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Certain recurrence relations

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Slater ((5), p. 27) and Bose ((1), p. 202) have obtained certain recurrence relations about Whittaker and hypergeometric functions. In this note an attempt has been made to obtain generalizations of these recurrence relations.

The first two recurrence relations proved in this note are

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}c_{r}\,x^{\frac{1}{2}n-\frac{1}{2}r}W_{k+\frac{1}{2}r,\,m+\frac{1}{2}r}(x)=\frac{\Gamma(m+n-k+\frac{1}{2})}{\Gamma(m-k+\frac{1}{2})}W_{k-\frac{1}{2}n,\,m+\frac{1}{2}n}(x) \tag{1·1}$$

and

$$\sum_{r=0}^{n}\frac{\Gamma(m-k-r+\frac{1}{2})}{\Gamma(m-k-n+\frac{1}{2})}(-1)^{n-r}\,{}^{n}c_{r}\,W_{k+r,\,m}(x)=x^{\frac{1}{2}n}W_{k+\frac{1}{2}n,\,m-\frac{1}{2}n}(x). \tag{1·2}$$

Two other recurrence relations about hypergeometric functions obtained with the help of the above two relations are

$$\sum_{r=0}^{n}\frac{\Gamma(c-r-\beta+1)}{\Gamma(c-r+1)}(-1)^{n-r}\,{}^{n}c_{r}\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ c-r+1\end{matrix};x\right)=\frac{\Gamma(n+\beta)\,\Gamma(c-\beta-n+1)}{\Gamma(\beta)\,\Gamma(c+1)}\,{}_2F_1\left(\begin{matrix}\alpha,\beta+n\\ c+1\end{matrix};x\right) \tag{1·3}$$

and

$$\sum_{r=0}^{n}\frac{\Gamma(\beta-r)}{\Gamma(c-r)}(-1)^{n-r}\,{}^{n}c_{r}\,{}_2F_1\left(\begin{matrix}\alpha,\beta-r\\ c-r\end{matrix};x\right)=\frac{\Gamma(\beta-n)\,\Gamma(c-\beta+n)}{\Gamma(c)\,\Gamma(c-\beta)}\,{}_2F_1\left(\begin{matrix}\alpha,\beta-n\\ c\end{matrix};x\right). \tag{1·4}$$

In section II, the relation (1·1) is proved with the help of the theorem on Laplace transforms which is mentioned as a lemma in section I and relation (1·2) is proved with the help of the relation (1·1).

In section III, another two recurrence relations for the hypergeometric function have been obtained with the help of the relations (1·1) and (1·2).

Throughout this paper we shall use the notation for the Laplace transform

$$L\{f(t);p\}=\int_0^\infty e^{-pt}f(t)\,dt.$$

Section I

Lemma. *If* $\lambda > n-1$, $\Re(\sigma+\lambda+1) > 0$ *and* $\Re(p+a) > 0$ *then*

$$L[t^n(t+a)^{-\lambda-1}f(t);p]=\sum_{r=0}^{n}\frac{(-1)^{n-r}a^{n-r}}{\Gamma(\lambda-r+1)}\,{}^{n}c_{r}\int_0^\infty x^{\lambda-r}e^{-ax}L[f(t);x+p]\,dx, \tag{2·1}$$

where $f(t)=O(t^{\sigma})$ *for small* t *and* $f(t)=O(e^{-at}t^{\mu})$ *for large* t.

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Proof. Since ((2), p. 137)

$$L[t^{\lambda} e^{-bt}; p] = \Gamma(\lambda+1)(p+b)^{-\lambda-1} \quad (\Re(\lambda) > -1, \Re(p+b) > 0),$$

and by virtue of Leibnitz's theorem

$$\frac{d^n}{dt^n}(t^{\lambda} e^{-bt}) = e^{-bt} \sum_{r=0}^{n} \frac{(-1)^{n-r}\Gamma(\lambda+1)}{\Gamma(\lambda-r+1)} {}^nc_r b^{n-r} t^{\lambda-r},$$

therefore, if we start with $f(t) = t^{\lambda} e^{-bt}$ in the well-known property of Laplace transform ((2), p. 129), viz.

$$p^n L[f(t); p] = L[f^n(t); p],$$

where $f(0) = f'(0) = f''(0) = \ldots, f^{n-1}(0) = 0$ and $f^n(t)$ stands for $(d^n/dt^n)[f(t)]$ then

$$\Gamma(\lambda+1)p^n(p+b)^{-\lambda-1} = L\left[e^{-bt} \sum_{r=0}^{n} \frac{(-1)^{n-r}\Gamma(\lambda+1)}{\Gamma(\lambda-r+1)} {}^nc_r b^{n-r} t^{\lambda-r}; p\right] \quad (\Re(\lambda) > n-1). \tag{2·2}$$

Now using (2·2) and the result

$$L[e^{-at} f(t); p] = L[f(t); p+a]$$

in the Parseval–Goldstein theorem ((4), p. 105), we get

$$\int_0^{\infty} e^{-ap} p^n (p+b)^{-\lambda-1} f(p)\, dp = \sum_{r=0}^{n} \frac{(-1)^{n-r} b^{n-r}}{\Gamma(\lambda-r+1)} {}^nc_r \int_0^{\infty} x^{\lambda-r} e^{-bx} L[f(t); x+a]\, dx.$$

Finally, replacing a by p and b by a, we get the required result.

Now uniform convergence of (2·1) can easily be proved when $f(t) = O(t^{\sigma})$ and $f(t) = O(e^{-at} t^{\mu})$ for small and large t, under the conditions stated along with the theorem.

SECTION II

In this section, we shall make use of the above lemma to get some recurrence relations.

If we start with $f(t) = t^v$, then ((2), p. 137)

$$L[t^v; p] = \Gamma(v+1) p^{-v-1} \quad (\Re(v) > -1, \Re(p) > 0).$$

Substituting this value in the result (2·1), we get

$$L[t^{n+v}(t+a)^{-\lambda-1}; p] = \Gamma(v+1) \sum_{r=0}^{n} \frac{(-1)^{n-r} a^{n-r}}{\Gamma(\lambda-r+1)} {}^nc_r \int_0^{\infty} e^{-ax} x^{\lambda-r} (p+x)^{-v-1}\, dx$$
$$(\Re(v) > -1, \lambda > n-1, \Re(p+a) > 0).$$

Solving the right-hand side integral with the help of the result ((2), p. 294)

$$L[t^{\lambda-1}(t+a)^{-v}; p] = \frac{p^{-\lambda} a^{-v}}{\Gamma(v)} E(\lambda, v::ap) \quad (\Re(\lambda) > 0, \Re(p) > 0, |\arg a| < \pi),$$

we get

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}(ap)^n}{\Gamma(\lambda-r+1)} {}^nc_r E(\lambda-r+1, v+1::ap) = \frac{1}{\Gamma(\lambda+1)} E(n+v+1, \lambda+1::ap)$$
$$(\Re(p+a) > 0, \Re(v) > -1, \lambda > n-1). \tag{3·1}$$

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Now with the help of the result

$$E[\mu, \lambda :: x] = \Gamma(\mu)\,\Gamma(\lambda)\,e^{\frac{1}{2}x}\,x^{-\frac{1}{2}(1-\mu-\lambda)}W_{\frac{1}{2}(1-\mu-\lambda),\,\frac{1}{2}(\mu-\lambda)}(x)$$

(3·1) can be written by making suitable adjustment in parameters

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}c_r\,x^{\frac{1}{2}n-\frac{1}{2}r}W_{k+\frac{1}{2}r,\,m+\frac{1}{2}r}(x) = \frac{\Gamma(m+n-k+\frac{1}{2})}{\Gamma(m-k+\frac{1}{2})}W_{k-\frac{1}{2}n,\,m+\frac{1}{2}n}(x). \tag{3·2}$$

On taking $n = 1$, (3·2) gives rise to result ((5), p. 27)

$$(m-k+\tfrac{1}{2})\,W_{k-\frac{1}{2},\,m+\frac{1}{2}}(x) + x^{\frac{1}{2}}W_{k,\,m}(x) = W_{k+\frac{1}{2},\,m+\frac{1}{2}}(x). \tag{3·3}$$

Now substituting in the result (3·2), $x = \frac{1}{2}y^2$ and multiplying both the sides by $e^{\frac{1}{4}y^2}\,y^{v-2m-n-\frac{1}{2}}$, we get

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}c_r\,\frac{e^{\frac{1}{4}y^2}\,y^{v-2m-r-\frac{1}{2}}}{2^{\frac{1}{2}n-\frac{1}{2}r}}W_{k+\frac{1}{2}r,\,m+\frac{1}{2}r}(\tfrac{1}{2}y^2)$$

$$= \frac{\Gamma(m+n-k+\frac{1}{2})}{\Gamma(m-k+\frac{1}{2})}e^{\frac{1}{4}y^2}\,y^{v-2m-n-\frac{1}{2}}W_{k-\frac{1}{2}n,\,m+\frac{1}{2}n}(\tfrac{1}{2}y^2). \tag{3·4}$$

Taking images in Hankel transform ((3), p. 84) of both the sides of (3·4) and substituting

$$\frac{k}{2}+\frac{3m}{2}-\frac{v}{2}-\frac{1}{4}, \quad \frac{k}{2}-\frac{m}{2}+\frac{v}{2}+\frac{1}{4}$$

and $\frac{1}{2}y^2$ as k, m and x respectively, we get

$$\sum_{r=0}^{n}\frac{\Gamma(m-k-r+\frac{1}{2})}{\Gamma(m-k-n+\frac{1}{2})}(-1)^{n-r}\,{}^{n}c_r\,W_{k+r,\,m}(x) = x^{\frac{1}{2}n}W_{k+\frac{1}{2}n,\,m-\frac{1}{2}n}(x). \tag{3·5}$$

On taking $n = 1$, (3·5) gives rise to result ((5), p. 27).

Section III

In this section, with the help of the above two recurrence relations for Whittaker functions (3·2) and (3·5), we will get two more recurrence relations for hypergeometric functions.

Since we have ((2), p. 216)

$$L\left[\frac{\Gamma(m+n-k+\frac{1}{2})}{\Gamma(m-k+\frac{1}{2})}W_{k-\frac{1}{2}n,\,m+\frac{1}{2}n}(at);\,p\right]$$

$$= \frac{\Gamma(m+n-k+\frac{1}{2})\,\Gamma(m+\frac{1}{2}n+\frac{3}{2})\,\Gamma(\frac{3}{2}-m-\frac{1}{2}n)}{\Gamma(m-k+\frac{1}{2})\,\Gamma(\frac{1}{2}n-k+2)\,(p+\frac{1}{2}a)^{m+\frac{1}{2}n+\frac{3}{2}}}a^{m+\frac{1}{2}n+\frac{1}{2}}$$

$$\times\,{}_2F_1\left(\begin{matrix}m+\frac{1}{2}n+\frac{3}{2},\ m+n-k+\frac{1}{2}\\ 2-k+\frac{1}{2}n\end{matrix};\frac{p-\frac{1}{2}a}{p+\frac{1}{2}a}\right)$$

$$(\Re\,[\tfrac{3}{2}\pm(m+\tfrac{1}{2}n)]>0, \quad \Re\,(p+\tfrac{1}{2}a)>0). \tag{3·6}$$

so that

$$L\left[\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}c_r(at)^{\frac{1}{2}n-\frac{1}{2}r}W_{k+\frac{1}{2}r,\,m+\frac{1}{2}r}(at);\,p\right]$$

$$= \sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}c_r\,\frac{\Gamma(m+\frac{1}{2}n+\frac{3}{2})\,\Gamma(\frac{1}{2}n-r-m+\frac{3}{2})}{\Gamma(\frac{1}{2}n-r-k+2)\,(p+\frac{1}{2}a)^{m+\frac{1}{2}n+\frac{3}{2}}}a^{\frac{1}{2}n+m+\frac{1}{2}}$$

$$\times\,{}_2F_1\left(\begin{matrix}m+\frac{1}{2}n+\frac{3}{2},\ m-k+\frac{1}{2}\\ \frac{1}{2}n-r-k+2\end{matrix};\frac{p-\frac{1}{2}a}{p+\frac{1}{2}a}\right)$$

$$(\Re(m+\tfrac{1}{2}n)<\tfrac{3}{2}, \quad \Re(m+\tfrac{1}{2}n)>-\tfrac{3}{2}, \quad \Re(p+\tfrac{1}{2}a)>0), \tag{3·7}$$

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Taking images of both the sides of (3·2) by the help of (3·6) and (3·7) in the Laplace transform and substituting $m+\frac{1}{2}n+\frac{3}{2}$, $m-k+\frac{1}{2}$, $1-k+\frac{1}{2}n$ and $(p-\frac{1}{2}a)/(p+\frac{1}{2}a)$ as α, β, c and x respectively, we get

$$\sum_{r=0}^{n} \frac{\Gamma(c-r-\beta+1)}{\Gamma(c-r+1)}(-1)^{n-r}\,{}^{n}c_{r}\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ c-r+1\end{matrix};x\right) = \frac{\Gamma(n+\beta)\,\Gamma(c-\beta-n+1)}{\Gamma(\beta)\,\Gamma(c+1)}\,{}_2F_1\left(\begin{matrix}\alpha,\beta+n\\ c+1\end{matrix};x\right). \quad (3\cdot8)$$

Particularly, on taking $n = 1$, it is reduced to the known result ((1), p. 202)

$$\beta\,{}_2F_1\left(\begin{matrix}\alpha,\beta+1\\ c+1\end{matrix};x\right)+(c-\beta)\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ c+1\end{matrix};x\right) = c\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ c\end{matrix};x\right). \quad (3\cdot9)$$

Similarly, taking images of both sides of (3·5) in the Laplace transform and substituting $m+\frac{3}{2}$, $m-k+\frac{1}{2}$, $2-k$ and $(p-\frac{1}{2}a)/(p+\frac{1}{2}a)$ as α, β, c and x respectively, we get

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}c_{r}\,\frac{\Gamma(\beta-r)}{\Gamma(c-r)}\,{}_2F_1\left(\begin{matrix}\alpha,\beta-r\\ c-r\end{matrix};x\right) = \frac{\Gamma(\beta-n)\,\Gamma(c-\beta+n)}{\Gamma(c)\,\Gamma(c-\beta)}\,{}_2F_1\left(\begin{matrix}\alpha,\beta-n\\ c\end{matrix};x\right). \quad (3\cdot10)$$

On taking $n = 1$, (3·10) reduces to the result (3·9).

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