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Backward Error Analysis for Eigenproblems Involving Conjugate Symplectic Matrices

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Abstract. Conjugate symplectic eigenvalue problems arise in solving discrete linear-quadratic optimal control problems and discrete algebraic Riccati equations. In this article, backward errors of approximate pairs of conjugate symplectic matrices are obtained from their properties. Several numerical examples are given to illustrate the results.

AMS subject classifications: 65F10, 65F15, 65H10, 65L15.

Key words: Backward error, approximate eigenpairs, conjugate symplectic matrix.

1. Introduction

An important conjugate symplectic eigenvalue problem that arises in the solution of discrete linear-quadratic optimal control problems and discrete algebraic Riccati equations is formulated as follows [18–21]. Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices. A matrix $A\in\mathbb{C}^{m\times m}$ is said to be conjugate symplectic if $m=2n$ and $A^*JA=J$, where

$$J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Numerical algorithms for this eigenproblem have been proposed by many authors [2]. To judge the stability of the computed solution, one may introduce the backward error of an approximate eigenpair $(\mathbf{x},\lambda)$ of the matrix A, as a measure of the smallest perturbation E such that

$$(A+E)\mathbf{x}=\lambda\mathbf{x}\,,$$

and by comparing the backward error with the size of any uncertainties in the data matrix A determine whether the pair $(\mathbf{x},\lambda)$ solves the perturbed problem. Let us denote the trace, the

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conjugate, the transpose, the conjugate transformation and the Moore-Penrose generalised inverse of the matrix A by $tr(A)$, $\bar{A}$, A^T, A^*, and $A^\dagger$, respectively. The identity matrix of order m is denoted by I_m, and the Frobenius norm of a matrix is defined by

$$\|A\|_F = \sqrt{tr(A^*A)}\,.$$

We denote by P_A and $P_A^\perp$ the orthogonal projection onto the range $\mathscr{R}(A)$ of the matrix A and the projection complementary to P_A, respectively.

The normwise backward error of an eigenpair $(\mathbf{x}, \lambda)$ may be defined as

$$\eta(\mathbf{x}, \lambda) = \min_E \left\{ \alpha^{-1} \|E\|_F : (A+E)\mathbf{x} = \lambda \mathbf{x} \right\}, \tag{1.1}$$

where α is a given positive parameter that allows us some freedom in how the perturbations are measured. Backward errors of approximate solutions are very important, because they can be used to assess the stability and quality of numerical algorithms. Let us denote $X_k = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k)$, $\Lambda_k = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k)$, and the set of approximate eigenpairs by $\{(\mathbf{x}_j, \lambda_j),\ j = 1, \cdots, k\}$. The following definition of the backward error [1]

$$\eta(X_k, \Lambda_k) = \min_E \left\{ \alpha^{-1} \|E\|_F : (A+E)X_k = X_k \Lambda_k \right\}, \tag{1.2}$$

is a natural generalisation of (1.1).

It is notable that this definition is only suitable for unstructured matrix eigenpairs. Since many applications involve structured eigenproblem computation, a definition of the corresponding structured backward error is needed. Thus if $\mathscr{K}$ is a set of some classes of structured matrices, the structured backward error of approximate eigenpairs for the structured matrix class $\mathscr{K}$ may be defined by

$$\eta_{\mathscr{K}}(X_k, \Lambda_k) = \min_E \left\{ \alpha^{-1} \|E\|_F : (A+E)X_k = X_k \Lambda_k,\ A,\ A+E \in \mathscr{K} \right\}. \tag{1.3}$$

From Eqs. (1.2) and (1.3) it is easy to see that

$$\eta(X_k, \Lambda_k) \le \eta_{\mathscr{K}}(X_k, \Lambda_k).$$

Explicit expressions for structured backward error $\eta_{\mathscr{K}}(X_k, \Lambda_k)$ have been given for some particular classes $\mathscr{K}$ — viz.

- the set of unitary matrices [3];
- the set of Hermitian unitary matrices [1];
- the set of symplectic unitary matrices [1];
- the set of symmetric orthogonal matrices [1]; and
- the set of symmetric centrosymmetric and symmetric skew-centrosymmetric matrices, respectively [22].

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However, no explicit formula of $\eta_{\mathscr{K}}(X_k, \Lambda_k)$ has previously been given for the set of conjugate symplectic matrices

$$\mathscr{K}_1 = \left\{ A \in \mathbb{C}^{2n\times 2n} : A^* J A = J \right\}, \tag{1.4}$$

so the backward error of the approximate eigenpairs for this structured matrix class is considered here and an explicit formula of $\eta_{\mathscr{K}_1}(X_k, \Lambda_k)$ is presented. In Section 2, we give some useful lemmas to deduce the main results. In Section 3, the backward error of eigenproblem of conjugate symplectic matrices is discussed. Several numerical examples to illustrate our results are presented in Section 4, followed by our conclusion in Section 5.

2. Preliminary

The following lemmas are basic for the derivation of our main results presented in the next section.

Lemma 2.1 (cf. Ref. [16]). *Let matrices $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times q}$ and $D \in \mathbb{C}^{p\times m}$ be given and let*

$$S = \left\{ X \in \mathbb{C}^{m\times n} : \|AXB - C\|_F = \min \right\},$$
$$G_1 = \left\{ X \in \mathbb{C}^{m\times n} : AXB = C \right\}, \qquad G_2 = \left\{ X \in \mathbb{C}^{p\times p} : XA = D,\ X^* = X \right\}.$$

Then $G_1 \neq \emptyset$ if and only if $P_A C P_{B^*} = C$; $G_2 \neq \emptyset$ if and only if $DP_{A^*} = D$ and $(P_A D A^\dagger)^* = P_A D A^\dagger$ and if $G_1 \neq \emptyset$ then

$$G_1 = S = \left\{ A^\dagger C B^\dagger + Z - P_{A^*} Z P_B : Z \in \mathbb{C}^{m\times n} \ \ arbitrary \right\}.$$

Lemma 2.2. *For matrices $C \in \mathbb{C}^{m\times m}$, $X_1, X_2 \in \mathbb{C}^{p\times n}$ and $B_1, B_2 \in \mathbb{C}^{m\times n}$, there exists $A_{ls} \in \mathbb{C}^{m\times p}$ such that*

$$\|CA_{ls}X_1 - B_1\|_F^2 + \|CA_{ls}X_2 - B_2\|_F^2 = \min_{\forall A\in\mathbb{C}^{m\times p}} \left(\|CAX_1 - B_1\|_F^2 + \|CAX_2 - B_2\|_F^2 \right)$$

and

$$A_{ls} = C^\dagger (B_1, B_2)(X_1, X_2)^+ + Z - P_{C^*} Z P_{(X_1, X_2)},\ Z \in \mathbb{C}^{m\times p}\ arbitrary\,.$$

Proof. It is easy to see that

$$\|CAX_1 - B_1\|_F^2 + \|CAX_2 - B_2\|_F^2 = \|CA(X_1, X_2) - (B_1, B_2)\|_F^2\ ,$$

so the result follows immediately from Lemma 2.1. □

Lemma 2.3. *Suppose the matrices Y, $B \in \mathbb{C}^{2n\times m}$ have the block forms*

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

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where $Y_1,\ Y_2,\ B_1,\ B_2 \in \mathbb{C}^{n\times m}$. *Let*

$$\begin{aligned}\Phi =&\left\{A \in \mathscr{K}_1 : AY = B\right\},\\ \Psi =&\left\{A \in \mathbb{C}^{2n\times 2n} : A_1Y = B_1,\ A_2Y = B_2,\ Y^*(A_1^*A_2 + P)Y = B_1^*B_2 + Y_2^*Y_1,\right.\\ &\left.(A_1^*A_2 + P)^* = A_1^*A_2 + P\right\}.\end{aligned}$$

Then $\Phi = \Psi$, *where* $A_{ii} \in \mathbb{C}^{n\times n},\ i = 1,2$ *and*

$$P = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}, \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \equiv \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Proof. Assuming $A \in \Phi$, from $AY = B$ and the block forms of A, B and Y we have

$$A_1Y = B_1, \qquad A_2Y = B_2\ ; \tag{2.1}$$

and from the equation $A^*JA = J$ we have

$$\begin{aligned}&A_{11}^*A_{21} = A_{21}^*A_{11}\ ,\\ &A_{12}^*A_{22} = A_{22}^*A_{12}\ ,\\ &A_{11}^*A_{22} - A_{21}^*A_{12} = I_n\ ,\\ &A_{12}^*A_{21} - A_{22}^*A_{11} = -I_n\ .\end{aligned} \tag{2.2}$$

It is easy to see that the equalities in (2.2) can be formulated as

$$(A_1^*A_2 + P)^* = A_1^*A_2 + P\ ; \tag{2.3}$$

and from (2.1) we have

$$Y^*(A_1^*A_2 + P)Y = B_1^*B_2 + Y_2^*Y_1\ . \tag{2.4}$$

Combining (2.1), (2.3) and (2.4) yields $A \in \Psi$, and hence $\Phi \subseteq \Psi$. Now assume $A \in \Psi$. From the equalities $A_1Y = B_1$, $A_2Y = B_2$, $(A_1^*A_2 + P)^* = A_1^*A_2 + P$, one may deduce that $AY = B$ and $A^*JA = J$, implying $A \in \Phi$ and hence $\Psi \subseteq \Phi$ such that $\Psi = \Phi$. □

Lemma 2.4. *Let matrices* $Y \in \mathbb{C}^{2n\times m}$ *and* $B \in \mathbb{C}^{m\times m}$. *Then*

$$\Upsilon = \left\{Q \in \mathbb{C}^{2n\times 2n} : Y^*QY = B,\ Q^* = Q\right\} \neq \emptyset$$

if and only if $P_{Y^*}BP_{Y^*} = B$ *and* $Y^{*\dagger}BY^{\dagger} = Y^{*\dagger}B^*Y^{\dagger}$.

Proof. Let Y have the singular value decomposition (SVD):

$$Y = Udiag(\Sigma_Y, 0)V^*,$$

where $U \in \mathbb{C}^{2n\times 2n}$ and $V = (V_1, V_2) \in \mathbb{C}^{m\times m}$ are unitary, $V_1 \in \mathbb{C}^{m\times r}$, $V_2 \in \mathbb{C}^{m\times(m-r)}$ and $\Sigma_Y = \mathrm{diag}(\sigma_1, \cdots, \sigma_r), \sigma_i > 0,\ i = 1 : r,\ r = rank(Y)$.

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We first prove the necessity. If $\Upsilon \neq \emptyset$, assume that there is $Q_0 \in \mathbb{C}^{2n\times 2n}$ satisfying $Y^*Q_0Y = B$ and $Q_0^* = Q_0$. Let

$$X = U^*Q_0U = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad V^*BV = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{2.5}$$

where $X_{11},\ D_{11} \in \mathbb{C}^{r\times r}$. Substituting (2.5) into $Y^*Q_0Y = B$ and $Q_0^* = Q_0$ gives

$$\begin{pmatrix} \Sigma_Y X_{11}\Sigma_Y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

Then $D_{12} = D_{21} = D_{22} = 0\,,\ D_{11} = V_1^*BV_1 = \Sigma_Y X_{11}\Sigma_Y$ and hence

$$V^*BV = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

therefore

$$\begin{aligned} B =& V\begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}V^* = (V_1, V_2)\begin{pmatrix} V_1^*BV_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} \\ =& V_1V_1^*BV_1V_1^* = P_{Y^*}BP_{Y^*}\,. \end{aligned}$$

Since $Q_0^* = Q_0$, we have $X^* = X$ such that $(\Sigma_Y^{-1}D_{11}\Sigma_Y^{-1})^* = X_{11}^* = X_{11} = \Sigma_Y^{-1}D_{11}\Sigma_Y^{-1}$, so one may deduce that

$$Y^{*\dagger}BY^{\dagger} = Y^{*\dagger}B^*Y^{\dagger}\,.$$

Conversely, suppose that $P_{Y^*}BP_{Y^*} = B$ and $Y^{*\dagger}BY^{\dagger} = Y^{*\dagger}B^*Y^{\dagger}$ hold. Let

$$Q^{'} = U\begin{pmatrix} \Sigma_Y^{-1}D_{11}\Sigma_Y^{-1} & F \\ F^* & G \end{pmatrix}U^*, \tag{2.6}$$

where $F \in \mathbb{C}^{r\times(2n-r)}$ and $G \in \mathbb{C}^{(2n-r)\times(2n-r)}$ with $G^* = G$. From $P_{Y^*}BP_{Y^*} = B$ we have

$$B = V\begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}V^*.$$

Now substituting $Y = U\mathrm{diag}(\Sigma_Y, 0)V^*$ and (2.6) into $Y^*Q^{'}Y$ gives

$$Y^*Q^{'}Y = V\begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}V^* = B\,,$$

which with the assumption $Y^{*\dagger}BY^{\dagger} = Y^{*\dagger}B^*Y^{\dagger}$ gives $(\Sigma_Y^{-1}D_{11}\Sigma_Y^{-1})^* = \Sigma_Y^{-1}D_{11}\Sigma_Y^{-1}$, and hence $Q^{'*} = Q^{'}$ such that $Q^{'} \in \Upsilon$ — *i.e*, $\Upsilon \neq \emptyset$. □

Lemma 2.5. *Let matrices $B \in \mathbb{C}^{2n\times 2n}$. Then*

$$\Xi = \left\{(X, Y) :\ X \in C^{n\times 2n},\ Y \in C^{n\times 2n},\ X^*Y = B\right\} \neq \emptyset\,.$$

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Proof. Let B have the singular value decomposition (SVD)

$$B = Udiag(\Sigma_B, 0)V^*,$$

where $U \in \mathbb{C}^{2n\times 2n}$ and $V \in \mathbb{C}^{2n\times 2n}$ are unitary, $\Sigma_B = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$, $\sigma_i > 0$, $i = 1 : r$, where $r = rank(Y)$. Thus there exists $\Sigma_{B_1}, \Sigma_{B_2} \in \mathscr{C}^{r\times r}$ such that $\Sigma_B = \Sigma_{B_1}\Sigma_{B_2}$. Let

$$X = W\begin{pmatrix} \Sigma_{B_1} & 0_{r\times(2n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(2n-r)} \end{pmatrix}U^*, \qquad Y = W\begin{pmatrix} \Sigma_{B_2} & 0_{r\times(2n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(2n-r)} \end{pmatrix}V^*,$$

where $W \in \mathbb{C}^{n\times n}$ is unitary. Then it is easy to see that $X^*Y = B$, which implies the desired result. □

Lemma 2.6 (cf. Ref. [1]). *Let matrices X, $B \in \mathbb{C}^{n\times k}$, Y, $C \in \mathbb{C}^{m\times k}$ be given, and*

$$S_k = \left\{A \in \mathbb{C}^{n\times m} : AY = B,\ A^*X = C\right\}.$$

If $C^*Y = X^*B$, then S_k can be expressed as

$$S_k = \left\{BY^\dagger + (CX^\dagger)^*P_Y^\perp + P_X^\perp H P_Y^\perp,\ H \in \mathbb{C}^{n\times m}\right\}.$$

Lemma 2.7. *Let matrices Φ and Ψ be given by Lemma 2.3. Then $\Phi = \Psi \neq \emptyset$ if and only if*

$$\begin{aligned} &BP_{Y^*} = B\,, \\ &P_{Y^*}(B_1^*B_2 + Y_2^*Y_1)P_{Y^*} = B_1^*B_2 + Y_2^*Y_1\,, \end{aligned} \tag{2.7}$$

and

$$Y^{*\dagger}(B_1^*B_2 + Y_2^*Y_1)Y^\dagger = Y^{*\dagger}(B_2^*B_1 + Y_1^*Y_2)Y^\dagger. \tag{2.8}$$

Proof. Let

$$\begin{aligned} \tilde{\Psi} = \Big\{(A_1, A_2) : Y^*(A_1^*A_2 + P)Y = B_1^*B_2 + Y_2^*Y_1\,, (A_1^*A_2 + P)^* = A_1^*A_2 + P\,, \\ A_1 \in \mathbb{C}^{n\times 2n},\ A_2 \in \mathbb{C}^{n\times 2n}\Big\}, \\ \Psi_1 = \left\{A_1 \in \mathbb{C}^{n\times 2n} : A_1Y = B_1\right\}, \qquad \Psi_2 = \left\{A_2 \in \mathbb{C}^{n\times 2n} : A_2Y = B_2\right\}, \end{aligned}$$

where Y, A_1, A_2, B_1, B_2, Y_1, Y_2 and P are given in Ψ by Lemma 2.3.
We first prove the necessity. Assuming $\Psi \neq \emptyset$, then $\tilde{\Psi} \neq \emptyset, \Psi_1 \neq \emptyset$ and $\Psi_2 \neq \emptyset$. It follows from Lemmas 2.1 and 2.4 that (2.7) and (2.8) hold.
Conversely, assume that (2.7) and (2.8) are true. It follows from Lemma 2.1 that $\Psi_1 \neq \emptyset$ and $\Psi_2 \neq \emptyset$; and from Lemmas 2.4 and 2.5 that $\tilde{\Psi} \neq \emptyset$. From Lemma 2.1 we then have

$$A_1^*A_2 + P = Y^{*\dagger}(B_1^*B_2 + Y_2^*Y_1)Y^\dagger + Z - P_YZP_Y\,, \tag{2.9}$$

where $Z \in \mathbb{C}^{2n\times 2n}$ satisfies $Z - P_YZP_Y = Z^* - P_YZ^*P_Y$. Multiplying (2.9) on the right by Y yields

$$A_1^*B_2 = Y^{*\dagger}(B_1^*B_2 + Y_2^*Y_1)P_{Y^*} + P_Y^\perp ZY - PY\,. \tag{2.10}$$

Since $[Y^{*\dagger}(B_1^*B_2 + Y_2^*Y_1)P_{Y^*} + P_Y^\perp ZY - PY]^*Y = B_2^*B_1$, from Lemma 2.6, (2.10) and $A_1Y = B_1$ one can deduce $\Psi \neq \emptyset$. □

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3. Expression for $\eta_{\mathscr{K}_1}(X_k,\Lambda_k)$

We now present the explicit formula for the structured backward error $\eta_{\mathscr{K}_1}(X_k,\Lambda_k)$ when $\mathscr{K}_1$ is given by (1.5). Consider the matrices $A\in\mathbb{C}^{2n\times 2n}$, $X_k\in\mathbb{C}^{2n\times k}$, $\Lambda_k=\mathrm{diag}(\lambda_1,\cdots,\lambda_k)$, $2n\geq k$ and $S_{\mathscr{K}_1}=\{A+E\in\mathscr{K}_1:(A+E)X_k=X_k\Lambda_k\}$. We partition X_k and $X_k\Lambda_k$ as

$$X_k=\begin{pmatrix} W_1\\ W_2\end{pmatrix},\ X_k\Lambda_k=\begin{pmatrix} E_1\\ E_2\end{pmatrix}, \tag{3.1}$$

where $W_1,\ W_2,\ E_1,\ E_2\in\mathbb{C}^{n\times k}$. In order to deduce $\eta_{\mathscr{K}_i}(X_k,\Lambda_k)$, it is necessary to obtain $S_{\mathscr{K}_1}\neq\emptyset$, so the first call is to provide a necessary and sufficient condition such that $S_{\mathscr{K}_1}\neq\emptyset$.

Lemma 3.1. *$S_{\mathscr{K}_1}\neq\emptyset$ if and only if the following equalities hold:*

$$X_k\Lambda_kP_{X_k}^*=X_k\Lambda_k\,, \tag{3.2}$$

$$P_{X_k^*}(E_1^*E_2+W_2^*W_1)P_{X_k^*}=E_1^*E_2+W_2^*W_1\,, \tag{3.3}$$

$$(X_k\Lambda_kX_k^\dagger)^*JX_k\Lambda_kX_k^\dagger=P_{X_k}JP_{X_k}\,. \tag{3.4}$$

Proof. Let X_k and $X_k\Lambda_k$ have the block form (3.1). It follows from Lemma 2.7 that $S_{\mathscr{K}_1}\neq\emptyset$ if and only if (3.2), (3.3) and

$$X_k^{*\dagger}(E_1^*E_2+W_2^*W_1)X_k^\dagger=X_k^{*\dagger}(E_2^*E_1+W_1^*W_2)X_k^\dagger \tag{3.5}$$

hold. From (3.1), we then have that (3.5) holds if and only if (3.4) holds. □

Let us now consider

$$P=\begin{pmatrix}0&0\\ I_n&0\end{pmatrix}\in\mathbb{C}^{2n\times 2n},\qquad A=\begin{pmatrix}A_{11}&A_{12}\\ A_{21}&A_{22}\end{pmatrix}\equiv\begin{pmatrix}A_1\\ A_2\end{pmatrix}, \tag{3.6}$$

where $A_{ij}\in\mathbb{C}^{n\times n}$, $A_i\in\mathbb{C}^{n\times 2n}$, $i,j=1,2$. The following theorem provides the explicit formula for $\eta_{\mathscr{K}_1}(X_k,\Lambda_k)$.

Theorem 3.1. *Let $A\in\mathscr{K}_1$, $X_k\in\mathbb{C}^{2n\times k}$ and $\Lambda_k\in\mathbb{C}^{k\times k}$, $2n\geq k$. Suppose* (3.2), (3.3) *and* (3.4) *are satisfied. Then*

$$E_{opt}=P_{X_k}^{\perp}P_{X_k}^{\perp^\dagger}(\Omega_1,\Omega_2)(X_kE_2^\dagger,X_kE_1^\dagger)^\dagger(X_kE_2^\dagger,X_kE_1^\dagger)-(\Omega_1,\Omega_2)\,; \tag{3.7}$$

and for any $\alpha>0$,

$$\eta_{\mathscr{K}_1}(X_k,\Lambda_k)=\alpha^{-1}\|E_{opt}\|_F\,, \tag{3.8}$$

where

$$\Omega_1=A_1^*+P_{X_k}^{\perp}PX_kE_2^\dagger-X_k^{*\dagger}E_1^*-P_{X_k}^{\perp}P_{X_k}^{\perp\dagger}A_1^*P_{E_2}^{\perp}P_{E_2}^{\perp^\dagger}\,,$$

$$\Omega_2=A_2^*+P_{X_k}^{\perp}P^*X_kE_1^\dagger-X_k^{*\dagger}E_2^*-P_{X_k}^{\perp}P_{X_k}^{\perp\dagger}A_2^*P_{E_1}^{\perp}P_{E_1}^{\perp^\dagger}\,,$$

with P and A_i given by (3.6).

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Proof. We partition $A+E$ into the block form

$$A+E=\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}=\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix},$$

where $\tilde{A}_1, \tilde{A}_2 \in \mathbb{C}^{n\times 2n}$ and $\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}, \tilde{A}_{22} \in \mathbb{C}^{n\times n}$. Let

$$\widetilde{S}_{\mathscr{K}_1}=\Big\{A+E\in\mathbb{C}^{2n\times 2n}: \tilde{A}_1X_k=E_1,\ \tilde{A}_2X_k=E_2, X_k^*(\tilde{A}_1^*\tilde{A}_2+P)X_k=E_1^*E_2+W_2^*W_1,\\ (\tilde{A}_1^*\tilde{A}_2+P)^*=\tilde{A}_1^*\tilde{A}_2+P\Big\}, \tag{3.9}$$

where P is given by (3.6). It follows from Lemmas 2.3 and 3.1 that $S_{\mathscr{K}_1}=\widetilde{S}_{\mathscr{K}_1}\neq\emptyset$, hence

$$\mathscr{L}_1=\Big\{\tilde{A}_1\in\mathbb{C}^{n\times 2n}:\tilde{A}_1X_k=E_1\}\neq\emptyset\,,\qquad \mathscr{L}_2=\{\tilde{A}_2\in\mathbb{C}^{n\times 2n}:\tilde{A}_2X_k=E_2\Big\}\neq\emptyset\,,$$
$$\mathscr{H}=\Big\{\tilde{A}_1^*\tilde{A}_2+P\in\mathbb{C}^{2n\times 2n}: X_k^*(\tilde{A}_1^*\tilde{A}_2+P)X_k=E_1^*E_2+W_2^*W_1\,,(\tilde{A}_1^*\tilde{A}_2+P)^*=\tilde{A}_1^*\tilde{A}_2+P\Big\}$$
$$\neq\emptyset.$$

Applying Lemma 2.1 to the equation $X_k^*(\tilde{A}_1^*\tilde{A}_2+P)X_k=E_1^*E_2+W_2^*W_1$ yields

$$\tilde{A}_1^*\tilde{A}_2+P=X_k^{*\dagger}(E_1^*E_2+W_2^*W_1)X_k^{\dagger}+Z-P_{X_k}ZP_{X_k}\,,\quad Z\in\mathbb{C}^{2n\times 2n}\text{ arbitrary}\,, \tag{3.10}$$

which implies that

$$\tilde{A}_1^*\tilde{A}_2=X_k^{*\dagger}(E_1^*E_2+W_2^*W_1)X_k^{\dagger}+Z-P_{X_k}ZP_{X_k}-P\,. \tag{3.11}$$

From (3.4) and $(\tilde{A}_1^*\tilde{A}_2+P)^*=\tilde{A}_1^*\tilde{A}_2+P$, one may deduce that

$$(Z-P_{X_k}ZP_{X_k})^*=Z-P_{X_k}ZP_{X_k}\,,$$

and hence

$$(Z^*-P_{X_k}Z^*P_{X_k})X_k=(Z-P_{X_k}ZP_{X_k})X_k\,,$$

which implies that

$$P_{X_k}^{\perp}Z^*X_k=P_{X_k}^{\perp}ZX_k\,. \tag{3.12}$$

Right-multiplying both sides of (3.11) by X_k then yields

$$\tilde{A}_1^*E_2=X_k^{*\dagger}(E_1^*E_2+W_2^*W_1)P_{X_k^*}-PX_k+ZX_k-P_{X_k}ZX_k\,. \tag{3.13}$$

Applying Lemma 2.6 to (3.13) and $\tilde{A}_1X_k=E_1$ gives

$$\tilde{A}_1^*=T_1+P_{X_k}^{\perp}ZX_kE_2^{\dagger}+P_{X_k}^{\perp}Z_2P_{E_2}^{\perp}\,, \tag{3.14}$$

where $T_1=-P_{X_k}^{\perp}PX_kE_2^{\dagger}+X_k^{*\dagger}E_1^*$, $Z_2\in\mathbb{C}^{2n\times n}$ arbitrary.
Using a similar technique, we may deduce that

$$\tilde{A}_2=T_2+E_1^{*\dagger}X_k^*ZP_{X_k}^{\perp}+P_{E_1}^{\perp}Z_2'P_{X_k}^{\perp}, \tag{3.15}$$

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where $T_2 = E_2 X_k^{\dagger} - E_1^{*\dagger} X_k^* P P_{X_k}^{\perp}$, $Z_2' \in \mathbb{C}^{n\times 2n}$ arbitrary. Hence

$$\begin{aligned}\|E\|_F^2 =& \|A+E-A\|_F^2 = \left\| \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} - \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \right\|_F^2 \\ =& \|\tilde{A}_1 - A_1\|_F^2 + \|\tilde{A}_2 - A_2\|_F^2 . \end{aligned} \tag{3.16}$$

Substituting (3.14) and (3.15) into (3.16) gives

$$\begin{aligned}\min_{Z_2,Z_2',Z} \|E\|_F^2 =& \min_{Z} \left\{ \min_{Z_2} \left\| P_{X_k}^{\perp} Z_2 P_{E_2}^{\perp} - (A_1^* - T_1 - P_{X_k}^{\perp} Z X_k E_2^{\dagger}) \right\|_F^2 \right. \\ & \left. + \min_{Z_2'} \left\| P_{E_1}^{\perp} Z_2' P_{X_k}^{\perp} - (-E_1^{*\dagger} X_k^* Z P_{X_k}^{\perp} - T_2 + A_2) \right\|_F^2 \right\} . \end{aligned} \tag{3.17}$$

Applying Lemma 2.1 to (3.17) then yields

$$\begin{aligned}\min \|E\|_F^2 =& \min_{Z} \left\| P_{X_k}^{\perp} P_{X_k}^{\perp\dagger} (A_1^* - T_1 - P_{X_k}^{\perp} Z X_k E_2^{\dagger}) P_{E_2}^{\perp\dagger} P_{E_2}^{\perp} - A_1^* + T_1 + P_{X_k}^{\perp} Z X_k E_2^{\dagger}) \right\|_F^2 \\ & + \min_{Z} \left\| P_{E_1}^{\perp} P_{E_1}^{\perp\dagger} (-E_1^{*\dagger} X_k^* Z P_{X_k}^{\perp} - T_2 + A_2) P_{X_k}^{\perp\dagger} P_{X_k}^{\perp} + E_1^{*\dagger} X_k^* Z P_{X_k}^{\perp} + T_2 - A_2) \right\|_F^2 \\ =& \min_{Z} \left[\left\| P_{X_k}^{\perp} Z X_k E_2^{\dagger} - \Omega_1 \right\|_F^2 + \left\| P_{X_k}^{\perp} Z^* X_k E_1^{\dagger} - \Omega_2 \right\|_F^2 \right] , \end{aligned} \tag{3.18}$$

where

$$\begin{aligned}\Omega_1 =& A_1^* + P_{X_k}^{\perp} P X_k E_2^{\dagger} - X_k^{*\dagger} E_1^* - P_{X_k}^{\perp} P_{X_k}^{\perp +} A_1^* P_{E_2}^{\perp} P_{E_2}^{\perp^{\dagger}} , \\ \Omega_2 =& A_2^* + P_{X_k}^{\perp} P^* X_k E_1^{\dagger} - X_k^{*\dagger} E_2^* - P_{X_k}^{\perp} P_{X_k}^{\perp\dagger} A_2^* P_{E_1}^{\perp} P_{E_1}^{\perp^{\dagger}} . \end{aligned}$$

From (3.12) and (3.18) we have

$$\min \|E\|_F^2 = \min_{Z} \left[\left\| P_{X_k}^{\perp} Z X_k E_2^{\dagger} - \Omega_1 \right\|_F^2 + \left\| P_{X_k}^{\perp} Z X_k E_1^{\dagger} - \Omega_2 \right\|_F^2 \right] , \tag{3.19}$$

so applying Lemma 2.2 to (3.19) gives (3.7) and hence (3.8). □

4. Numerical Examples

We now consider several numerical examples in order to assess the stability of the symplectic Jacobi algorithm [2], by comparing the backward error $\eta_{\mathscr{K}_1}(X_k, \Lambda_k)$ with the unstructured backward error $\tilde{\eta}(X_k, \Lambda_k)$ for

$$A = \begin{pmatrix} \mathbf{i} \cdot I_r & -\mathbf{i} \cdot I_r \\ \mathbf{i} \cdot I_r & O_{r\times r} \end{pmatrix} \in \mathscr{K}_1 \subseteq \mathbb{C}^{2r\times 2r}$$

where $\mathbf{i} \equiv \sqrt{-1}$.

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Example 4.1. Let $r = 2$. Then the symplectic Jacobi algorithm in Ref. [2] gives

$$\Lambda_4 = \mathrm{diag}(1.0010 + 0.0001i, 1.0008 - 0.0002i, 0.9860 + 0.0010i, 0.9902 - 0.0021i)$$

and

$$X_4 = \begin{pmatrix} a & a & 0 & 0 \\ 0 & 0 & a & b \\ b & c & 0 & 0 \\ 0 & 0 & b & a \end{pmatrix},$$

where $a = 0.1983 + 0.2011i$, $b = 0.8126$ and $c = 0.6946 - 0.0195i$. From (3.1),

$$X_4 = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \qquad X_4\Lambda_4 = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}.$$

It is easy to verify that

$$X_4\Lambda_4 P_{X_4}^* = X_4\Lambda_4$$
$$= \begin{pmatrix} 0.1985 + 0.2013i & 0.1985 + 0.2013i & 0 & 0 \\ 0 & 0 & 0.1985 + 0.2012i & 0.8133 - 0.0002i \\ 0.8012 + 0.0008i & 0.6849 - 0.0185i & 0 & 0 \\ 0 & 0 & 0.8046 - 0.0017i & 0.1968 + 0.1987i \end{pmatrix},$$
$$P_{X_4^*}(E_1^*E_2 + W_2^*W_1)P_{X_4^*} = E_1^*E_2 + W_2^*W_1$$
$$= \begin{pmatrix} 0.3203 + 0.0023i & 0.2934 + 0.0219i & 0 & 0 \\ 0.2930 - 0.0175i & 0.2660 + 0.0020i & 0 & 0 \\ 0 & 0 & 0.3205 + 0.0012i & 0.7394 - 0.0002i \\ 0 & 0 & 0.7341 - 0.0012i & 0.3212 - 0.0018i \end{pmatrix},$$

and

$$(X_4\Lambda_4 X_4^\dagger)^* J X_4\Lambda_4 X_4^\dagger = P_{X_4} J P_{X_4} \doteq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

so (3.2)-(3.4) are satisfied and from Theorem 3.1 we obtain

$$\eta_{\mathscr{K}_1}(X_4, \Lambda_4) = \frac{2.39278300279382 \times 10^{-8}}{\alpha}.$$

It is also easy to compute

$$\tilde{\eta}(X_4, \Lambda_4) = \frac{2.39278300279263 \times 10^{-8}}{\alpha},$$

so that

$$\tilde{\eta}(X_4, \Lambda_4) \le \eta_{\mathscr{K}_1}(X_4, \Lambda_4) \le \sqrt{2}\tilde{\eta}(X_4, \Lambda_4).$$

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Example 4.2. Let $r = 3$, when by the symplectic Jacobi algorithm [2] we have

$$\Lambda_6 = \mathrm{diag}(1.0071 + 0.0001i, 0.9800 - 0.0002i, 0.9927 + 0.0021i,$$
$$1.0009 - 0.0006i, 0.9894 - 0.0001i, 1.0004),$$
$$X_6 = \begin{pmatrix} a & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & a & a \\ b & c & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & c \end{pmatrix},$$

where $a = 1.7026, b = 0.0216 - 0.0185i$ and $c = 0.8208 + 0.3117i$. From (3.1),

$$X_6 = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \qquad X_6\Lambda_6 = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix},$$

and it is easy to verify that

$$X_6\Lambda_6 P_{X_6}^* = X_6\Lambda_6 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1 & b_2 \\ b_3 & b_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & c_4 \end{pmatrix},$$

$a_1 = 1.7147 + 0.0002i$, $a_2 = 1.6685 - 0.0003i$, $a_3 = 1.6902 + 0.0036i$, $a_4 = 0.0216 - 0.0185i$, $b_1 = 1.6846 - 0.0002i$, $b_2 = 1.7033$, $b_3 = 0.0218 - 0.0186i$, $b_4 = 0.8044 + 0.3053i$, and $c_1 = 0.0215 - 0.0183i$, $c_2 = 1.7041 - 0.0010i$, $c_3 = 0.0214 - 0.0183i$, $c_4 = 0.8211 + 0.3118i$;

$$P_{X_6^*}(E_1^*E_2 + W_2^*W_1)P_{X_6^*} = E_1^*E_2 + W_2^*W_1, = \begin{pmatrix} d_1 & d_2 & 0 & 0 & 0 & 0 \\ d_3 & d_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & e_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & f_2 \\ 0 & 0 & 0 & 0 & f_3 & f_4 \end{pmatrix},$$

$d_1 = 0.0742 - 0.0004i$, $d_2 = 1.4161 + 0.5548i$, $d_3 = 1.4339 - 0.5617i$, $d_4 = 2.7391 - 0.0212i$, $e_1 = 0.0730 + 0.0005i$, $e_2 = 2.8811 - 0.0078i$, $e_3 = 2.8996 + 0.0000i$, $e_4 = 0.0736 + 0.0000i$, $f_1 = 0.0728 + 0.0007i$, $f_2 = 1.4199 + 0.5569i$, $f_3 = 1.4339 - 0.5619i$, $f_4 = 2.7961 + 0.0004i$; and

$$(X_6\Lambda_6 X_6^\dagger)^* J X_6\Lambda_6 X_6^\dagger = P_{X_6} J P_{X_6} \doteq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

This implies that (3.2)-(3.4) are satisfied, and from Theorem 3.1 we obtain

$$\eta_{\mathscr{K}_1}(X_6, \Lambda_6) = \frac{4.48920492728992 \times 10^{-7}}{\alpha}.$$

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It is also easy to compute

$$\tilde{\eta}(X_6, \Lambda_6) = \frac{4.48920492728940 \times 10^{-7}}{\alpha},$$

so that

$$\tilde{\eta}(X_6, \Lambda_6) \le \eta_{\mathcal{K}_1}(X_6, \Lambda_6) \le \sqrt{2}\tilde{\eta}(X_6, \Lambda_6).$$

Example 4.3. Let $r = 4$, when by the symplectic Jacobi algorithm [2] we have

$$\Lambda_8 = \mathrm{diag}(0.9970 + 0.0002i, 1.0061 - 0.0003i, 0.9667 + 0.0011i, 0.9846 - 0.0026i, 0.9081 + 0.0015i, 0.9882 + 0.0010i, 0.9754 - 0.0005i, 1.006 + 0.0040i)$$

and

$$X_8 = \begin{pmatrix} a & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & a \end{pmatrix},$$

where $a = 0.6846$, $b = 0.5908 + 0.3023i$, $c = 0.7849 - 0.3170i$. From (3.1) we have

$$X_8 = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \qquad X_8\Lambda_8 = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix},$$

and it is easy to verify that

$$X_8\Lambda_8 P_{X_8}^* = X_8\Lambda_8 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & c_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_3 & d_4 \end{pmatrix},$$

$a_1 = 0.6825 + 0.0001i$, $a_2 = 0.6888 - 0.0002i$, $a_3 = 0.6618 + 0.0008i$, $a_4 = 0.5825 + 0.2961i$, $b_1 = 0.6217 + 0.0010i$, $b_2 = 0.6765 + 0.0007i$, $b_3 = 0.6678 - 0.0003i$, $b_4 = 0.5931 + 0.3065i$, $c_1 = 0.5890 + 0.3015i$, $c_2 = 0.7896 - 0.3192i$, $c_3 = 0.5708 + 0.2929i$, $c_4 = 0.6741 - 0.0018i$, $d_1 = 0.5361 + 0.2754i$, $d_2 = 0.7760 - 0.3125i$, $d_3 = 0.5764 + 0.2946i$, $d_4 = 0.6887 + 0.0027i$;

$$P_{X_8^*}(E_1^*E_2 + W_2^*W_1)P_{X_8^*} = E_1^*E_2 + W_2^*W_1, = \begin{pmatrix} e_1 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_3 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_5 & f_1 & 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 & f_4 & 0 & f_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & g_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_3 & g_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_4 \end{pmatrix},$$

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$e_1 = 0.8065-0.0012i, e_2 = 0.9433-0.4249i, e_3 = 0.9430+0.4248i, e_4 = 1.0813-0.0027i$, $e_5 = 0.7825 - 0.0136i$, $f_1 = 0.4404$, $f_2 = 0.3780 + 0.1934i$, $f_3 = 0.8879 + 0.0016i$, $f_4 = 0.4045 + 0.2070i$, $f_5 = 0.4192 + 0.0016i$, $g_1 = 0.7380 - 0.0363i$, $g_2 = 0.8866 - 0.4020i$, $g_3 = 0.9002 + 0.4030i$, $g_4 = 1.0621 + 0.0051i$, $h_1 = 0.7893 - 0.0100i$, $h_2 = 0.9003 + 0.0020i$, $h_3 = 0.9008 - 0.0019i$, $h_4 = 0.8138 - 0.0025i$; and

$$(X_8\Lambda_8X_8^\dagger)^*JX_8\Lambda_8X_8^\dagger = P_{X_8}JP_{X_8} \doteq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

This implies that (3.2)-(3.4) are satisfied, and from Theorem 3.1 we obtain

$$\eta_{\mathscr{K}_1}(X_8, \Lambda_8) = \frac{1.805211578430026 \times 10^{-5}}{\alpha}.$$

It is also easy to compute

$$\tilde{\eta}(X_8, \Lambda_8) = \frac{1.805211578430019 \times 10^{-5}}{\alpha}.$$

It follows that

$$\tilde{\eta}(X_8, \Lambda_8) \le \eta_{\mathscr{K}_1}(X_8, \Lambda_8) \le \sqrt{2}\tilde{\eta}(X_8, \Lambda_8).$$

Finally, when A is some stochastic conjugate symplectic matrix, from the symplectic Jacobi algorithm [2] and a simple calculation we see that the above inequalities still hold, and hence we have the structured stability and stability of the numerical algorithm.

All of the above runs were done in MATLAB 7.0 on a 1.86 GHZ CPU and 1022 MB memory computer, in floating-point arithmetic with machine precision 10^{-16}.

5. Conclusion

Explicit formulae on backward errors for approximate eigenpairs of conjugate symplectic matrices have been derived. In general, we consider

$$\tilde{\eta}(X_k, \Lambda_k) \le \eta_{\mathscr{K}_1}(X_k, \Lambda_k),$$

where $\tilde{\eta}(X_k, \Lambda_k)$ and $\eta_{\mathscr{K}_1}(X_k, \Lambda_k)$ denote the unstructured backward error and the structured backward error, respectively. If $\eta_{\mathscr{K}_1}(X_k, \Lambda_k)$ is small, then the approximate solution (X_k, Λ_k) derived by the numerical algorithm is structurally stable. If there exists a small constant c such that $\eta_{\mathscr{K}_1}(X_k, \Lambda_k) \le c\tilde{\eta}(X_k, \Lambda_k)$, then this implies backward stability. However, it is theoretically difficult to find a constant c such that $\eta_{\mathscr{K}_1}(X_k, \Lambda_k) \le c\tilde{\eta}(X_k, \Lambda_k)$ — cf. expressions (3.7) and (3.8). This is an issue to be studied in future. If we assess the backward stability of the approximate solutions of a numerical algorithm, we shall need to compare $\tilde{\eta}(X_k, \Lambda_k)$ and $\eta_{\mathscr{K}_1}(X_k, \Lambda_k)$.

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