Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2018.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

101.I (Michel Bataille)

Let a, b, c be the side-lengths of a triangle ABC, r its inradius and R its circumradius. Prove that

$$\sqrt{\frac{2r}{R}} \leqslant \frac{\sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)}}{a+b+c} \leqslant 1.$$

101.J (Isaac Sofair)

Squares are described externally on the sides of a convex quadrilateral. By Van Aubel's theorem, the line segments joining the centres of opposite squares are equal and perpendicular to each other.

(a) Prove that the length of each line segment is $\sqrt{2S + \frac{1}{2}(x^2 + y^2)}$, where S is the area of the quadrilateral and x, y are the lengths of its diagonals.

(b) Show that $\dfrac{A}{S} \geqslant 2$, where A is the area of the quadrilateral whose vertices are the centres of the squares. When does equality hold?

101.K (Abdurrahim Yilmaz)

Consider a square $ABCD$ with centre O, circumradius r and a point P in the plane of the square with $|OP| = R$. Let $\alpha, \beta, \gamma, \delta$ denote the areas of the triangles with side-lengths $|PA|, |PB|, |PC|$; $|PA|, |PB|, |PD|$; $|PA|, |PC|, |PD|$ and $|PB|, |PC|, |PD|$ respectively, whenever these triangles exist. Prove that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \tfrac{1}{4}\left[3(R^2 + r^2)^2 - 10R^2r^2\right].$$

101.L (Finbarr Holland)

Suppose G and K denote the centroid and the Lemoine point of a triangle ABC. Prove that, unless ABC is equilateral, at least one of the ratios

$$\frac{|AG|}{|AK|}, \frac{|BG|}{|BK|}, \frac{|CG|}{|CK|}$$

exceeds 1.

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Solutions and comments on **101.A, 101.B, 101.C, 101.D** (March 2017).

101.A (Martin Lukarevski)

Let P be a point inside the triangle ABC and let x, y, z denote the perpendicular distances from P to the sides BC, AC, AB respectively. Let R and r denote the circumradius and inradius of triangle ABC.

Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} \leqslant (R + r)\sqrt{\dfrac{2}{R}}$.

Solutions to this popular geometrical inequality generally proceeded along the following lines.

With the usual notation, let a, b, c be the side-lengths of the triangle ABC and Δ its area. By the Cauchy-Schwarz inequality

$$\sqrt{x}+\sqrt{y}+\sqrt{z} = \sqrt{\frac{ax}{a}} + \sqrt{\frac{by}{b}} + \sqrt{\frac{cz}{c}} \leqslant \sqrt{ax+by+cz}\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

so that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leqslant \sqrt{\frac{2\Delta}{abc}}\sqrt{ab + bc + ca} = \frac{1}{\sqrt{2R}}\sqrt{ab + bc + ca} \quad (*)$$

since $\Delta = \dfrac{abc}{4R}$.

By expanding the brackets in Heron's formula and using $abc = 4Rrs$, we quickly obtain $ab + bc + ca = s^2 + r^2 + 4rR$. We now invoke Gerretsen's inequality, $s^2 \leqslant 4R^2 + 4rR + 3r^2$, which also featured in the solution to **98.B** (November 2014, p. 539). This gives

$$ab + bc + ca \leqslant 4R^2 + 8rR + 4r^2 = 4(R + r)^2$$

and substituting into (*) then gives the required result.

Since $R \geqslant 2r$ (Euler), the inequality in **101.A** implies that $\sqrt{x} + \sqrt{y} + \sqrt{z} \leqslant 3\sqrt{\dfrac{R}{2}}$. As Peter Nüesch observed, this was Problem 959 proposed by L. Carlitz in *Mathematics Magazine* **48** (1975), p. 294.

Correct solutions were received from: M. Bataille, S. Dolan, GCHQ Problem Solving Group, P. F. Johnson, P. Kitchenside, P. Nüesch, Z. Retkes, L. Wimmer and the proposer Martin Lukarevski.

101.B (Joseph Tonien)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f\big(f(n) + 2017\big) + 2017 = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

Several respondents gave specific examples of functions satisfying this functional equation, quickly appreciating the proposer's intention that there is considerable freedom in generating solutions.

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Let $g(n) = f(n) + 2017$. Then $g\big(g(n)\big) = 0$ (*) and it suffices to find all such functions g. Clearly the zero function is a solution. Otherwise, let $A = g^{-1}(0)$ and $B = g(\mathbb{Z})$. Then we have:

- $0 \in A$, since $g\big(g(0)\big) = 0$ so that $g(0) = g\big(g\big(g(0)\big)\big) = 0$ by (*);
- $B \subseteq A$, by(*);
- $g(\mathbb{Z} - A) \subseteq A - \{0\}$, for if $y \notin A$, then $g(y) \in B - \{0\} \subseteq A - \{0\}$.

Conversely, for any proper subset A with $\{0\} \subset A \subset \mathbb{Z}$, define

$$g(n) = \begin{cases} 0, \text{ if } n \in A, \\ \text{any non-zero element of } A, \text{ if } n \in \mathbb{Z} - A. \end{cases}$$

Then g is not the zero function and
either $n \in A$ so that $g(n) = 0$ and, since $0 \in A$, $g\big(g(n)\big) = 0$
or $n \in \mathbb{Z} - A$ so that $g(n) \in A$ and $g\big(g(n)\big) = 0$.
Thus $g\big(g(n)\big) = 0$ for all $n \in \mathbb{Z}$ as required.

As the proposer notes, the more general equation $f\big(f(n) + a\big) + b = 0$ succumbs to a similar analysis.

Correct solutions were received from: R. G. Bardelang, M. Bataille, M. V. Channakeshava, C. Davis, S. Dolan, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, G. B. Trustrum and the proposer Joseph Tonien.

101.C (Michael Fox)

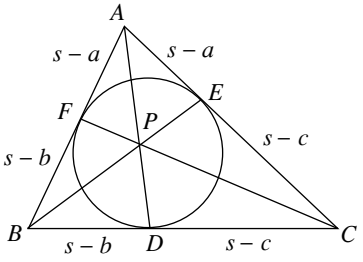
An ellipse inscribed in triangle ABC touches BC at D, CA at E and AB at F.

(i) Prove that AD, BE and CF pass through a common point P.

(ii) Given that the areas PBC, PCA and PAB are in the ratios $\alpha : \beta : \gamma$, determine what fraction of area ABC is occupied by the ellipse.

Almost all solvers applied an affine transformations to transform the ellipse into a circle. Such a transformation preserves concurrency, tangency and the ratio of corresponding lengths on lines and areas. It thus suffices to deal with the case of a triangle and its incircle.

(i)



With the usual notation, lengths are as shown in the diagram. Then

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA} = \frac{s-a}{s-b}\cdot\frac{s-b}{s-c}\cdot\frac{s-c}{s-a} = 1$$

so, by the converse of Ceva's theorem, AD, BE, CF are concurrent at P (the Gergonne point of triangle ABC).

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(ii) The area ratios $ABD : ADC = s - b : s - c = PBD : PDC$ give, by subtraction, $PAB : PCA = s - b : s - c$. Similarly $PCA : PBC = s - a : s - b$ so that $\alpha : \beta : \gamma = \dfrac{1}{s-a} : \dfrac{1}{s-b} : \dfrac{1}{s-c}$ and

$$s - a : s - b : s - c : s = \frac{1}{\alpha} : \frac{1}{\beta} : \frac{1}{\gamma} : \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

The fraction of a triangle ABC occupied by its incircle is $\dfrac{\pi r^2}{\Delta} = \dfrac{\pi\Delta}{s^2}$ where $\Delta = rs$ is the area of triangle ABC. By Heron's formula,

$$\frac{\pi\Delta}{s^2} = \frac{\pi\sqrt{(s-a)(s-b)(s-c)}}{s^{3/2}} = \frac{\pi\left(\frac{1}{\alpha\beta\gamma}\right)^{1/2}}{\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^{3/2}} = \frac{\pi\alpha\beta\gamma}{(\beta\gamma+\gamma\alpha+\alpha\beta)^{3/2}}.$$

Correct solutions were received from: R. G. Bardelang, M. V. Channakeshava, S. Dolan, G. Howlett, P. F. Johnson, J. A. Mundie, P. Nüesch, C. Starr, G. B. Trustrum and the proposer Michael Fox.

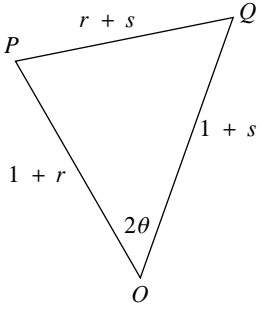
101.D (Paul Stephenson)

A circle of unit radius is surrounded by a ring of $n \geqslant 3$ other circles (possibly of differing radii), each of which touches the unit circle and two others. The centres of adjacent circles in the ring are joined by straight lines to form a polygon with n sides – a *ring n-gon*.

Prove that a regular p-gon has the smallest perimeter of all ring n-gons with $n \leqslant p$.

Stan Dolan found the following succinct and elegant resolution of this intriguing problem.

Let the unit circle have centre O and let P, Q be the centres of adjacent circles in the ring with radii r, s and $\angle POQ = 2\theta < \pi$, as shown in the diagram. Then $r + s \geqslant (1 + r)\sin\theta + (1 + s)\sin\theta$ gives $r + s \geqslant \dfrac{2\sin\theta}{1 - \sin\theta}$.



Applying this result to the n circles in the ring whose adjacent centres subtend angles of $2\theta_1, \ldots, 2\theta_n$ at O with $0 < \theta_i < \frac{1}{2}\pi$ and $\sum_{i=1}^{n} \theta_i = \pi$ shows that the perimeter of the ring n-gon, L, satisfies

$$L \geqslant 2\sum_{i=1}^{n} \frac{\sin\theta_i}{1 - \sin\theta_i}.$$

Since $\dfrac{\sin\theta}{1 - \sin\theta}$ is convex for $0 < \theta < \frac{1}{2}\pi$, Jensen's inequality shows that

$$L \geqslant \frac{2n\sin\frac{\pi}{n}}{1 - \sin\frac{\pi}{n}},$$

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where the right-hand side is the perimeter of a regular ring n-gon. But $\dfrac{\sin\theta}{\theta(1-\sin\theta)}$ is increasing for $0 < \theta < \frac{1}{2}\pi$ so that

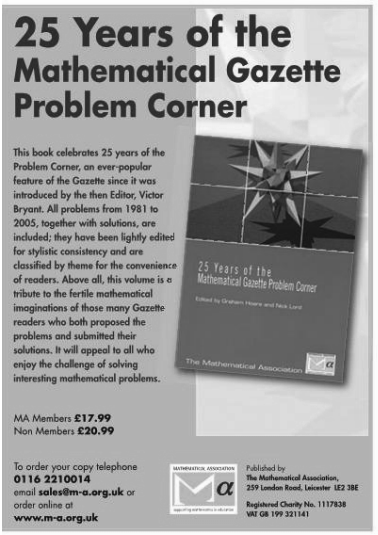
$$L \geqslant \frac{2n\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}} \geqslant \frac{2p\sin\frac{\pi}{p}}{1-\sin\frac{\pi}{p}}$$

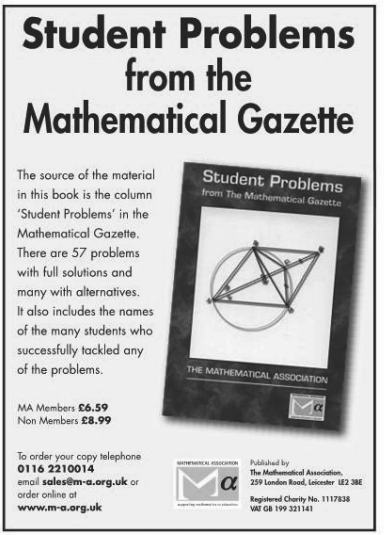
for $3 \leqslant n \leqslant p$ which clinches the result.

Correct solutions were received from: S. Dolan, C. Starr and the proposer Paul Stephenson.

Readers will be saddened by the news of the death of John Chick in June 2017. Since the 1970s he contributed to the *Gazette* in many ways — as author, referee, book reviewer, problem setter and problem solver. His Note 'Another angle on angle?' appears in this issue (pp. 502-508) and typifies his careful and meticulous style. He also regularly attended the M. A. Annual Conference where his genial charm and erudition will be greatly missed.

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