i.e. for $\mathbf{E}_0 \parallel \mathbf{B}_0$), the existence of a perturbation wave-vector component $k_z \neq 0$ is very important even at $k_* \ll 1$, because at $k_z = 0$ the effect under investigation vanishes.

The field $\mathbf{E}_0$ can practically always be considered as weak, despite its very large absolute value in the solar atmosphere. This means that the following condition has to be satisfied:

$$E_R \equiv \frac{E_0}{E_D} = \frac{e|E_0|}{m_e \nu_{ei} V_{T_e}} < 1. \tag{7}$$

In the present paper the mechanism of direct initiation of instability by the electric field is used together with a supposition about the next development of instability in the framework of stationary scenario (Kryshtal and Kucherenko 1995). It should be noted that for $E_R \leqslant 1$ the fraction of runaway electrons becomes inadmissibly high, and the instability becomes a marginal one. This means that, for mechanism and scenario to be appropriate it is necessary to change the condition (7) to the condition

$$E_R \ll 1. \tag{8}$$

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Only in this case will the corrections due to consideration of terms proportional to $|\mathbf{E}_0|$ in the MDR not influence on the form of the dispersion law (2), and they can be taken into account only in the expression for the growth rate. At the same time, a similar expression for the growth rate in the case of a shifted Maxwellian distribution function with shift velocity

$$(\mathbf{u}_e)_z = u_e = \frac{e|\mathbf{E}|}{m_e \nu_{ei}} \tag{9}$$

can easily be obtained from the analogous expression for a pure Maxwellian distribution (Kryshtal 1992*a*, *b*, 1997). From the calculational point of view, this means a trivial replacement of ω by $\omega' = \omega - k_z u_e$ in the MDR and in the expression for the growth rate. Here ω is the (cyclic) frequency of plasma waves. Only for weak fields, where the condition (8) is satisfied, is it possible to approximate the real electron equilibrium velocity distribution function by the shifted Maxwellian one

$$f_{0e} = \left(\frac{m_e}{2\pi k_B T_e}\right)^{3/2} \exp\left[-\frac{(v-u_e)^2}{2V_{T_e}^2}\right], \tag{10}$$

where k_B is Boltzmann's constant. Other, more delicate, problems of this approximation have been discussed in detail in Krall and Trivelpiece (1973) and Alexandrov *et al.* 1989).

2. Plasma model and basic equations

The plasma under investigation is assumed to be unrestricted, homogeneous and fully ionized. This means that, in the studied frequency ranges and time scales (i.e. the instability development times), the influence of gradient-drift instabilities, which occur in a plasma owing to boundary conditions and density heterogeneities, can be neglected. The assumption of complete ionization of plasma means satisfaction of the conditions

$$\nu_{ne}, \nu_{ni} \ll \nu_{ii}, \nu_{ei}, \nu_{ee}, \tag{11}$$

where ν_{ne} and ν_{ni} are the frequencies of charged-particle collisions with neutral atoms, whereas ν_{ii}, ν_{ei} and ν_{ee} are the frequencies of the mutual charged-particle collisions. The conditions (11) allow one to consider the relatively cool and dense plasma of lower layers of flare atmosphere with a certain small percentage of neutrals as being fully ionized. The legitimacy of these approximations has been discussed in Kryshtal and Kucherenko (1995), where practically the same plasma model is used.

In the calculation, taking into account the known models of the solar atmosphere (Machado *et al.* 1980; Vernazza *et al.* 1981; Alikajeva *et al.* 1991), it is assumed that the main physical parameters of plasma (i.e. the electron density n_e, temperature T_e and amplitude of the external magnetic field $B_0 = |\mathbf{B}_0|$) vary in the following ranges:

$$9.70 \times 10^{15}\,\mathrm{m}^{-3} \leqslant n_e \leqslant 1.30 \times 10^{19}\,\mathrm{m}^{-3}, \tag{12}$$

$$4000\,\mathrm{K} \leqslant T_e \leqslant 35000\,\mathrm{K}, \tag{13}$$

$$0.01\,\mathrm{T} \leqslant B_0 \leqslant 0.15\,\mathrm{T}. \tag{14}$$

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(It should be noted here that one model with $T_e = 447\,000$ K has also been investigated; see Table 1 below). Throughout the calculations, all of the values ν_{ee} and ν_{ii} were expressed in terms of ν_{ei}, making use of known formulae (Krall and Trivelpiece 1973; Somov, 1994). The ion–electron collision frequency is given by (Chen 1984)

$$\nu_{ei} = 2 \times 10^{-6} \frac{Z n_e \ln \Lambda}{T_{e,\mathrm{eV}}^{3/2}} \mathrm{s}^{-1}, \tag{15}$$

where $T_{e,\mathrm{eV}}$ is the electron temperature expressed in electron volts. Throughout this paper, it is assumed that $z = 1$ (singly charged ions) and $\ln \Lambda = 9.8$.

As in the case of high-frequency Langmuir waves, in this paper the case is considered where the ion–electron collisions dominate. The expression for the growth rate that will be obtained below demonstrates that such a situation is the most favourable for the development of instability. In the phenomenological approach, with consideration of collisions of different types (Kryshtal and Kucherenko 1995), the quantity $\tilde{\nu}_{ei}$ appears in the DR, and is related to ν_{ei} as follows:

$$\tilde{\nu}_{ei} = \sigma_k \nu_{ei}, \tag{16}$$

where $\sigma_k = \sigma_k(k_*, E_R, \omega_*)$ is a known function of the plasma parameters (Alexandrov *et al.* 1989; Chen 1984). The specific form of this function depends on the specific dispersion law $\omega = \omega(\mathbf{k})$ of the plasma waves. If the condition (4) can be replaced by

$$\tilde{\nu}_{ei}/\Omega_e \ll 1 \tag{17}$$

then, for

$$\omega_*^2 < 1 \tag{18}$$

and under the conditions (5) and (8), it can easily be shown that

$$1 \leqslant |\sigma_k| \leqslant 3.5, \tag{19}$$

when all of the quantities n_e, T_e, B_0 vary in the intervals (12)–(14). The value of the upper limit in (19) depends on the form of the function $\omega = \omega(\mathbf{k})$. Satisfaction of the conditions (5), (8), (11) and (17)–(19) allows us to use the relatively simple BGK integral (Bhatnagar *et al.* 1954) to take account of collisions in the plasma model.

The classical condition for the weakness of the electric field (Pines and Schrieffer 1961),

$$\frac{e|\mathbf{E}_0|}{k_z k_B T_e} \ll 1, \tag{20}$$

in the present paper is more rigid than the condition of (8) because of the smallness of the value of k_z.

For the growth rate $\gamma \equiv \mathrm{Im}\, \omega_k$ (where $\omega_k \equiv \omega(\mathbf{k})$) the following condition is satisfied:

$$|\mathrm{Im}\, \omega_k| \ll |\mathrm{Re}\, \omega_k|, \tag{21}$$

which is standard in the framework of linear theory.

The quasistatic condition for the electric field amplitude E_0 and time of instability development t_{inst} has the form

$$\left| \frac{1}{E_0} \frac{\partial E_0}{\partial t} \right| \ll \frac{1}{t_{\mathrm{inst}}} = \gamma, \tag{22}$$

and provides, together with the conditions (5), (6), (17), (18), (20) and (21), the

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TABLE 1. Values of absolute maxima of growth rates for different plasma equations of state.

Case	$n_e\,(\mathrm{cm}^{-3})$	$T_e\,(\mathrm{K})$	$B_0\,(\mathrm{T})$	σ_k	$(\omega_*)_M$	$(z)_M$	$(k_*)_M$	$(k_*)_{\max}$	$(E_R)_M$	$(\gamma/\Omega_e)_M$
A	1.37×10^{12}	5290	0.987	1	0.37	0.20	0.0015	0.0049	0.20	0.591
B	3.58×10^{11}	5620	0.321	1	0.51	0.25	0.0050	0.0132	0.15	0.757
C	1.00×10^{11}	6500	0.154	1	0.66	0.05	0.0065	0.0172	0.08	0.0318
D	6.89×10^{10}	447000	0.116	1	0.73	0.01	0.0020	0.0110	0.20	2.26×10^{-8}
E	6.89×10^{10}	447000	0.1145	2.5	0.74	0.01	0.0020	0.0115	0.20	2.06×10^{-8}
F	6.89×10^{10}	447000	0.1145	3.5	0.74	0.01	0.0020	0.0015	0.20	1.94×10^{-8}
G	1.00×10^{11}	6500	0.1454	2.5	0.70	0.05	0.0010	0.0206	0.10	0.89
H	1.00×10^{11}	6500	0.1395	3.5	0.73	0.15	0.0160	0.0348	0.17	4.68

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possibility of instability development in the framework of the stationary scenario. The growth of this instability is provided by the mechanism of direct initiation.

For the pure electron Bernstein modes, the longitudinal dielectric permeability ε_l from (1), with the equilibrium distribution function (9), (10), has the form

$$\varepsilon_l = 1+\frac{1}{k^2 d_e^2}\left[1+i\frac{\pi^{1/2}\omega'}{k_z V_{Te}}\frac{\sum\limits_{m=-\infty}^{\infty}\tilde{A}_m(z_e)\,W(\beta_{vm})}{1-\dfrac{\pi^{1/2}\tilde{\nu}_{ei}}{k_z V_{Te}}\sum\limits_{m=-\infty}^{\infty}\tilde{A}_m(z_e)\,W(\beta_{vm})}\right], \tag{23}$$

where

$$d_e = \left(\frac{\varepsilon_0 k_B T_e}{ne^2}\right)^{1/2}$$

is the electron Debye radius, $\omega' = \omega - k_z u_e$,

$$\beta_{vm} = \frac{\omega' + i\nu_{ei} - m\Omega_e}{k_z V_{Te}}$$

(where $m = 1, 2, 3, \ldots$ is the number of harmonics), and $W(x)$ is the known function (Krall and Trivelpiece 1973)

$$W(x) = \exp(-x^2)\left[1+\frac{2i}{\pi^{1/2}}\int_0^x \exp(t^2)\,dt\right]. \tag{24}$$

The value of σ_k in the expression (16) for $\tilde{\nu}_{ei}$ is equal to unity when the ion–electron collisions dominate.

At $\nu_{ei} = 0$ and $k_z = 0$, the known DR for the classical Bernstein modes (Chen 1984) can be easily obtained from (23):

$$\varepsilon_l(\omega, k_\perp) = 0 = 1+\frac{\omega_P^2}{k_\perp^2 V_T^2}\left[1-\sum_{m=-\infty}^{\infty}\frac{\omega}{\omega - m\Omega}\tilde{A}_m(z)\right]$$

$$= 1-\frac{\omega_*^2}{z}A_1(\omega, k_\perp). \tag{25}$$

Here and later in the text, the following notation is used:

$$A_N(\omega, k_\perp) = \sum_{m=1}^{\infty}\frac{\tilde{A}_m(z)}{(\omega^2/m^2\Omega^2 - 1)^N}, \qquad N = 1, 2, 3, \ldots \quad . \tag{26}$$

The physical characteristics of the lower-layer plasmas in the flare atmosphere, which are determined by (12)–(14), provide satisfaction of the conditions (8) and (17)–(20). In the framework of the plasma model used here, these conditions can be supplemented by two additional ones.

(i) The relatively low temperature and relatively high density of the plasma in the region under consideration allow the perpendicular wavelength $\lambda_\perp = 2\pi/k_\perp$ to become of the order of the electron Debye radius d_e. When $\lambda_\perp \leqslant d_e$, the classical plasma approximation becomes incorrect, as well as the DR (23), which has been obtained in the framework of this approximation. (For λ_z such a danger does not exist, if the condition (5)

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is satisfied.) That is why in the present paper the 'long-wave' approximation has been used:

$$z \ll 1. \tag{27}$$

(ii) Unlike coronal plasmas, where the inequality

$$\omega_P \geqslant \Omega \tag{28}$$

can be considered as typical, in the region investigated here the strong magnetic fields with the amplitudes from the interval (14) determine, together with n_e and T_e from (12) and (13), the interval for the ratio of the frequencies (ω_p and Ω):

$$0.31 \leqslant \omega_* \leqslant 0.75. \tag{29}$$

Such a plasma can be called a moderately rarefied one by analogy with a rarefied plasma (Krall and Trivelpiece 1973) with $\omega_*^2 \ll 1$.

The satisfaction of the conditions (5), (6), (8) and (27) and the equations (17)–(20) and (29) allows one to simplify the form of the transcendental (in general) DR (23), as well as the process of its solution.

In the framework of the plasma model used here, in the DR

$$\varepsilon_l(\omega', \mathbf{k}) = 0 \tag{30}$$

the function ε_l from (23) can be expanded in power series in the small parameters k_* and ν_*, where

$$\nu_* \equiv \tilde{\nu}_{ei}/\Omega. \tag{31}$$

Moreover, the asymptotic representation of the functions $I_m(z)$ at $z \ll 1$ can be used to simplify the functions $\tilde{A}_m(z)$. Then ε_l can be represented in the form

$$\varepsilon_l(\omega', \mathbf{k}) = \varepsilon_{WL}(\omega, \mathbf{k}, E_0) + \varepsilon_L(\omega', \mathbf{k}), \tag{32}$$

where

$$\varepsilon_{WL} = E_{WL}(\omega, \mathbf{k}) + \sum_{m=1}^{\infty} P_m(\omega, \mathbf{k}), \tag{33}$$

$$P_m(\omega, \mathbf{k}) = \frac{2m-1}{(2m)!} i k_z \frac{eE_0}{m_e} \frac{\partial^{2m} E_{WL}(\omega, k_*, \nu_*)}{\partial \omega^{2m}}, \tag{34}$$

$$E_{WL} = \operatorname{Re} E_{WL} + i \operatorname{Im} E_{WL}, \tag{35}$$

$$\operatorname{Re} E_{WL} = 1 + \frac{\omega_*}{z}\left(1 - \frac{1+A_1}{M}\right) - \omega_P^2 \frac{k_*^2(1-k_*^2)}{M\omega^2}\left(1 - \frac{\tilde{\nu}_{ei}^2}{\omega^2} H_1\right)(H_1 + H_2 + 24A_1A_2), \tag{36}$$

$$\operatorname{Im} E_{WL} = -\frac{\omega_*^2}{z}\frac{L}{M}\nu_* - \tilde{\nu}_{ei}\omega_P^2 \frac{k_*^2(1-k_*^2)}{2M\omega^3}(1+A_1)(H_1 + H_2 + 24A_1A_2), \tag{37}$$

$$M(\omega, k_\perp) = 1 - \frac{\tilde{\nu}_{ei}^2}{\omega^2}(1 - A_1^2 + 4A_1 + 4A_2) + \frac{\tilde{\nu}_{ei}^4}{\omega^4} H_1^2, \tag{38}$$

$$L(\omega, k_\perp) = A_1^2 - A_1 - 2A_2 + \frac{\tilde{\nu}_{ei}^2}{\omega^2} H_1^2, \tag{39}$$

$$H_1 = 1 + 3A_1 + 2A_2, \tag{40}$$

$$H_2 = 3A_1 + 7A_2 + 4A_3 - 24A_1A_3. \tag{41}$$

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In (34) the function $E_{WL} = E_{WL}(\omega, \ldots)$ (i.e. it depends on ω but not on ω') only in the case when u_z has the form (9) and $E_0 = 0$. But if $u_z \not\equiv u_z(\mathbf{E}_0)$, i.e. when the shift velocity has a non-electric nature, then $E_{WL} = E_{WL}(\omega', \ldots)$. The representation of the function ε_{WL} in the form of a sum of two terms is the distinctive feature of the method of Brinca and Dysthe (1983) used in the present paper. In accordance with this method, the function E_{WL} has to be, by definition, the term of zeroth order in the asymptotic expansion of the function ε_{WL} in a power series in the small parameter

$$E_* = \frac{k_z e|\mathbf{E}_0|}{m_e \Omega^2} = k_* z^{1/2} \nu_* E_R \ll 1. \tag{42}$$

In the plasma model used here the condition (42) is fulfilled within a wide margin. Therefore only the first term in the sum (33) need be kept. The asymptotic expansion of the function

$$Z(x) = -i\pi^{1/2} x W(x), \tag{43}$$

for $|x| \gg 1$ and $|\mathrm{Re}\, x| \gg |\mathrm{Im}\, x|$ has the form (Chen 1984)

$$Z(x) = 1 + \frac{1}{x^2} + \ldots - i(\tfrac{1}{2}\pi)^{1/2} x \exp(-\tfrac{1}{2}x^2). \tag{44}$$

Landau damping appears in the DR immediately, since the value $k_z \not\equiv 0$ appears in it. However, it is clear from general considerations that, for $k_z \to 0$ and $|x| \to \infty$ in (44), it is possible, starting from a certain value $k_* = (k_*)_0$, to neglect the imaginary term in the expansion for $Z(x)$ because of its exponential factor. In the situation when $k_* \ll 1$ but $k_* > (k_*)_0$, the function $\varepsilon_l(\omega', \mathbf{k})$ can be represented in the form (32), where the function $\varepsilon_L(\omega', k_*, \nu_*)$ is completely determined by consideration of the imaginary part in (44). Obviously, in the present case this is nothing more then standard mathematical practice. In the case when $\varepsilon_l(\omega', \mathbf{k})$ can be represented in the form (32), and $|\varepsilon_L| \ll |\varepsilon_{WL}|$ in the field of definition of both of these functions, the expression for the value of $(k_*)_{\max}$ can be obtained easily enough. This value, when the condition (5) is satisfied, corresponds to the angle of maximum deviation of the wave vector $\mathbf{k}$ from the normal to the direction of $\mathbf{B}_0$. At $k_* < (k_*)_{\max}$ the corrections to the DR (25) due to consideration of the terms with $\nu_* \neq 0$ and $k_* \neq 0$ will be less than the discrepancy that arises on substitution of the approximate dispersion law (2), (3) into the exact DR (23) (Kryshtal 1997). If one takes into account from the very outset the contribution of the function ε_L to $\varepsilon_l(\omega', \mathbf{k})$ then the restriction on the value of the discrepancy can be replaced by the weaker condition

$$\left| \frac{\omega' - m\Omega}{k_z V_T} \right| \gg 1, \tag{45}$$

which allows one to determine exactly the value of $(k_*)_{\max}$ for the harmonics of number m.

Considering the conditions

$$\tilde{\nu}_{ei} \ll |\omega' - m\Omega| \qquad (\text{at } m = 1, 2, \ldots, 6), \tag{46}$$

$$k_z u_e \ll \omega, \tag{47}$$

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which are satisfied in the plasma model used here, the satisfaction of the condition (45) does not need any additional restrictions to be imposed on the plasma parameters in the region under investigation.

The function $\varepsilon_L(\omega', \mathbf{k})$ in (32) has the form

$$\varepsilon_L(\omega', \mathbf{k}) = \varepsilon_L(\omega, \mathbf{k}) + \frac{i}{2} k_z \frac{eE_0}{m_e} \frac{\partial^2 \varepsilon_L(\omega, \mathbf{k})}{\partial \omega^2}, \tag{48}$$

$$\varepsilon_L(\omega, \mathbf{k}) = \operatorname{Re} \varepsilon_L + i \operatorname{Im} \varepsilon_L, \tag{49}$$

where

$$\operatorname{Re} \varepsilon_L(\omega, \mathbf{k}) = \frac{F_0}{F_4}\left[-\frac{\tilde{\nu}_{ei}}{\omega}(F_1 + F_2) + F_3 \right], \tag{50}$$

$$\operatorname{Im} \varepsilon_L(\omega, \mathbf{k}) = \frac{F_0}{F_4}\left[\frac{\tilde{\nu}_{ei}}{\omega} F_3 + F_1 + F_2 \right], \tag{51}$$

$$F_0 = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega}{k_z V_T} \exp(-z)\,(1 - k_*^2) \frac{\omega_*^2}{z}, \tag{52}$$

$$F_1 = \operatorname{Re} S\,(1 + \operatorname{Im} \sigma) - \operatorname{Im} S \operatorname{Re} \sigma, \tag{53}$$

$$F_2 = \left(\frac{\pi}{2}\right)^{1/2} \frac{\tilde{\nu}_{ei}}{k_z V_T} [(\operatorname{Re} S)^2 + (\operatorname{Im} S)^2], \tag{54}$$

$$F_3 = -\operatorname{Re} S \operatorname{Re} \sigma - \operatorname{Im} S\,(1 + \operatorname{Im} \sigma), \tag{55}$$

$$F_4 = \left[1 + \operatorname{Im} \sigma + \left(\frac{\pi}{2}\right)^{1/2} \frac{\tilde{\nu}_{ei}}{k_z V_T} \operatorname{Re} S \right]^2 + \left[\operatorname{Re} \sigma - \left(\frac{\pi}{2}\right)^{1/2} \frac{\tilde{\nu}_{ei}}{k_z V_T} \operatorname{Im} S \right\}^2, \tag{56}$$

and the functions $S(\omega, \mathbf{k})$ and $\sigma(\omega, \mathbf{k})$ are defined by

$$S(\omega, \mathbf{k}) = \sum_{m=-\infty}^{\infty} I_m(z) \exp\left[-\frac{(\omega + i\tilde{\nu}_{ei} - m\Omega)^2}{2k_z^2 V_T^2} \right], \tag{57}$$

$$\sigma(\omega, \mathbf{k}) = \sum_{m=-\infty}^{+\infty} \frac{\tilde{\nu}_{ei} \exp(-z)\, I_m(z)}{(\omega + i\tilde{\nu}_{ei} - m\Omega)} \left[1 + \frac{k_z^2 V_T^2}{(\omega + i\tilde{\nu}_{ei} - m\Omega)^2} \right]. \tag{58}$$

The conditions (5), (17), (18) and (20) allow one to obtain in a relatively simple way the real and imaginary parts of the functions $S(\omega, \mathbf{k})$ and $\sigma(\omega, \mathbf{k})$, i.e. to represent them in the standard form

$$S = \operatorname{Re} S + i \operatorname{Im} S, \tag{59}$$

$$\sigma = \operatorname{Re} \sigma + i \operatorname{Im} \sigma. \tag{60}$$

For the solution of the problem formulated above, the well-known resonance method has been used (Alexandrov *et al.* 1989; Krall and Trivelpiece, 1973), with a stability analysis being carried out for each harmonic separately. In this method the investigated mode (harmonic) makes the main contribution to the expression for the almost-singular functions A_N from (26), whereas the

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contributions of the others reduce to small corrections. The instability growth rate is calculated from the formula

$$\gamma_{\omega_m} = -\frac{\operatorname{Im}\varepsilon_1}{\partial/\partial\omega \operatorname{Re}\varepsilon_1}\bigg|_{\omega'=\omega_m}, \tag{61}$$

i.e. at $\omega' = \omega_m(\mathbf{k})$, where $\omega_m(\mathbf{k})$ is the dispersion function of the harmonic of number m. This permits one to continually control the order of smallness of different terms during the calculation. Owing to the conditions (5), (6), (17), (18) and (42) the calculational accuracy, i.e. the order of smallness of the minimum terms in the expressions, can be set a priori.

Considering (12)–(14), all the calculations in the present paper have been carried out taking into account terms of order of smallness $O(z\tau)$, where

$$\tau = \frac{z\omega_*^2}{4e^z} \tag{62}$$

and τ is the small parameter by definition.

3. Instability of second harmonics

In Kryshtal (1992*a*, *b*) suitable formulae were obtained for calculating the growth rates of harmonics of arbitrary number. A pure Maxwellian equilibrium electron velocity distribution function was used there. However, it can easily be shown that for the plasma model studied here the results obtained can be used for a function f_{0e} in the form (10), because $\omega \approx \omega'$ with accuracy up to terms $o(z\tau)$ everywhere in the calculations. Only in the final expression (61) does the difference between ω and ω' have to be taken into account. A similar situation occurs in the investigation of Langmuir-wave instability (Kryshtal and Kucherenko 1995).

Current observational data (Melrose 1991; Benka 1994) do not provide reliable evidence for the possibility of radiation of electron cyclotron harmonics higher than the second in the region under investigation. At the same time, a stability analysis of the first harmonics of Bernstein modes for the plasma in the lower levels in the flare atmosphere has shown that the mode $m = 1$ (close to the upper-hybrid frequency) is not excited (Kryshtal 1997). In such a situation the problem of the possibility of Bernstein-wave generation is reduced in a narrow sense to the investigation of second-harmonic instability.

In the present plasma model the DR for second harmonics $m = 2$ has the form

$$\omega'^2 = 4\Omega^2(1+\tau) \equiv \omega_2^2. \tag{63}$$

In this case the value $(k_*)_{\max}$ is defined by

$$(k_*)_{\max} = \frac{[z\exp(-3z)]^{1/2}}{2\sqrt{3}n}\omega_*^2, \tag{64}$$

where n is the free parameter in the equation

$$|\operatorname{Re}\varepsilon_l(\omega_2, k) - \operatorname{Re}E_{WL}(\omega_2, k_\perp, k_z = 0)| \leqslant \frac{1}{n}|\varepsilon_l(\omega_2, k_\perp)|. \tag{65}$$

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At $n \geqslant 3$ the difference between the two values of $(k_*)_{\max}$ obtained from (65) and (45) becomes of the next order of smallness with respect to the terms $O(z\tau)$.

For a laboratory plasma with parameter values in the intervals (12)–(14) under the conditions (8) and (20), z, k_* and E_* have to be considered as variables and ω_*, E_R and ν_* as constants. However, for the solar plasma an equation of state the type that is necessary for our purposes is not available, and so ω_*^2 and E_R were considered as the variables in the calculation. It was supposed that, according to (29),

$$0.1 \leqslant \omega_*^2 \leqslant 0.57, \tag{66}$$

and that E_R varied in the interval

$$0 \leqslant E_R \leqslant 0.20. \tag{67}$$

The upper limit in (67) has been chosen intentionally too high to take into account the possibility of existence of very strong large-scale electric fields in the subphotosphere area (Gopasyuk and Gopasyuk 1995). For $E_R \leqslant 0.20$, the ratio of the runaway electron number to the total electron number in the plasma does not exceed about 5%.

For second harmonics of Bernstein modes, determined by (63), the expression for the growth rate has the form

$$\frac{\gamma_{\omega_2}}{\Omega_e} = \frac{z\left[\left(\frac{2E_R}{zp\tau}G_1 - \sigma_k G_2\right)\nu_* - G_3\right]}{G_4 - G_6 - 16\dfrac{\sigma_k E_R \nu_*^2}{p\omega_*^2\tau^4}G_5}, \tag{68}$$

where

$$p = (z^{1/2}k_*)^{-1}, \tag{69}$$

$$G_1 = (1+\tau)^{-1/2}\left[1 + \tfrac{7}{4}\tau + 48\frac{2+7\tau}{zp^2\tau\omega_*^2}\right], \tag{70}$$

$$G_2 = 1 - \frac{z\tau}{2\omega_*^2} - \frac{2(1+\frac{9}{4}\tau)}{zp^2\tau\omega_*^2}\left[1 + \frac{\omega_*^2}{z} + \tfrac{1}{3}\omega_*^2\exp(-z)\right], \tag{71}$$

$$G_3 = 2\Phi\left[1 + \frac{\sigma_k E_R \nu_*^2 \exp(z)}{4z^2k_*^4}(p\tau - 2\sqrt{2}\pi)\right], \tag{72}$$

$$G_4 = 1 + \tau[1 + \tfrac{1}{3}\omega_*^2\exp(-z)] + \frac{3}{p^2\tau^2}(1 + \tfrac{17}{6}\tau), \tag{73}$$

$$G_5 = (1+\tau)^{3/2}\left[1 + \tfrac{15}{4}\tau\left(1 - \frac{2z}{5\omega_*^2}\right) + 16\frac{\tau\omega_*^2}{z^2}\right], \tag{74}$$

$$G_6 = \tfrac{1}{4}z\nu_* p^4\Phi\Big[\sigma_k\frac{\omega_*^2\exp(-z)}{z}[1 + (2\pi)^{1/2}\sigma_k p\nu_*] + 4p^3(1+\tau)^{1/2}E_R Q[4(2\pi)^{1/2}p\nu_*\sigma_k - 1]\Big], \tag{75}$$

$$Q = \operatorname{Re} S = \tfrac{1}{8}z^2\exp(-\tfrac{1}{2}p^2\tau^2), \tag{76}$$

$$\Phi = [\tfrac{1}{2}\pi(1+\tau)]^{1/2}(1-k_*^2)\,Q\frac{\omega_*^2 p}{z}\exp(-z). \tag{77}$$

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The specific form (9), (10) of the function f_{0e} and all the approximations made above in the plasma model used here result in the absence of streaming terms in the MDR, where only direct-effect terms are present (Kryshtal and Kucherenko 1995, 1996). Obviously, this is also the case for the growth-rate expression.

Even a rough (i.e. qualitative) analysis of the obtained expressions (68)–(77) shows that Landau damping is small when $z \ll 1$ and $k_* \ll 1$, and increases when z and k_* increase. But, in contrast to the situation for the growth rate γ_{ω_1} of the first harmonics $m = 1$ (Kryshtal 1997), the local maxima of the function γ_{ω_2} for $\gamma_{\omega_2} > 0$ appear long before the values z, k_* and E_R reach their maxima, determined by (42), (64) and (67). This shows that in the present case a real, non-marginal instability takes place. However, it will be shown below that this non-marginal character of the instability depends strongly on the physical properties of the medium, i.e. on the specific plasma equation of state. In the framework of the model studied here, these characteristics (density n, temperature T, magnetic field B_0 and electric field E_0) determine the numerical values of the 'constants' ν_* and σ_k in (68)–(77).

4. Numerical simulation and discussion

The results of numerical analysis of the expressions (68)–(77) when the plasma parameters vary in the intervals (12)–(14) and z varies in the interval

$$0.01 \leqslant z \leqslant 0.25 \tag{78}$$

can be briefly summarized as follows.

(i) For a given equation of state, i.e. for fixed values of n, T and ω_* and a fixed value of σ_k, there are areas of instability $\gamma_{\omega_2} > 0$ for all values of z in the interval (78).

(ii) The width (i.e. the interval of change for k_*) of each such 'zone of instability' and the maximum value $(\gamma_{\omega_2})_{\max}$ of the growth rate in it for

$$(k_*)_{\min} \leqslant k_* \leqslant (k_*)_{\max} \tag{79}$$

depend on z, and this width increases when E_R increases. The initial value $(k_*)_{\min}$, which must be zero in the ideal situation, is determined from (50)–(58). Everywhere in the calculations it was supposed that $(k_*)_{\min} = 0.0005$, whereas $(k_*)_{\max}$ was determined for each pair (ω_*, z) from (64).

(iii) The function $\gamma_{\omega_2} = \gamma_{\omega_2}(k_*)$ inside each zone (79) has only one clear maximum – which certainly proves the existence of an instability.

(iv) Each zone (79) has its own value of $(E_R)_{\min}$, from which the instability starts. For some equations of state this value is zero (see Table 1 below and comments on it). This means that the 'first' local maximum $(\gamma_{\omega_2})_{\max}$ in the region $\gamma_{\omega_2} > 0$ appears at $|\mathbf{E}_0| = 0$. This, of course, is evidence that the present model is inapplicable for a plasma with such an equation of state. The values of the absolute maxima of the growth rates have been obtained through (68)–(77) with the help of a standard method. The results of the calculation for eight different triples of values (n, T, σ_k), i.e. for eight different equations of state (cases A–H) are presented in Table 1.

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In Table 1 $(\omega_*)_M, (z)_M, (k_*)_M$ and $(E_R)_M$ denote the values of ω_*, z, k_* and E_R respectively at which the function γ/Ω reaches its absolute maximum in the region under consideration. The results of the calculation shown in this table demonstrate that the instabilities in cases A, B, D, E, F and H, in accordance with the terminology of previous papers (Kryshtal and Kucherenko 1995, 1996), are the marginal ones. In cases A and B the values of $(\gamma/\Omega)_M$ are too large, as are the values of $(z)_M$ and $(E_R)_M$. In these cases the corresponding magnetic field amplitude $|\mathbf{B}_0|$ appears to be extremely large in the neighbourhood of the growth-rate maximum. Furthermore, the first positive values of γ_{ω_2} appear in these cases at $E_R = 0$. This of course indicates the presence of some pathology in such solutions. In cases D, E and F the values of $(\gamma/\Omega_e)_M$ are, in contrast, too small, and the values of $(z)_M$ and $(E_R)_M$ lie at the ends of the appropriate intervals. In case H the specific value $(\gamma/\Omega_e)_M = 4.68$ indicates a severely pathological solution, because in the linear approximation the condition $\gamma \ll \omega$ has to be satisfied a priori. Thus it is clear that, in this sense, case G, where

$$\frac{\gamma}{\Omega} \approx \frac{1}{2}\left(\frac{\gamma}{\Omega_e}\right)_M \approx 0.45, \tag{80}$$

is a fairly marginal case too. Thus we come by a process of elimination to case C, which is in all senses an optimal and nonmarginal one. The values of n_e and T_e for case C and all the other cases have been taken from Avrett *et al.* (1981). Making use of these results and the results of the present paper, it is possible to determine approximately the interval of heights in the flare atmosphere where the investigated instability can arise and the mechanism of direct initiation can be realized. From the point of view of this mechanism, the value of $(E_R)_M$ is not so important as the value of $(E_R)_{\min}$ at which the surface of local maxima of γ_{ω_2} enters the half-space $\gamma_{\omega_2} > 0$. However, the results given in Table 1 allow one to base the choice of the fixed values $z = z_0 = (z)_M$ and $\omega_* = (\omega_*)_0 = (\omega_*)_M$ (at which the behaviour of the function $\gamma_{\omega_2} = \gamma_{\omega_2}(k_*, E_R)$ has to be investigated) and to determine the value of $(E_R)_{\min}$ too. The values $z_0 = 0.05$ and $(\omega_*)_0 = 0.66$ have been used in the calculation. The behaviour of the function $\gamma_{\omega_2} = \gamma_{\omega_2}(k_*, E_R)$ at chosen values of z_0 and $(\omega_*)_0$ is shown in Figure 1.

For the present plasma model, for example, we have obtained the value $(E_R)_{\min} = 5 \times 10^{-6}$ at $z_0 = 0.05$ and $\omega_* = 0.31$. In this case $\gamma/\Omega = 4.86 \times 10^{-11}$, and if $\Omega = \Omega_e = 2.39 \times 10^{10}$ s^{-1} (which corresponds to $|\mathbf{B}_0| = 0.322$ T) then time of instability development $\tau \approx 0.84$ s (assuming that $\tau \approx \gamma^{-1}$). The calculation demonstrates that $(E_R)_{\min}$ depends strongly on the value of ω_*, i.e. in practice on the value of the magnetic field amplitude $|\mathbf{B}_0|$ in the area of instability development. Therefore, if the investigated harmonics $m = 2$ can be considered as being generated during the time $\tau \approx \gamma^{-1}$ and transformed into electromagnetic radiation then the time of observation of this radiation can be used for estimation of the value of $(E_R)_{\min}$. In the same manner, the preferred range (from the point of view of instability development) of variation for the amplitude B_0 can be obtained. However, this is true only for the case when the necessary condition for the escape of radiation from the plasma in the area of generation (Somov 1994) is satisfied. It should be noted in particular that the development of second-harmonic instability does not need any additional restrictions on the physical characteristics of the plasma, such as a high value

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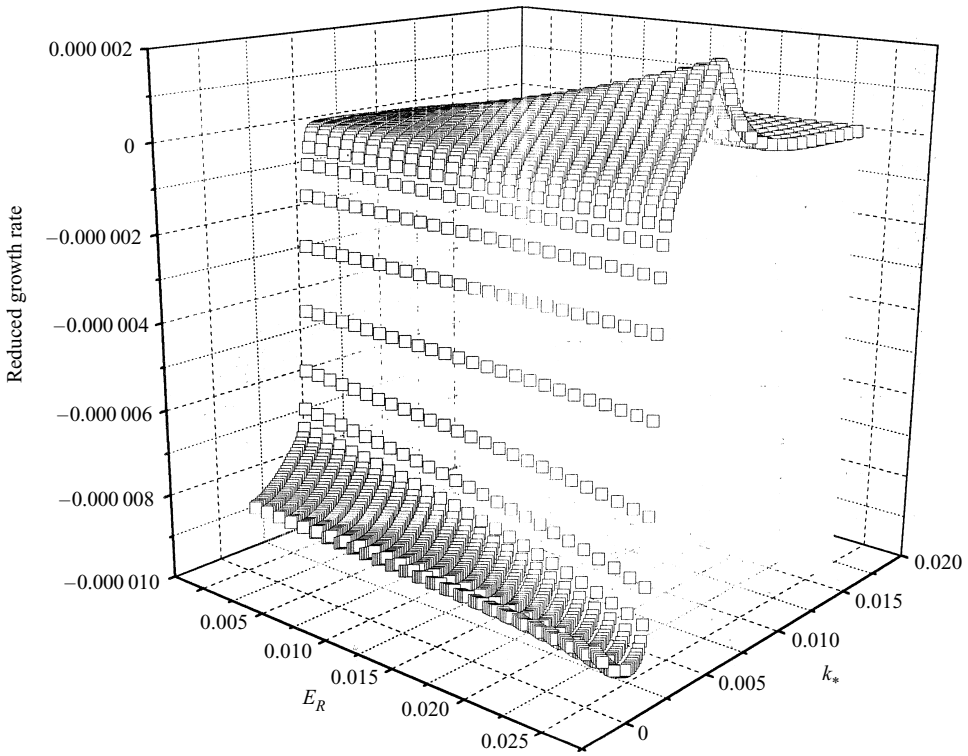


Figure 1. Growth rate of the second-Bernstein-mode instability as a function of the perturbation wave vector's tilt angle and the amplitude of the external electric field for $z_0 = 0.05$ and $(\omega_*)_0 = 0.66$.

of the ratio T_e/T_i in the case of ion-acoustic waves (Kryshtal and Kucherenko 1996).

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References

Alexandrov, A. F., Bogdankevich, L. S. and Rukhadze, A. A. 1988 *Principles of Plasma Electrodynamics*. Vys'shaya shkola, Moscow.

Alikajeva, K. V., Kondrashova, N. N., Redjuk, T. I. and Rudnikova, E. G. 1995*a* *Kinem. Phys. Celest. Bodies* **11**(2), 3.

Alikajeva, K. V., Baranovsky, E. A., Kondrashova, N. N., Redjuk, T. I. and Rudnikova, E. G. 1995*b* *Kinem. Phys. Celest. Bodies* **11**(2), 11.

Baranovsky, E. A., Lozitskaya, N. I., Lozitskij, V. G. 1991 *Kinem. Phys. Celest. Bodies* **7**(3), 52.

Benka, S. G. 1994 *Proceedings of Kofu Symposium* (ed. S. Enome and T. Hirayama), p. 225. NRO Report No. 360.

Bhatnagar, P. L., Gross, E. P. and Krook, M. 1954 *Phys. Rev.* **94**, 511.

Brinca, A. L. 1980 *Geophys. Res. Lett.* **7**, 901.

Brinca, A. L. and Dysthe, K. B. *J. Plasma Phys.* **29**, 35.

https://doi.org/10.1017/S0022377898007004 Published online by Cambridge University Press

Chen, F. F. 1984 *Introduction to Plasma Physics and Controlled Fusion*, 2nd edn, Vol. 1: *Plasma Physics*. Plenum, New York.

Foukal, P. and Hinata, S. 1991 *Solar Phys.* **132**, 307.

Foukal, P., Miller, P. and Gilliam, L. 1983 *Solar Phys.* **83**, 83.

Gopasyuk, S. I. 1987 *Res. Sci. Tech., Ser. Astron. (USSR)* **34**, 6.

Gopasyuk, S. I. and Gopasyuk, O. S. 1995 *Kinem. Phys. Celest. Bodies* **11**(3), 73.

Krall, N. A. and Trivelpiece, A. W. 1973 *Principles of Plasma Physics*. MacGraw-Hill, New York (reprinted 1986 San Francisco Press).

Kryshtal, A. N. 1992*a* *Preprint* ГАО-92-4Р, *Institute of Theoretical Physics*, *Kiev*.

Kryshtal, A. N. 1992*b* *Proceedings of the 26th ESLAB Symposium (Killarney, Ireland, 16–19 June 1992)*, p. 301. ESA SP-346.

Kryshtal, A. N. 1997 *Kinem. Phys. Celest. Bodies* **13**(1), 24.

Kryshtal, A. N. and Kucherenko, V. P. 1995 *J. Plasma Phys.* **53**, 169.

Kryshtal, A. N. and Kucherenko, V. P. 1996 *Solar Phys.* **165**, 139.

Machado, M. E., Avrett, E. H., Vernazza, J. E. and Noyes, R. W. 1980 *Astrophys. J.* **242**, 336.

Melrose, D. B. 1991 *Astrophys. J.* **380**, 256.

Melrose, D. B. and Dulk, G. A. 1982 *Astrophys. J.* **259**, 844.

Misra, K. D., Singh, B. D. and Mishra, S. P. 1979 *J. Geophys. Res.* **84**, 5923.

Pines, D. and Schrieffer, J. R. 1961 *Phys. Rev.* **124**, 1387.

Poletto, G. and Kopp, R. A. 1986 *The Lower Atmosphere of Solar Flares* (ed. D. F. Neideg), p. 453. Sacramento Peak.

Priest, E. R. and Forbes, T. G. 1990 *Solar Phys.* **126**, 319.

Somov, B. V. 1994 *Fundamentals of Cosmic Electrodynamics*. Kluwer, Dordrecht.

Vernazza, J. E., Avrett, E. H. and Loeser, R. 1981 *Astrophys. J. Suppl.* **45**, 635.

Willes, A. J. and Robinson, P. A. 1996 *Astrophys. J.* **467**, 465.

Zheleznyakov, V. V. and Zlotnik, E. Ya. 1974 *Solar Phys.* **43**, 431.

Zheleznyakov, V. V. and Zlotnik, E. Ya. 1975 *Solar Phys.* **44**, 461.

https://doi.org/10.1017/S0022377898007004 Published online by Cambridge University Press