J. Appl. Prob. **53,** 1001–1018 (2016)
*doi:*10.1017/jpr.2016.60
© *Applied Probability Trust* 2016

THE BOOLEAN MODEL IN THE SHANNON REGIME: THREE THRESHOLDS AND RELATED ASYMPTOTICS

VENKAT ANANTHARAM,* *University of California, Berkeley*
FRANÇOIS BACCELLI,** *The University of Texas at Austin*

Abstract

Consider a family of Boolean models, indexed by integers $n \geq 1$. The nth model features a Poisson point process in $\mathbb{R}^n$ of intensity $\mathrm{e}^{n\rho_n}$, and balls of independent and identically distributed radii distributed like $\bar{X}_n\sqrt{n}$. Assume that $\rho_n \to \rho$ as $n \to \infty$, and that $\bar{X}_n$ satisfies a large deviations principle. We show that there then exist the three deterministic thresholds τ_{d}, the degree threshold, τ_{p}, the percolation probability threshold, and τ_{v}, the volume fraction threshold, such that, asymptotically as n tends to ∞, we have the following features. (i) For $\rho < \tau_{\mathrm{d}}$, almost every point is isolated, namely its ball intersects no other ball; (ii) for $\tau_{\mathrm{d}} < \rho < \tau_{\mathrm{p}}$, the mean number of balls intersected by a typical ball converges to ∞ and nevertheless there is no percolation; (iii) for $\tau_{\mathrm{p}} < \rho < \tau_{\mathrm{v}}$, the volume fraction is 0 and nevertheless percolation occurs; (iv) for $\tau_{\mathrm{d}} < \rho < \tau_{\mathrm{v}}$, the mean number of balls intersected by a typical ball converges to ∞ and nevertheless the volume fraction is 0; (v) for $\rho > \tau_{\mathrm{v}}$, the whole space is covered. The analysis of this asymptotic regime is motivated by problems in information theory, but it could be of independent interest in stochastic geometry. The relations between these three thresholds and the Shannon–Poltyrev threshold are discussed.

Keywords: Point process; Boolean model; high-dimensional stochastic geometry; information theory; large deviations theory

2010 Mathematics Subject Classification: Primary 60G55; 94A15
Secondary 60D05; 60F10

.

1. Introduction

The Boolean model in high dimensions has been considered in a few papers, both within the framework of stochastic geometry [5], [8] and within the framework of information theory [1]. In the present paper we discuss three thresholds and some asymptotics related to these thresholds in a setting analogous to that in [1]. In this setting, the intensity of the underlying Poisson point process scales exponentially with dimension and the radii of the balls in the Boolean model scale with the dimension n of the ambient space like $\bar{X}_n\sqrt{n}$, where $(\bar{X}_n, n \geq 1)$ is a sequence of random variables. satisfying a large deviations principle (LDP). To understand the reason for selecting the $\sqrt{n}$ scaling on the radius, note that the volume of a ball of radius $R\sqrt{n}$ in $\mathbb{R}^n$ grows like $\mathrm{e}^{n\log(2\pi\mathrm{e}R^2)/2+o(n)}$.

The first threshold is the volume fraction threshold, which is the threshold at which the probability of coverage of the origin by the Boolean model switches from asymptotically

Received 17 August 2015; revision received 12 January 2016.
* Postal address: Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA. Email address: ananth@eecs.berkeley.edu
** Postal address: Department of Mathematics, The University of Texas at Austin, Austin, TX 78712-1202, USA. Email address: baccelli@math.utexas.edu

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

vanishing to asymptotically approaching 1 as the dimension n tends to ∞. The second threshold is the percolation probability threshold; it was first studied in detail in [8] in the particular case where $\bar{X}_n$ is a constant. The case with random $\bar{X}_n$ was also discussed in [5]. This is the threshold at which the probability of percolation in the Boolean model switches from asymptotically approaching 0 to being asymptotically nonzero as $n \to \infty$. The last is the degree threshold. This is the threshold at which the mean number of grains of the Palm version of the Boolean model that intersect the grain of the origin switches from asymptotically being finite to asymptotically approaching ∞ as $n \to \infty$. It is not hard to see that these three thresholds are decreasing in the order in which they were presented.

The main new contributions of the present paper are (a) representations of these three thresholds in terms of optimization problems based on the rate function of the LDP; (b) explicit asymptotics for various rates of convergence in the neighborhood of these thresholds; and (c) relations between these thresholds and the Shannon–Poltyrev threshold of information theory in the Gaussian case.

The paper is structured as follows. We start with a description of the setup. We then give a summary of results and heuristic explanations in Section 3. We provide proofs in Section 4. For ease of exposition, we first discuss the volume fraction threshold, then the degree threshold, and finally the percolation probability threshold in each of these sections. Some concluding remarks, making connections between the issues addressed here and the information theoretic concerns of [1], are given in Section 5, where in particular the instantiation of our general results in the case of Gaussian grains is worked out in detail.

2. Setup

In each dimension $n \geq 1$ we have a homogeneous Poisson process of intensity $\mathrm{e}^{n\rho_n}$ (i.e. normalized logarithmic intensity ρ_n). Assume that $\rho_n \to \rho$ as $n \to \infty$. Note that ρ is a real number (which can be negative). It is called the *asymptotic normalized logarithmic intensity* of this sequence of Poisson processes. We will assume that the processes are defined on a single probability space $(\Omega, \mathcal{F}, \mathbb{P})$, although the coupling between the different dimensions is not relevant for the issues we consider. We will denote by $\mathbb{P}_n^0$ the Palm probability of the Poisson point process in dimension n. See [2, Chapter 13] for the definition of and basic facts about Palm probabilities.

To each point $T_n^{(k)}$, $k \geq 1$, of the Poisson process in dimension n (enumerated in some way), we associate a mark $X_n^{(k)} \in \mathbb{R}^+$. The $X_n^{(k)}$, $k \geq 1$, are assumed to be independent and identically distributed (i.i.d.), and independent of the points. For each dimension n, let $X_n^{(k)} \stackrel{\mathrm{D}}{=} \bar{X}_n$ for all k. Let R_n^* denote $\mathbb{E}[\bar{X}_n]$, and let $R^* := \lim_{n\to\infty} R_n^*$, where the last limit is assumed to exist. We assume that $0 < R^* < \infty$. We assume that the sequence $(\bar{X}_n, n \geq 1)$ satisfies an LDP, with good and convex rate function $I(\cdot)$ [3], e.g. by assuming the Gärtner-Ellis conditions. We also assume that the following condition holds:

$$\limsup_{n\to\infty} \mathbb{E}[(\bar{X}_n)^{\gamma n}]^{1/n} < \infty \quad \text{for some } \gamma > 1. \tag{2.1}$$

By the deterministic setting we mean that $\bar{X}_n$ is deterministic and equal to R_n^* for each $n \geq 1$, with $R_n^* \to R^*$ as $n \to \infty$. The deterministic setting is a special case of the general setting, but we will separately highlight the results in this case, since it is of particular interest.

To the marked point process in dimension n, we associate a Boolean model where the grain of point $T_n^{(k)}$ is a closed ball of radius $X_n^{(k)}\sqrt{n}$. Let

$$\mathcal{C}_n := \bigcup_k B(T_n^{(k)}, X_n^{(k)}\sqrt{n}) \tag{2.2}$$

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

denote this Boolean model, with $B(t, r)$ denoting the closed ball of center $t \in \mathbb{R}^n$ and radius $r \geq 0$. Here, and in the rest of the paper, ':=' denotes equality by definition.

From Slivnyak's theorem [2, Chapter 13], the Palm version of the process in each dimension n (i.e. its law under $\mathbb{P}_n^0$) is equivalent in law to the superposition of a stationary version of the process and a process with a single point at the origin carrying a ball with radius having law $\bar{X}_n\sqrt{n}$, and independent of the stationary version (which will be called the reduced process of the Palm version below).

Our motivations for the analysis of this setting came from related problems in information theory that we studied recently [1]. More specifically, in the study of error probabilities for coding over an additive white Gaussian noise channel [4, Section 7.4], it is natural to consider a sequence of Poisson processes, one in each dimension $n \geq 1$, with well-defined asymptotic normalized logarithmic intensity, as was done in [1], motivated by the ideas in [9]. Such a process in dimension n corresponds to a codebook at block length n in the absence of power constraints, in the language of information theory [1]. The scenario considered in [1] goes beyond additive white Gaussian noise to consider a setting where the additive noise comes from sections of a stationary and ergodic process. The Boolean models that arise in the more general case involve grains, defined by the typicality sets of the noise process, which are not necessarily spherically symmetric. Even more generally, in [1] the underlying point process in each dimension is allowed to be an arbitrary stationary ergodic process (while still requiring the existence of an asymptotic normalized logarithmic intensity). The error exponent questions studied in [1] are related to the consideration of a Boolean model where the grains associated with the individual points are defined in terms of additive white Gaussian noise. For all $n \geq 1$ and $k \geq 1$, let $W_n^{(i,k)}$, $n \geq i \geq 1$, denote an i.i.d. sequence of Gaussian random variables, all centered and of variance $\sigma^2 > 0$. Let $W_n^{(k)}$ denote the n-dimensional vector with coordinates $W_n^{(i,k)}$, $n \geq i \geq 1$. Then $T_n^{(k)} + W_n^{(k)}$ belongs to the closed ball of center $T_n^{(k)}$ and radius $X_n^{(k)}\sqrt{n}$, with

$$X_n^{(k)} := \left(\frac{1}{n}\sum_{i=1}^{n}(W_n^{(i,k)})^2\right)^{1/2} \stackrel{\mathrm{D}}{=} \bar{X}_n. \tag{2.3}$$

The sequence $\{\bar{X}_n\}$ satisfies an LDP and all the additional technical assumptions listed above. For each $\sigma^2 > 0$, we call such a family of Boolean models (parameterized by $n \geq 1$) the case with *Gaussian grains*.

For the error exponent problem, what is of interest is not this Gaussian grain Boolean model, but the related Boolean model where the grain associated to each point is an associated *typicality region*, which is the set

$$\{T_n^{(k)} + v \colon v \in \mathbb{R}^n,\ \|v\|_2 \leq \sigma\sqrt{n} + \alpha_n\}. \tag{2.4}$$

In this definition, $\|v\|_2$ denotes the usual Euclidean length of v and the real numbers $0 < \alpha_n = O(\sqrt{n})$ are chosen such that

$$\frac{\alpha_n}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty, \tag{2.5a}$$

$$\mathbb{P}(\|W_n^{(k)}\|_2 \leq \sigma\sqrt{n} + \alpha_n) \to 1 \quad \text{as } n \to \infty \text{ (for each } 1 \leq k \leq n, \text{ of course)}, \tag{2.5b}$$

$$\frac{1}{n}\log \mathrm{Vol}\{v \in \mathbb{R}^n \colon \|v\|_2 \leq \sigma\sqrt{n} + \alpha_n\} \to \frac{1}{2}\log(2\pi\mathrm{e}\sigma^2) \quad \text{as } n \to \infty. \tag{2.5c}$$

It is straightforward to check that it is possible to choose $(\alpha_n,\ n \geq 1)$ satisfying these requirements. This model will be referred to as the *truncated Gaussian grain* model below. It fits within the class of deterministic Boolean models considered in this paper.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

Having carried out the analysis in [1], it was therefore natural for us to be curious about the asymptotic properties in dimension of the sequence of Boolean models with the grains being balls whose radii obey a large deviations principle in the sense described above, and the current paper may be viewed as a start in that direction. In particular, it is to be hoped that this particular asymptotic regime, which is so natural from an information theoretic viewpoint, will also be of value in the applications of stochastic geometry in other domains of science and engineering.

3. Results

3.1. Volume fraction threshold

Consider the stationary version of the marked Poisson process in each dimension. We are interested in the asymptotic behavior of the probability with which the origin is covered, namely, $\mathbb{P}(\underline{0} \in \mathcal{C}_n)$. By stationarity, for any Borel set of $\mathbb{R}^n$, this probability is also the mean fraction of the volume of the Borel set which is covered by the Boolean model. We claim that there is a number τ_{v}, called the *volume fraction threshold*, such that, for $\rho < \tau_{\mathrm{v}}$, this probability asymptotically approaches 0 as n tends to ∞, while, for $\rho > \tau_{\mathrm{v}}$, it asymptotically approaches 1. The value of τ_{v} depends on the large deviations rate function $I(\cdot)$ associated to the sequence of distributions of the normalized radii of the marks.

The idea of the proof is based on the fact that most of the volume of a ball is at the boundary. Hence, for all $R > 0$, the mean number of points at distance roughly $R\sqrt{n}$ from the origin grows like

$$\mathrm{e}^{n\rho_n}\mathrm{e}^{n\log(2\pi\mathrm{e})/2+o(n)}\mathrm{e}^{n\log R}.$$

Each such point covers the origin with probability $\mathbb{P}(\bar{X}_n \geq R)$. For $R < R^*$, this probability is asymptotically 1. For $R > R^*$, this probability decays like $\mathrm{e}^{-nI(R)+o(n)}$, where $I(\cdot)$ denotes the rate function for the convergence $\bar{X}_n \xrightarrow{\mathbb{P}} R^*$.

Let $\underline{0}$ denote the origin in $\mathbb{R}^n$. We should therefore have

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\underline{0}\in\mathcal{C}_n) = \rho + \frac{1}{2}\log(2\pi\mathrm{e}) + \sup_{R\geq R^*}(\log R - I(R)),$$

as long as

$$\rho + \tfrac{1}{2}\log(2\pi\mathrm{e}) + \sup_{R\geq R^*}(\log R - I(R)) < 0,$$

where we used the fact that $I(R^*) = 0$, which implies that $\log(R) \leq \log(R^*) - I(R^*)$ for $R \leq R^*$.

Also, $\lim_{n\to\infty}\mathbb{P}(\underline{0}\in\mathcal{C}_n) = 1$, if

$$\rho + \tfrac{1}{2}\log(2\pi\mathrm{e}) + \sup_{R\geq R^*}(\log R - I(R)) > 0.$$

This gives a heuristic explanation of the value of the threshold in the following theorem.

Theorem 3.1. *Under the abovementioned assumptions, the volume fraction threshold is equal to*

$$\tau_{\mathrm{v}} = -\tfrac{1}{2}\log(2\pi\mathrm{e}) + \inf_{R\geq R^*}(I(R) - \log R). \tag{3.1}$$

More precisely, for $\rho < \tau_{\mathrm{v}}$, as n tends to ∞, the volume fraction in dimension n, namely, $\mathbb{P}(\underline{0}\in\mathcal{C}_n)$, tends to 0 exponentially fast with

$$\lim_{n\to\infty}\frac{1}{n}\log(\mathbb{P}(\underline{0}\in\mathcal{C}_n)) = \rho - \tau_{\mathrm{v}}, \tag{3.2}$$

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

whereas, for $\rho > \tau_{\mathrm{v}}$, it tends to 1 *with*

$$\lim_{n\to\infty} \frac{1}{n} \log(-\log \mathbb{P}(\underline{0} \notin \mathcal{C}_n)) = \rho - \tau_{\mathrm{v}}. \tag{3.3}$$

Note that

$$\tau_{\mathrm{v}} \le -\tfrac{1}{2}\log(2\pi\mathrm{e}) - \log R^*. \tag{3.4}$$

In the case of deterministic radii, i.e. when $\bar{X}_n$ equals the deterministic value R_n^* for each $n \ge 1$, with $R_n^* \to R^*$ as $n \to \infty$, we have equality in (3.4). The right-hand side of (3.4) is identical to what is called the Shannon–Poltyrev threshold in [1], where it in effect arose in the context of Boolean models with Gaussian grains truncated to their typicality regions, as described earlier. In Section 5 we will discuss the connection between the questions addressed in this paper and the information theoretic questions studied in [1] in more depth.

This threshold can also be described as follows: the volume of the n-ball of random radius $\bar{X}_n\sqrt{n}$ scales like $\mathrm{e}^{nV+o(n)}$ as n tends to ∞ for some constant V; we have $\tau_{\mathrm{v}} = -V$, or, equivalently, the critical density $\mathrm{e}^{n\tau_{\mathrm{v}}+o(n)}$ scales like the inverse of the volume of this n-ball.

3.2. Degree threshold

We are interested in the number D_n of points other than $\underline{0}$ whose ball intersects the ball of the point at the origin under $\mathbb{P}_n^0$.

We claim that there is a number τ_{d}, which we will call the *degree threshold*, such that if $\rho < \tau_{\mathrm{d}}$ then $\mathbb{E}_n^0[D_n]$ asymptotically goes to 0 when n tends to ∞, while, for $\rho > \tau_{\mathrm{d}}$, it asymptotically goes to ∞.

We argue as follows: condition on the radius of the ball of the point at the origin, call it $s\sqrt{n}$. Every point that lands in the ball of radius $s\sqrt{n}$ will have its ball meeting the ball at the origin. The number of such points grows like

$$\mathrm{e}^{n\rho_n}\mathrm{e}^{n\log(2\pi\mathrm{e})/2+o(n)}\mathrm{e}^{n\log s}.$$

Next consider points at a distance roughly $R\sqrt{n}$ from the (point at the) origin, with $R > s$. The number of such points grows like

$$\mathrm{e}^{n\rho_n}\mathrm{e}^{n\log(2\pi\mathrm{e})/2+o(n)}\mathrm{e}^{n\log R}.$$

Each such point has its ball intersecting the ball of the point at the origin with probability asymptotically equal to 1 if $R - s < R^*$ and with probability decaying like $\mathrm{e}^{-nI(R-s)+o(n)}$ if $R - s > R^*$. The number of points meeting the ball of the origin, conditioned on this ball having radius $s\sqrt{n}$, therefore grows like

$$\exp\Big(n\Big(\rho + \tfrac{1}{2}\log(2\pi\mathrm{e}) + \sup_{R\ge s+R^*}(\log R - I(R-s))\Big) + o(n)\Big).$$

The probability that the ball of the point at the origin has radius roughly $s\sqrt{n}$ scales like $\mathrm{e}^{-nI(s)+o(n)}$. Thus, the overall growth rate of the number of points whose ball meets the ball of

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

the point at the origin is

$$\begin{aligned}
&\sup_{s>0}\Big(-I(s)+\rho+\tfrac{1}{2}\log(2\pi\mathrm{e})+\sup_{R\geq s+R^*}(\log R-I(R-s))\Big)\\
&\quad=\rho+\frac{1}{2}\log(2\pi\mathrm{e})+\sup_{s>0}\sup_{R\geq s+R^*}(-I(s)+\log R-I(R-s)),\\
&\quad=\rho+\frac{1}{2}\log(2\pi\mathrm{e})+\sup_{R>R^*}\Big(\log R+\sup_{0<s\leq R-R^*}(-I(s)-I(R-s))\Big),\\
&\quad\overset{(a)}{=}\rho+\frac{1}{2}\log(2\pi\mathrm{e})+\max\Big(\sup_{R^*\leq R<2R^*}(\log R-I(R-R^*)),\\
&\qquad\qquad\sup_{R\geq 2R^*}\Big(\log R-2I\Big(\frac{R}{2}\Big)\Big)\Big),\\
&\quad=\rho+\frac{1}{2}\log(2\pi\mathrm{e})+\sup_{R\geq 2R^*}\Big(\log R-2I\Big(\frac{R}{2}\Big)\Big),
\end{aligned}$$

where in step (a) we used the convexity of the rate function $I(\cdot)$ and the fact that $I(R^*)=0$, and in the last step we observed that the maximum in the first of the terms in the overall maximum occurs at $R=2R^*$. This gives the intuition behind the value of the threshold in the following theorem.

Theorem 3.2. *Under the conditions of Theorem 3.1, the degree threshold is*

$$\tau_{\mathrm{d}}=-\frac{1}{2}\log(2\pi\mathrm{e})+\inf_{R\geq 2R^*}\Big(2I\Big(\frac{R}{2}\Big)-\log R\Big). \tag{3.5}$$

That is, for $\rho<\tau_{\mathrm{d}}$, as n tends to ∞, $\mathbb{E}_n^0[D_n]$ tends to 0 exponentially fast, whereas, for $\rho>\tau_{\mathrm{d}}$, it tends to ∞ exponentially fast. In both cases,

$$\lim_{n\to\infty}\frac{1}{n}\log(\mathbb{E}_n^0[D_n])=\rho-\tau_{\mathrm{d}}.$$

It is sometimes more convenient to write the degree threshold as

$$\tau_{\mathrm{d}}=-\tfrac{1}{2}\log(2\pi\mathrm{e})+\inf_{R\geq R^*}(2I(R)-\log(2R)). \tag{3.6}$$

Note that

$$\tau_{\mathrm{d}}\leq-\tfrac{1}{2}\log(2\pi\mathrm{e})-\log(2R^*),$$

and that the right-hand side of the last inequality is the degree threshold in the case of deterministic radii [8].

In the general case, the degree threshold can be described as follows. Let $\bar{X}_n'$ be an independent random variable with the same law as $\bar{X}_n$. The volume of the n-ball of random radius $(\bar{X}_n+\bar{X}_n')\sqrt{n}$ scales like $\mathrm{e}^{nV+o(n)}$ as n tends to ∞ for some constant V; we have $\tau_{\mathrm{d}}=-V$, or, equivalently, the critical density $\mathrm{e}^{n\tau_{\mathrm{d}}+o(n)}$ scales like the inverse of the volume of this n-ball.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

3.3. Percolation probability threshold

Consider the Palm version of the process in dimension n. Consider the connected component of $\mathcal{C}_n$ that contains the origin, called the *cluster of the origin*, and denote the set of points of the underlying Poisson process that lie in this connected component by K_n. The *percolation probability* in dimension n is denoted by

$$\theta_n := \mathbb{P}_n^0(|K_n| = \infty),$$

with $|A|$ the cardinality of set A. This is one of the standard definitions for percolation probability in continuum percolation theory; see [7, Section 1.4].

We are interested in the asymptotics of the percolation probability as $n \to \infty$. We claim that there is a number τ_{p}, called the *percolation probability threshold*, such that, for $\rho < \tau_{\mathrm{p}}$, we have $\theta_n \to 0$ as $n \to \infty$, while, for $\rho > \tau_{\mathrm{p}}$, we have $\liminf_n \theta_n > 0$.

Proposition 3.1. *In the case of deterministic radii, the percolation probability and the degree thresholds coincide, i.e.* $\tau_{\mathrm{p}} = \tau_{\mathrm{d}}$.

To see that $\tau_{\mathrm{p}} \geq \tau_{\mathrm{d}}$, note that if $\rho < \tau_{\mathrm{d}}$ then $\mathbb{E}_n^0[D_n] \to 0$ as $n \to \infty$ from Theorem 3.2. It follows that $\mathbb{P}_n^0(D_n = 0) \to 1$ as $n \to \infty$. Hence, $\mathbb{P}_n^0(|K_n| = 1) \to 1$ as $n \to \infty$, from which it follows that $\theta_n \to 0$ as $n \to \infty$. This means that $\rho < \tau_{\mathrm{p}}$. This argument actually works in the general case, i.e. it does not require the assumption of deterministic radii.

To see that $\tau_{\mathrm{p}} \leq \tau_{\mathrm{d}}$ in the case of deterministic radii, we need to prove that if $\rho > \tau_{\mathrm{d}}$ then $\liminf_n \theta_n > 0$. To this end, let us recall the main result of [8]. In our notation, Penrose [8] considered the sequence of Poisson Boolean models with deterministic radii $R_n^* = R^*$ for each $n \geq 1$, and with normalized logarithmic intensities ρ_n^y defined via

$$\mathrm{e}^{n\rho_n^y} \frac{(\pi n)^{n/2}}{\Gamma(n/2+1)} (2R^*)^n = y \quad \text{for all } n \geq 1,$$

where $y > 0$ is a fixed real number. Let θ_n^y denote the percolation probability in dimension n with these choices. The main result [8, Theorem 1] is that $\lim_{n\to\infty} \theta_n^y$ exists and equals the survival probability of a branching process with offspring distribution Poisson with mean y, and started with a single individual. In particular, this means that if $y > 1$ then $\liminf_{n\to\infty} \theta_n^y > 0$.

In our scenario with deterministic radii, the degree threshold (see (3.6)) is

$$\tau_{\mathrm{d}} = -\tfrac{1}{2}\log(2\pi\mathrm{e}) - \log(2R^*) \quad \text{(deterministic radii)}.$$

It suffices to observe that if $\rho > \tau_{\mathrm{d}}$ then

$$\lim_{n\to\infty} \mathrm{e}^{n\rho_n} \frac{(\pi n)^{n/2}}{\Gamma(n/2+1)} (2R_n^*)^n = \infty.$$

That $\liminf_{n\to\infty} \theta_n > 0$ then follows from the result of [8] cited above.

The main result in the case with random radii is as follows.

Theorem 3.3. *The percolation probability threshold is given by*

$$\tau_{\mathrm{p}} = -\tfrac{1}{2}\log(2\pi\mathrm{e}) + \inf_{R \geq R^*} (I(R) - \log(R + R^*)). \tag{3.7}$$

That is, for $\rho < \tau_{\mathrm{p}}$, $\theta_n \to 0$ *when n tends to* ∞, *whereas, for* $\rho > \tau_{\mathrm{p}}$, *we have* $\liminf_n \theta_n > 0$.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

Note that

$$\tau_{\mathrm{p}} \leq -\tfrac{1}{2}\log(2\pi\mathrm{e}) - \log(2R^*).$$

In the case of deterministic radii the minimum in the expression for the percolation probability threshold in (3.7) is achieved at $R = R^*$, and so we have

$$\tau_{\mathrm{p}} = -\tfrac{1}{2}\log(2\pi\mathrm{e}) - \log(2R^*) \quad \text{(deterministic radii)}.$$

This also equals the value of the degree threshold in the case of deterministic radii.

The volume of the n-ball of random radius $(\bar{X}_n + R^*)\sqrt{n}$ scales like $\mathrm{e}^{nV+o(n)}$ as n tends to ∞ for some constant V; we have $\tau_{\mathrm{p}} = -V$, or, equivalently, the critical density $\mathrm{e}^{n\tau_{\mathrm{p}}+o(n)}$ scales like the inverse of the volume of this n-ball. The intuition behind this result is that what matters for percolation is the mean number of balls that intersect a ball with typical radius (namely, roughly $R^*\sqrt{n}$); if $\rho < \tau_{\mathrm{p}}$ then, on an event whose probability tends to 1 as n tends to ∞, namely the event that the ball of the point at $\underline{0}$ has a radius in the interval $(R^*\sqrt{n} - \alpha_n, R^*\sqrt{n} + \alpha_n)$ for appropriate $0 < \alpha_n = O(\sqrt{n})$, no other ball intersects the latter ball asymptotically and, hence, there is no percolation. Conversely, for $\rho > \tau_{\mathrm{p}}$, when the ball at $\underline{0}$ is at typicality, i.e. its radius lies in an interval of the kind defined above, we can consider a thinned version of the reduced process where we only retain points whose balls have radii that are at least above a threshold slightly less than the value of R achieving the infimum in the definition of τ_{p} (assume for the moment that this infimum is achieved), and it will still hold that the mean number of balls intersecting the ball at the origin tends to ∞ like $\mathrm{e}^{n\delta}$ with some $\delta > 0$. Since these balls themselves have radii at least as large as the typical ball of the unconditional distribution, this scenario propagates via a supercritical branching process, implying asymptotic percolation. For technical reasons, the formal proof looks slightly different from this sketch, but this is the basic intuition.

3.4. Ordering of the thresholds

Theorem 3.4. *Under the abovementioned assumptions,*

$$\tau_{\mathrm{d}} \leq \tau_{\mathrm{p}} \leq \tau_{\mathrm{v}}. \tag{3.8}$$

Remark. The ordering relation of Theorem 3.4 is not limited to the Poisson case. The family of Boolean models considered here can naturally be extended to a family of particle processes [11], where the nth particle process features a stationary and ergodic point process μ_n in $\mathbb{R}^n$ with normalized logarithmic intensity ρ_n such that $\rho_n \to \rho$ as $n \to \infty$, and i.i.d. marks satisfying the same independence and LDP assumptions as above. This family of particle processes will be said to admit a volume fraction threshold τ_{v} if the associated $\mathcal{C}_n$, still defined by (2.2), is such that $\mathbb{P}(\underline{0} \in \mathcal{C}_n)$ asymptotically approaches 0 as n tends to ∞ for $\rho < \tau_{\mathrm{v}}$, while, for $\rho > \tau_{\mathrm{v}}$, it asymptotically approaches 1. Similarly, it will be said to admit a degree threshold τ_{d} if the Palm expectation of D_n tends to 0 as n tends to ∞ for $\rho < \tau_{\mathrm{d}}$, while, for $\rho > \tau_{\mathrm{d}}$, it tends to ∞. The definition of the percolation probability threshold can also be extended verbatim. Assuming that these three thresholds exist, then they must satisfy (3.8). This follows from first principles. If the volume fraction asymptotically tends to 1 then percolation must hold asymptotically; hence, $\tau_{\mathrm{p}} \leq \tau_{\mathrm{v}}$. If the mean number of balls that intersect the ball at the origin tends to 0 then percolation cannot hold asymptotically; hence, $\tau_{\mathrm{d}} \leq \tau_{\mathrm{p}}$.

Returning to the Poisson case, to better understand the thresholds, we first need to recall some facts from basic convex analysis [10]. Since it is a good convex rate function, $I(\cdot)$ is *proper*, as defined in [10, p. 24]. Furthermore, since it is lower semicontinuous, its epigraph is

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

closed [10, Theorem 7.1], so $I(\cdot)$ is *closed* in the sense of [10, p. 52]. Recall that the domain of $I(\cdot)$, defined as the set of R for which $I(R)$ is finite, is an interval, which is nonempty because $I(R^*) = 0$. Since $I(\cdot)$ is closed and proper, the right and left derivatives, $I'_+(\cdot)$ and $I'_-(\cdot)$ respectively, are well defined as functions on $\mathbb{R}$ (both defined to be $+\infty$ to the right of the domain of $I(\cdot)$ and $-\infty$ to the left of the domain of $I(\cdot)$). These are nondecreasing functions, each of which is finite on the interior of the domain of $I(\cdot)$, and satisfy ([10, Theorem 24.1])

$$I'_+(z_1) \le I'_-(x) \le I'_+(x) \le I'_-(z_2) \quad \text{if } z_1 < x < z_2,$$

and, for all $x \in \mathbb{R}$,

$$\lim_{z \uparrow x} I'_-(z) = \lim_{z \uparrow x} I'_+(z) = I'_-(x) \quad \text{and} \quad \lim_{z \downarrow x} I'_-(z) = \lim_{z \downarrow x} I'_+(z) = I'_+(x).$$

We further note that $0 \in [I'_-(R^*), I'_+(R^*)]$, since $I(\cdot)$ is a nonnegative function with $I(R^*) = 0$.

This means that we can define the following radii.

- $R_{\mathrm{v}} \ge R^*$, as a value of R satisfying $1/R_{\mathrm{v}} \in [I'_-(R_{\mathrm{v}}), I'_+(R_{\mathrm{v}})]$. Such an R_{v} achieves the infimum in (3.1) for the volume fraction threshold. Furthermore, since $R \mapsto 1/R$ is strictly decreasing and decreases to 0 as $R \to \infty$, it follows that R_{v} is uniquely defined and finite.
- $R_{\mathrm{d}} \ge R^*$, as a value of R satisfying $1/2R_{\mathrm{d}} \in [I'_-(R_{\mathrm{d}}), I'_+(R_{\mathrm{d}})]$. Such an R_{d} achieves the minimum in (3.6) for the degree threshold. Furthermore, since $R \mapsto 1/2R$ is strictly decreasing and decreases to 0 as $R \to \infty$, it follows that R_{d} is uniquely defined and finite.
- $R_{\mathrm{p}} \ge R^*$, as a value of R satisfying $1/(R_{\mathrm{p}} + R^*) \in [I'_-(R_{\mathrm{p}}), I'_+(R_{\mathrm{p}})]$. Such an R_{p} achieves the minimum in (3.7) for the percolation probability threshold. Furthermore, since $R \mapsto 1/(R + R^*)$ is strictly decreasing and decreases to 0 as $R \to \infty$, it follows that R_{p} is uniquely defined and finite.

Theorem 3.5. *With the abovementioned definitions, we have*

$$R^* \le R_{\mathrm{d}} \le R_{\mathrm{p}} \le R_{\mathrm{v}} \le R_{\mathrm{p}} + R^* \le 2R_{\mathrm{d}}.$$

4. Proofs

4.1. Proof of Theorem 3.1

Below we will use the *directed random geometric graph* built as follows: its vertices are the nodes of the point process and there is an edge from $T_n^{(k)}$ to $T_n^{(l)}$, $l \ne k$, if $T_n^{(l)} \in B(T_n^{(k)}, X_n^{(k)}\sqrt{n})$.

Let d_n^- denote the in-degree of the node at the origin in this random directed graph under $\mathbb{P}^0$, namely the number of points whose ball contains the origin. Let d_n^+ denote the out-degree of the origin, namely the number of points which fall in the ball of the point at the origin. From the mass transport principle [6], or by straightforward elementary arguments based on an ergodic theorem for spatial averages, we have $\mathbb{E}_n^0[d_n^+] = \mathbb{E}_n^0[d_n^-]$. Now, we have

$$\begin{aligned}\mathbb{E}_n^0[d_n^+] &= \mathrm{e}^{n\rho_n}\mathbb{E}\left[\frac{(\pi n)^{n/2}\bar{X}_n^n}{\Gamma(n/2+1)}\right] \\ &= \mathrm{e}^{n\rho_n}\mathbb{E}[\mathrm{e}^{n\log(2\pi\mathrm{e})/2+o(n)}\mathrm{e}^{n\log\bar{X}_n}] \\ &= \mathrm{e}^{n(\rho+\log(2\pi\mathrm{e})/2)+o(n)}\mathbb{E}[\mathrm{e}^{n\log\bar{X}_n}].\end{aligned}$$

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

The first equality follows from Slivnyak's theorem. Then we have the following result.

Lemma 4.1. *It holds that*

$$\frac{1}{n}\log \mathbb{E}_n^0[d_n^-] \to \rho + \frac{1}{2}\log(2\pi \mathrm{e}) + \sup_{R\geq R^*}(\log R - I(R)) \quad \text{as } n \to \infty. \tag{4.1}$$

Proof. From what precedes,

$$\mathbb{E}_n^0[d_n^-] = \mathrm{e}^{n(\rho + n\log(2\pi \mathrm{e})/2)+o(n)}\mathbb{E}[\mathrm{e}^{n\log \bar{X}_n}].$$

It follows from assumption (2.1) and from Varadhan's lemma [12] that

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{E}[\mathrm{e}^{n\log \bar{X}_n}] = \sup_{R\geq R^*}(\log R - I(R)),$$

where we have used the observation that $\log R - I(R) \leq \log R^*$ for $0 \leq R \leq R^*$. This completes the proof. □

Now, from the independent thinning theorem [2, Exercise 11.3.1], the distribution of d_n^- is Poisson. Thus,

$$\mathbb{P}(\underline{0} \in \mathcal{C}_n) = 1 - \exp(-\mathbb{E}_n^0[d_n^-]) \leq \mathbb{E}_n^0[d_n^-]. \tag{4.2}$$

For $\rho < \tau_{\mathrm{v}}$, we see from (4.1) that $\mathbb{E}_n^0[d_n^-] \to 0$ as $n \to \infty$, which implies that $\mathbb{P}(\underline{0} \in \mathcal{C}_n) \to 0$ as $n \to \infty$. In addition, for all $\alpha < 1$, we have $1 - \alpha x \geq \exp(-x) \geq 1 - x$ for all sufficiently small $x > 0$. Thus, from (4.2), for all $\alpha < 1$ and sufficiently large n,

$$\frac{1}{n}\log \mathbb{E}_n^0[d_n^-] \geq \frac{1}{n}\log \mathbb{P}(\underline{0} \in \mathcal{C}_n) \geq \frac{1}{n}\log \mathbb{E}_n^0[d_n^-] + \frac{1}{n}\log \alpha.$$

Thus, we have

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(\underline{0} \in \mathcal{C}_n) = \rho + \frac{1}{2}\log(2\pi \mathrm{e}) + \sup_{R\geq R^*}(\log R - I(R)),$$

which concludes the proof of (3.2).

Suppose now that $\rho > \tau_{\mathrm{v}}$. Since

$$\mathbb{P}(\underline{0} \notin \mathcal{C}_n) = \exp(-\mathbb{E}_n^0[d_n^-]),$$

we then immediately obtain (3.3).

4.2. Proof of Theorem 3.2

In each dimension n, consider the stationary version of the process. Given $s > 0$, let $N_n(\underline{0}, s)$ denote the number of points whose balls intersect the ball of radius $s\sqrt{n}$ centered at the origin $\underline{0}$ in $\mathbb{R}^n$. Then

$$\begin{aligned}\mathbb{E}[N_n(\underline{0}, s)] &= \mathrm{e}^{n\rho_n}\mathbb{E}\left[\frac{(\pi n)^{n/2}}{\Gamma(n/2+1)}(\bar{X}_n + s)^n\right] \\ &= \mathrm{e}^{n\rho_n}\mathbb{E}[\mathrm{e}^{n\log(2\pi \mathrm{e})/2+o(n)}\mathrm{e}^{n\log(\bar{X}_n+s)}] \\ &= \mathrm{e}^{n(\rho+\log(2\pi \mathrm{e})/2)+o(n)}\mathbb{E}[\mathrm{e}^{n\log(\bar{X}_n+s)}].\end{aligned}$$

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

The first equality again follows from Slivnyak's theorem and the mass transport principle applied to the directed graph with an edge from $T_n^{(k)}$ to $T_n^{(l)}$, $l \neq k$, if $T_n^{(l)} \in B(T_n^{(k)}, (X_n^{(k)} + s)\sqrt{n})$.

Now consider the Palm version of the point process. Recall that, from Slivnyak's theorem, ignoring the point at $\underline{0}$ and its mark leaves behind a stationary version of the marked point process, which is called the reduced process. We therefore have, $\mathbb{E}_n^0[D_n] = \mathbb{E}[\mathbb{E}[N_n(\underline{0}, S) \mid S]]$, where $S \stackrel{\mathrm{D}}{=} \bar{X}_n$, independent of the reduced process. Thus,

$$\mathbb{E}_n^0[D_n] = \mathrm{e}^{n(\rho + \log(2\pi\mathrm{e})/2) + o(n)} \mathbb{E}[\mathrm{e}^{n \log(\bar{X}_n + \bar{X}_n')}],$$

with $\bar{X}_n$ and $\bar{X}_n'$ being i.i.d. We now use the fact that $\bar{X}_n + \bar{X}_n'$ satisfies an LDP with good convex rate function $2I(u/2)$ to derive (3.5) from Varadhan's lemma [3], [12]. For this, we have to check that if $\bar{X}_n'$ is a variable with the same law as $\bar{X}_n$ and such that $\bar{X}_n'$ and $\bar{X}_n$ are independent, then

$$\limsup_{n \to \infty} \mathbb{E}[(\bar{X}_n + \bar{X}_n')^{\gamma n}]^{1/n} < \infty \quad \text{for some } \gamma > 1. \tag{4.3}$$

But this can be obtained from the convexity argument $\mathbb{E}[(\bar{X}_n + \bar{X}_n')^{\gamma n}] \leq 2^{\gamma n} \mathbb{E}[(\bar{X}_n)^{\gamma n}]$, which implies that $\mathbb{E}[(\bar{X}_n + \bar{X}_n')^{\gamma n}]^{1/n} \leq 2^{\gamma} \mathbb{E}[(\bar{X}_n)^{\gamma n}]^{1/n}$. Hence, (4.3) follows from (2.1).

This completes the proof of the results on the degree threshold.

4.3. Proof of Theorem 3.3

We need to prove two things: (i) if $\rho < \tau_{\mathrm{p}}$ then $\lim_{n \to \infty} \theta_n = 0$, and (ii) if $\rho > \tau_{\mathrm{p}}$ then $\liminf_{n \to \infty} \theta_n > 0$.

To prove (i), we follow the lines of the proof in Section 4.2. Rather than considering $\mathbb{E}_n^0[D_n]$, we consider $\mathbb{P}_n^0(D_n > 0)$. As in Section 4.2, we write $\mathbb{P}_n^0(D_n > 0) = \mathbb{E}_n^0[\mathbb{P}_n^0(D_n > 0 \mid S)]$, where $S\sqrt{n}$ now refers to the radius of the ball of the point at the origin. Now, if

$$\rho < -\tfrac{1}{2} \log(2\pi\mathrm{e}) + \inf_{R \geq R^*} (I(R) - \log(R + R^*))$$

then, because R_{p}, as defined in Section 3.4, is finite, for sufficiently small $\varepsilon > 0$, we also have

$$\rho < -\tfrac{1}{2} \log(2\pi\mathrm{e}) + \inf_{R \geq R^*} (I(R) - \log(R + R^* + \varepsilon)).$$

On the event $\{S \leq R^* + \varepsilon'\}$, with $\varepsilon > \varepsilon' > 0$, we have

$$\begin{aligned}
&\lim_n \frac{1}{n} \log(\mathbb{E}_n^0[D_n \mid S]) \\
&\quad = \rho + \tfrac{1}{2} \log(2\pi\mathrm{e}) + \sup_{R \geq R^* + S} (\log R - I(R - S)) \\
&\quad = \rho + \tfrac{1}{2} \log(2\pi\mathrm{e}) + \sup_{\tilde{R} \geq R^*} (\log(\tilde{R} + S) - I(\tilde{R})) \\
&\quad \leq \inf_{R \geq R^*} (I(R) - \log(R + R^* + \varepsilon)) + \sup_{\tilde{R} \geq R^*} (\log(\tilde{R} + S) - I(\tilde{R})) \\
&\quad = \sup_{\tilde{R} \geq R^*} (\log(\tilde{R} + S) - I(\tilde{R})) - \sup_{\tilde{R} \geq R^*} (\log(\tilde{R} + R^* + \varepsilon) - I(\tilde{R})) \\
&\quad < 0.
\end{aligned} \tag{4.4}$$

We also have

$$\begin{aligned}
\mathbb{P}_n^0(D_n > 0) &= \mathbb{P}_n^0(D_n > 0,\ S > R^* + \varepsilon') + \mathbb{E}_n^0[\mathbb{P}_n^0(D_n > 0 \mid S)\, \mathbf{1}(S \leq R^* + \varepsilon')] \\
&\leq \mathbb{P}(S > R^* + \varepsilon') + \mathbb{E}_n^0[\mathbb{E}_n^0[D_n \mid S]\, \mathbf{1}(S \leq R^* + \varepsilon')].
\end{aligned}$$

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

In the last expression, the first term has probability asymptotically approaches 0 as $n \to \infty$. From (4.4), for all s in the integration interval, the integrand in the second term tends pointwise to 0 as $n \to \infty$. From this and dominated convergence, we conclude that $\mathbb{P}_n^0(D_n > 0) \to 0$ as $n \to \infty$, which proves (i).

We now prove (ii), assuming that

$$\rho > -\tfrac{1}{2}\log(2\pi \mathrm{e}) + \inf_{R \geq R^*}(I(R) - \log(R + R^*)).$$

Let R_{p}, as defined in Section 3.4, achieve the minimum in the definition of τ_{p} in (3.7). We need to distinguish between the two cases $R_{\mathrm{p}} = R^*$ and $R_{\mathrm{p}} > R^*$.

Consider first the case $R_{\mathrm{p}} = R^*$. Then $\rho = -\frac{1}{2}\log(2\pi \mathrm{e}) - \log(2R^*) + \delta$ for some $\delta > 0$. This means that we can choose $\gamma > 0$ such that $\rho > -\frac{1}{2}\log(2\pi \mathrm{e}) - \log(2(R^* - \gamma)) + \delta/2$. For each dimension n, we consider the thinned version of the reduced process, where we only retain the points whose balls have radius at least $(R^* - \gamma)\sqrt{n}$. We also consider only the event that the ball of the point at the origin has radius at least $(R^* - \gamma)\sqrt{n}$.

Let $\tilde{\theta}_n$ denote the probability of percolation from the origin via its ball and through the balls of the thinned reduced point process, on the event that the ball of the point at the origin has radius at least $(R^* - \gamma)\sqrt{n}$. Since $\theta_n \geq \tilde{\theta}_n$, if we can show that $\liminf_{n\to\infty} \tilde{\theta}_n > 0$, then we will be done.

Let us now show that the following properties hold.

(P1) The probability that the ball at $\underline{0}$ has radius at least $(R^* - \gamma)\sqrt{n}$ tends to 1 as n tends to ∞.

(P2) The probability that the number J_n of balls of the thinned point process intersecting the ball at $\underline{0}$ (with radius at least $(R^* - \gamma)\sqrt{n}$) is positive tends to 1 as n tends to ∞.

(P3) The Boolean model with deterministic radii $(R^* - \gamma)\sqrt{n}$ for the points of the thinned reduced process percolates, i.e. the associated percolation probability has a positive liminf.

Property (P1) is immediate. For proving (P2), we show that $\mathbb{E}[J_n]$ tends to ∞ with n, which will complete the proof since J_n is Poisson. The probability that $\bar{X}_n$ exceeds $(R^* - \gamma)\sqrt{n}$ is asymptotically 1. Hence, using the result of Theorem 3.2 on the degree threshold with deterministic radii, we obtain

$$\lim_n \frac{1}{n}\log(\mathbb{E}[J_n]) \geq \rho + \frac{1}{2}\log(2\pi \mathrm{e}) + \log(2(R^* - \gamma)) > \frac{\delta}{2},$$

which completes the proof of (P2). For proving (P3), we use the results in [8]. The percolation probability threshold of the Boolean models with deterministic radii $(R^* - \gamma)\sqrt{n}$ is $-\frac{1}{2}\log(2\pi \mathrm{e}) - \log(2(R^* - \gamma))$. Since the normalized logarithmic intensity of the thinned reduced process is still asymptotically ρ, and since this exceeds $-\frac{1}{2}\log(2\pi \mathrm{e}) - \log(2(R^* - \gamma))$, the proof of (P3) follows from [8].

The proof of the desired result, in the case $R_{\mathrm{p}} = R^*$, now follows immediately from properties (P1)–(P3).

We next turn to the case $R_{\mathrm{p}} > R^*$. Note that in this case we must have $I(R_{\mathrm{p}}) < \infty$. Since

$$\rho = -\tfrac{1}{2}\log(2\pi \mathrm{e}) + I(R_{\mathrm{p}}) - \log(R_{\mathrm{p}} + R^*) + \delta$$

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

for some $\delta > 0$, we can again choose $\gamma > 0$ such that $R^* < R_{\mathrm{p}} - \gamma < R_{\mathrm{p}}$ (which implies that $I(R_{\mathrm{p}} - \gamma) < \infty$), and such that

$$\rho > -\frac{1}{2}\log(2\pi \mathrm{e}) + I(R_{\mathrm{p}} - \gamma) - \log(R_{\mathrm{p}} + R^* - 2\gamma) + \frac{\delta}{2}.$$

For each dimension n, we consider the thinned version of the point process, where we only retain the points whose balls have radius at least $(R_{\mathrm{p}} - \gamma)\sqrt{n}$, and the event that the ball at the origin has radius at least $(R^* - \gamma)\sqrt{n}$.

Let $\tilde{\theta}_n$ now denote the probability of percolation from the origin via its ball and through the balls of the thinned point process, on the event that the ball at the origin has radius at least $(R^* - \gamma)\sqrt{n}$. As above, it is enough to show that $\liminf_{n\to\infty} \tilde{\theta}_n > 0$, which in turn will be proved if we show that the following properties hold.

(P4) The probability that the ball at $\underline{0}$ has radius at least $(R^* - \gamma)\sqrt{n}$ tends to 1 as n tends to ∞.

(P5) The probability that the number J_n of balls of this thinned point process intersecting the ball of $\underline{0}$ (with radius at least $(R^* - \gamma)\sqrt{n}$) is positive tends to 1 as n tends to ∞.

(P6) The Boolean model with deterministic radii $(R_{\mathrm{p}} - \gamma)\sqrt{n}$ for the points of the thinned point process percolates.

For proving (P5), we use the fact that the probability that $\bar{X}_n$ exceeds $(R_{\mathrm{p}} - \gamma)\sqrt{n}$ scales like $\mathrm{e}^{-nI(R_{\mathrm{p}}-\gamma)+o(n)}$. Applying arguments similar to those used for the evaluation of the degree threshold, we obtain

$$\lim_n \frac{1}{n}\log(\mathbb{E}[J_n]) = \rho + \frac{1}{2}\log(2\pi \mathrm{e}) - I(R_{\mathrm{p}} - \gamma) + \log(R_{\mathrm{p}} + R^* - 2\gamma)) > \frac{\delta}{2}.$$

For proving (P6), we use the fact that the percolation probability threshold of the Boolean models with deterministic radii $(R_{\mathrm{p}} - \gamma)\sqrt{n}$ is $-\frac{1}{2}\log(2\pi \mathrm{e}) - \log(2(R_{\mathrm{p}} - \gamma))$. Since the normalized logarithmic intensity of the thinned reduced process is asymptotically $\rho - I(R_{\mathrm{p}} - \gamma)$, and since this exceeds $-\frac{1}{2}\log(2\pi \mathrm{e}) - \log(R_{\mathrm{p}} + R^* - 2\gamma))$ (because $R_{\mathrm{p}} > R^*$), the proof of (P5) follows from [8].

This also completes the overall proof.

Remark. The proof of the percolation probability threshold in the case of deterministic radii (i.e. when the radii in dimension n equal a constant $R_n^*\sqrt{n}$, with $R_n^* \to R^*$ as $n \to \infty$) can be completed in a much simpler way than the general proof. Since $\tau_{\mathrm{p}} = \tau_{\mathrm{d}}$ in the case of deterministic radii, the absence of percolation when $\rho < \tau_{\mathrm{p}}$ is an immediate consequence of the result proved earlier that $\mathbb{P}(D_n > 0) \to 0$ as $n \to \infty$ when $\rho < \tau_{\mathrm{d}}$. For the case $\rho > \tau_{\mathrm{p}}$, the proof can be done in a way exactly as the general case where $R_{\mathrm{p}} = R^*$ was handled above.

4.4. Proof of Theorem 3.4

We first prove that $\tau_{\mathrm{d}} \le \tau_{\mathrm{p}}$. By the convexity of the rate function $I(\cdot)$, and since $I(R^*) = 0$, we have, for all $R \ge R^*$, $2I((R + R^*)/2) \le I(R)$. Hence, for all $R \ge R^*$,

$$2I\left(\frac{R + R^*}{2}\right) - \log\left(2\left(\frac{R + R^*}{2}\right)\right) \le I(R) - \log(R + R^*).$$

This establishes the result.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

The fact that $\tau_{\mathrm{p}} \le \tau_{\mathrm{v}}$ immediately follows from

$$\inf_{R \ge R^*} (I(R) - \log(R + R^*)) \le \inf_{R \ge R^*} (I(R) - \log(R)).$$

4.5. Proof of Theorem 3.5

The fact that $R^* \le R_{\mathrm{d}}$ is immediate from the definition of R_{d}.

We first prove that $R_{\mathrm{d}} \le R_{\mathrm{p}}$. Recall that R_{d} is uniquely defined by $1/2R_{\mathrm{d}} \in [I'_-(R_{\mathrm{d}}), I'_+(R_{\mathrm{d}})]$, and R_{p} is uniquely defined by $1/(R_{\mathrm{p}} + R^*) \in [I'_-(R_{\mathrm{p}}), I'_+(R_{\mathrm{p}})]$. Now

$$R_{\mathrm{d}} > R_{\mathrm{p}} \implies 2R_{\mathrm{d}} > R_{\mathrm{p}} + R^* \implies \frac{1}{2R_{\mathrm{d}}} < \frac{1}{R_{\mathrm{p}} + R^*} \implies R_{\mathrm{d}} < R_{\mathrm{p}}.$$

This contradiction implies that $R_{\mathrm{d}} \le R_{\mathrm{p}}$.

We next prove that $R_{\mathrm{p}} \le R_{\mathrm{v}}$. Here we also need to recall that R_{v} is uniquely defined by $1/R_{\mathrm{v}} \in [I'_-(R_{\mathrm{v}}), I'_+(R_{\mathrm{v}})]$. Now

$$R_{\mathrm{p}} > R_{\mathrm{v}} \implies R_{\mathrm{p}} + R^* > R_{\mathrm{v}} \implies \frac{1}{R_{\mathrm{p}} + R^*} < \frac{1}{R_{\mathrm{v}}} \implies R_{\mathrm{p}} < R_{\mathrm{v}}.$$

This contradiction proves that $R_{\mathrm{p}} \le R_{\mathrm{v}}$.

We next prove that $R_{\mathrm{v}} \le R_{\mathrm{p}} + R^*$. For this, it suffices to observe the contradiction

$$R_{\mathrm{v}} > R_{\mathrm{p}} + R^* \implies \frac{1}{R_{\mathrm{v}}} < \frac{1}{R_{\mathrm{p}} + R^*} \implies R_{\mathrm{v}} < R_{\mathrm{p}} \implies R_{\mathrm{v}} < R_{\mathrm{p}} + R^*.$$

We finally prove that $R_{\mathrm{p}} + R^* \le 2R_{\mathrm{d}}$. For this we observe the contradiction

$$R_{\mathrm{p}} + R^* > 2R_{\mathrm{d}} \implies \frac{1}{R_{\mathrm{p}} + R^*} < \frac{1}{2R_{\mathrm{d}}} \implies R_{\mathrm{p}} < R_{\mathrm{d}} \implies R_{\mathrm{p}} + R^* < 2R_{\mathrm{d}}.$$

This completes the proof.

5. Remarks

5.1. Connections with the Shannon–Poltyrev threshold

In this subsection we make some remarks about the connections between the concerns of this paper and the problem of error exponents in channel coding over the additive white Gaussian noise channel [4, Section 7.4], as discussed in [1] in the Poltyrev regime.

The *Shannon–Poltyrev threshold* is a threshold for the asymptotic normalized logarithmic intensity of the Boolean models with truncated Gaussian grains that relates to a decoding problem motivated by information theory. For all $n \ge 1$ and $k \ge 0$, let $W_n^{(i,k)}$, $n \ge i \ge 1$, denote an i.i.d. sequence of Gaussian random variables, all centered and of variance σ^2, and let $W_n^{(k)}$ denote the n-dimensional vector with coordinates $W_n^{(i,k)}$, $n \ge i \ge 1$. Then $T_n^{(k)} + W_n^{(k)}$ belongs to the closed ball of center $T_n^{(k)}$ and radius $X_n^{(k)}\sqrt{n}$, and if $\bar{X}_n$ is defined as in (2.3) then the random variables $(\bar{X}_n, n \ge 1)$ satisfy an LDP and all the assumptions listed above. For Shannon's channel coding problem in the Poltyrev regime, as considered in [1], the focus is on the probability of error. As a result, we only want to associate those points in Euclidean space that have a high probability of being of the type $T_n^{(k)} + W_n^{(k)}$ to the point $T_n^{(k)}$. Therefore, we consider a Boolean model with grains based on an associated *typicality region* for the

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

noise vector, namely the set defined as in (2.4), where the $\alpha_n > 0$ in that equation are chosen to satisfy the conditions in (2.5). This now gives rise to a family of deterministic Boolean models with truncated Gaussian grains. The Shannon–Poltyrev threshold is the asymptotic normalized logarithmic intensity up to which it is possible to make such an association with an asymptotically vanishing probability of error. It is also the threshold up to which there is a vanishingly small probability that a point in Euclidean space is covered by multiple truncated Gaussian grains, which directly corresponds to what is called the volume fraction threshold in the present paper.

It turns out that the volume fraction threshold is smaller for Gaussian grains than for truncated Gaussian grains, even though the normalized radii of the grains in the two models have the same asymptotic limit R^*. To see this, consider Gaussian grains with per-coordinate variance σ^2, as above. Then $\bar{X}_n^2$ is distributed as the average of n i.i.d. squared Gaussian random variables of mean 0 and variance σ^2, so we have $\bar{X}_n^2 \xrightarrow{\mathbb{P}} \sigma^2$ as $n \to \infty$, which implies that $\bar{X}_n \xrightarrow{\mathbb{P}} \sigma$ as $n \to \infty$. This means that $\mathbb{E}[\bar{X}_n]$ (which is bounded above by $(\mathbb{E}[(\bar{X}_n)^2])^{1/2} = \sigma$) converges to σ as $n \to \infty$. This means that $R^* = \sigma$. The volume fraction threshold for deterministic grains with radii $R^*\sqrt{n}$ in dimension n is then given by the right-hand side of (3.4). The exponent of the growth rate in n of the volume of each Gaussian grain is strictly larger than this. That it is at least as large follows immediately from the convexity of the function $R \mapsto R^n$, defined for $R \geq 0$. To see the strict inequality, first note that the density of the radius of the Gaussian grain in dimension n can be written as $g_n^\sigma(r)$, $r \geq 0$, where $g_n^\sigma(r) = (1/\sigma)g_n^1(r/\sigma)$, with

$$g_1^\sigma(r) := \frac{nr^{n-1}\mathrm{e}^{-r^2/2}}{2^{n/2}\Gamma(n/2+1)}, \qquad r \geq 0,$$

where $\Gamma(\cdot)$ denotes the standard Euler gamma function. To determine the asymptotic growth rate of the expected volume of a grain, we need to compute

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty V_n r^n g_n^\sigma(r)\, \mathrm{d}r.$$

It is convenient to reparameterize r as $v\sigma\sqrt{n}$, giving

$$g_n^\sigma(v\sigma\sqrt{n}) = \mathrm{e}^{-n(v^2/2 - 1/2 - \log(v) + o(1))}.$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n g_n^\sigma(r)\, \mathrm{d}r \\
&\quad = \lim_{n\to\infty} \frac{1}{n} \log \left((2\pi\mathrm{e}\sigma^2)^{n/2} \int_0^\infty \mathrm{e}^{n(\log v + o(1))} \mathrm{e}^{-n(v^2/2 - 1/2 - \log(v) + o(1))}\, \mathrm{d}v \right) \\
&\quad \overset{\text{(a)}}{=} \frac{1}{2}\log(2\pi\mathrm{e}\sigma^2) + \frac{1}{2}(\log 4 - 1) \\
&\quad > \frac{1}{2}\log(2\pi\mathrm{e}\sigma^2),
\end{aligned}$$

where step (a) follows from Laplace's principle that the asymptotics is controlled by the exponential term in the integrand with the largest exponent. Laplace's principle as just applied is only a heuristic, of course, but this calculation makes the point that the volume fraction threshold for Gaussian grains is strictly smaller than the volume fraction threshold for the

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

TABLE 1.

ρ	τ_{d}		τ_{p}		τ_{v}		τ_{s}		
$\mathbb{E}_n^0[D_n] \to_{n\to\infty}$	0		∞	∞	∞	∞	∞	∞	∞
$\theta_n \to_{n\to\infty}$	0	0	0		>0	>0	>0	>0	>0
$f_n \to_{n\to\infty}$	0	0	0	0	0		1	1	1
$p_n \to_{n\to\infty}$	0	0	0	0	0	0	0		1

truncated Gaussian grains, i.e. the Shannon–Poltyrev threshold denoted by τ_{s} below. In Table 1 we provide a comparison of τ_{s} to the other thresholds in case of Gaussian grains. In this table, $\mathbb{E}_n^0[D_n]$ denotes the Palm expectation of the degree, θ_n the percolation probability, $f_n = \mathbb{P}(\underline{0} \in \mathcal{C}_n)$ the volume fraction, and p_n the Poltyrev probability of decoding error (see [1]) in dimension n.

5.2. Thresholds in the Gaussian grain case

It is interesting to consider the case of Gaussian grains in detail as an illustration of the general results in this paper, and we turn to this next, giving, in the process, a rigorous derivation of the volume fraction threshold for this case, as discussed in the preceding subsection.

We first need to determine the large deviations rate function for the sequence $(\bar{X}_n, n \geq 1)$, with each $\bar{X}_n$ defined as in (2.3). Here we think of $\sigma^2 > 0$ as being fixed. It is easy to do this by first observing that $(\bar{X}_n^2, n \geq 1)$ satisfies the large deviations principle with rate function $J(\cdot)$ given by

$$J(z) = \begin{cases} \dfrac{z}{2\sigma^2} - \dfrac{1}{2} - \dfrac{1}{2}\log\dfrac{z}{\sigma^2} & \text{if } z > 0, \\ \infty & \text{otherwise.} \end{cases}$$

This follows from the fact that if Z_n is a Gaussian random variable with mean zero and variance σ^2 then

$$\log \mathbb{E}[\mathrm{e}^{\theta Z_n^2}] = \begin{cases} -\frac{1}{2}\log(1 - 2\theta\sigma^2) & \text{if } \theta < 1/2\sigma^2, \\ \infty & \text{otherwise.} \end{cases}$$

The contraction principle [3, Theorem 4.2.1] then gives the rate function of the sequence $(\bar{X}_n, n \geq 1)$ as being $I(\cdot)$, where

$$I(R) = \begin{cases} \dfrac{R^2}{2\sigma^2} - \dfrac{1}{2} - \dfrac{1}{2}\log\dfrac{R^2}{\sigma^2} & \text{if } R > 0, \\ \infty & \text{otherwise.} \end{cases}$$

The volume fraction threshold for the case of Gaussian grains (with $\sigma^2 > 0$ being fixed) is then given by finding the solution $R_{\mathrm{v}} \geq \sigma$ to the equality $R/\sigma^2 - 1/R = 1/R$. There is a unique solution to this equation, namely, $R_{\mathrm{v}} = \sigma\sqrt{2}$, and this turns out to satisfy $R_{\mathrm{v}} \geq \sigma$, as it should. Here σ is playing the role of R^* in the general theory. Substituting back into $\tau_{\mathrm{v}} = -\frac{1}{2}\log(2\pi\mathrm{e}) + I(R_{\mathrm{v}}) - \log(R_{\mathrm{v}})$ for the volume fraction threshold gives $\tau_{\mathrm{v}} = -\frac{1}{2}\log(2\pi\mathrm{e}\sigma^2) - \frac{1}{2}(\log 4 - 1)$. This is the announced rigorous derivation of the formula that was found above by the heuristic application of Laplace's principle.

The degree threshold is given by finding the solution $R_{\mathrm{d}} \geq \sigma$ to the equality $R/\sigma^2 - 1/R = 1/2R$. There is a unique solution to this equation, namely, $R_{\mathrm{d}} = \sigma\sqrt{3/2}$, and this turns out to satisfy $R_{\mathrm{d}} \geq \sigma$, as it should. Substituting back into $\tau_{\mathrm{d}} = -\frac{1}{2}\log(2\pi\mathrm{e}) + 2I(R_{\mathrm{d}}) - \log(2R_{\mathrm{d}})$ for the degree threshold gives $\tau_{\mathrm{d}} = -\frac{1}{2}\log(2\pi\mathrm{e}\sigma^2) - \frac{1}{2}(\log\frac{27}{2} - 1)$.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

The percolation probability threshold is given by finding the solution $R_{\mathrm{p}} \geq \sigma$ to the equality $R/\sigma^2 - 1/R = 1/(R+\sigma)$. There is a unique solution to this equation, namely, $R_{\mathrm{p}} = \sigma c$, where c is the unique root of the equation $c^3 + c^2 - 2c - 1 = 0$, which satisfies $c \geq 0$. In fact, this root satisfies $c > 1$. That there is a unique such root and that it satisfies $c > 1$ can be verified by noting that the expression on the left-hand side of this equation equals -1 at $c = 0$ and at $c = 1$, and goes to ∞ as $c \to \infty$ and, furthermore, the derivative in c of the expression is $3c^2 + 2c - 2$, which equals -2 at $c = 0$ and is a convex function. Substituting back into $\tau_{\mathrm{p}} = -\frac{1}{2}\log(2\pi \mathrm{e}) + I(R_{\mathrm{p}}) - \log(R_{\mathrm{p}} + \sigma)$ for the percolation probability threshold gives

$$\tau_{\mathrm{p}} = -\tfrac{1}{2}\log(2\pi \mathrm{e}\sigma^2) - \tfrac{1}{2}(\log(c^2(1+c)^2) - c^2 + 1).$$

Numerical evaluation of c gives $1.246\,979\,6 < c < 1.246\,979\,7$. This approximation suffices to verify that

$$\log\left(\tfrac{27}{2}\right) - 1 > \log(c^2(1+c)^2) - c^2 + 1 > \log(4) - 1,$$

which confirms that $\tau_{\mathrm{d}} \leq \tau_{\mathrm{p}} \leq \tau_{\mathrm{v}}$ in the case of Gaussian grains, as required by Theorem 3.4 (note that in this case, the inequalities are strict). This approximation also suffices to verify that

$$1 < \sqrt{\tfrac{3}{2}} < c < \sqrt{2} < 1 + c < \sqrt{6},$$

which confirms that $R^* \leq R_{\mathrm{d}} \leq R_{\mathrm{p}} \leq R_{\mathrm{v}} \leq R_{\mathrm{p}} + R^* \leq 2R_{\mathrm{d}}$, in the case of Gaussian grains, as required by Theorem 3.5.

Returning to the Boolean model with truncated Gaussian grains discussed in the last subsection, since, for every $\sigma^2 > 0$, this family of Boolean models is a deterministic model with $R^* = \sigma$, the rate function for this model satisfies $I(\sigma) = 0$, and $I(R) = \infty$ for all $R \neq \sigma$. Thus, in this case the deterministic threshold equals the percolation probability threshold, and both are $\log 2$ below the volume fraction threshold. It is interesting to note, as observed in [1] and [9], that this threshold also has a meaning; it is the threshold for the asymptotic normalized logarithmic intensity up to which the truncated Gaussian grain of any given point of the Poisson process is so small that, with probability asymptotically equal to 1, it does not meet the grain of any other point of the Poisson process. This feature, which relates to a study of pairwise conflict in decoding between two codewords, is central to Gallager's analysis of error exponents in the power constrained channels that are of interest to engineers; for more details, see Section 7.4 of [4] and, in particular, the study therein of what is called Gallager's E_0 function.

Acknowledgements

The research of the first author was supported by the ARO MURI grant W911NF- 08-1-0233, 'Tools for the Analysis and Design of Complex Multi-Scale Networks', the NSF grants CNS-0910702 and ECCS-1343398, and the NSF Science and Technology Center grant CCF-0939370, 'Science of Information'. The work of the second author was supported by an award from the Simons Foundation (# 197982 to The University of Texas at Austin). The authors thank the reviewer and E. O'Reilly for their comments.

References

[1] Anantharam, V. and Baccelli, F. (2015). Capacity and error exponents of stationary point processes under random additive displacements. *Adv. Appl. Prob.* **47,** 1–26.

[2] Daley, D. J. and Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes*, Vol. II, 2nd edn. Springer, New York.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press

[3] Dembo, A. and Zeitouni, O. (1993). *Large Deviation Techniques and Applications*. Jones and Bartlett, Boston, MA.
[4] Gallager, R. G. (1968). *Information Theory and Reliable Communication*. John Wiley, New York.
[5] Gouéré, J.-B. and Marchand, R. (2011). Continuum percolation in high dimensions. Preprint. Available at http://arxiv.org/abs/1108.6133.
[6] Last, G. and Thorisson, H. (2009). Invariant transports of stationary random measures and mass-stationarity. *Ann. Prob.* **37,** 790–813.
[7] Meester, R. and Roy, R. (1996). *Continuum Percolation*. Cambridge University Press.
[8] Penrose, M. D. (1996). Continuum percolation and Euclidean minimal spanning trees in high dimensions. *Ann. Appl. Prob.* **6,** 528–544.
[9] Poltyrev, G. (1994). On coding without restrictions for the AWGN channel. *IEEE Trans. Inf. Theory* **40,** 409–417.
[10] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press.
[11] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.
[12] Varadhan, S. R. S. (1984). *Large Deviations and Applications*. SIAM, Philadelphia, PA.

https://doi.org/10.1017/jpr.2016.60 Published online by Cambridge University Press