J. Plasma Physics (2007), *vol.* 73, *part* 2, *pp.* 179–188. © 2006 Cambridge University Press
doi:10.1017/S0022377806004442 First published online 4 May 2006 Printed in the United Kingdom

Steady nonlinear electrostatic plasma wave in a weak transverse magnetic field

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(Received 16 December 2005)

Abstract. The structure of a stationary electrostatic plasma wave propagating at a right angle to a weak magnetic field is studied. It is shown that the periodic finite amplitude wave is close in its physical structure to Bernstein–Greene–Kruskal wave of a perfectly definite type. The distinguishing feature of such a nonlinear wave is the absence of the resonant particles trapped by the wave. The electron distribution function, density perturbation and the shape of the wave electrostatic potential are found. The nonlinear dispersion relation is derived to determine the frequency shift due to the perturbation of the distribution function in the resonant region.

1. Introduction

The presence of a weak transverse magnetic field modifies essentially electrostatic wave–particle interaction in a collisionless plasma (Sagdeev and Shapiro 1973). A search for exact solutions of the initial value problem of the wave damping in the magnetic field is accompanied by significant mathematical difficulties due to the rather complicated nature of the particle dynamics described by 'non-integrable' equations of motion (Karney 1979; Chernikov et al. 1987). Nevertheless, there has been a considerable increase in understanding of the physics of the wave damping (Karney 1979, Sugihara and Midzuno 1979; Dawson et al. 1983; Zaslavsky et al. 1986; Domier et al. 1989; Eliasson et al. 2005; Valentini et al. 2005). It has been established that at the early stage of the interaction, the major contribution to the damping is given by particles trapped by the wave and accelerated along the wave front under the action of Lorentz force. As time progresses these particles escape the potential energy wells of the wave, so that at a later stage the damping is caused by passing (untrapped) particles interacting efficiently with the wave during the short-term crossings of the resonant region in the course of their slow Larmor rotation (Sagdeev and Shapiro 1973; Zaslavsky et al. 1986). The damping of the finite amplitude wave due to the interaction with passing particles vanishes in the infinitesimal magnetic field limit. Consequently, steady nonlinear waves must be close in structure to equilibrium Bernstein–Greene–Kruskal (BGK) waves (Bernstein et al. 1957).

In this paper, the steady periodic electrostatic wave in a weak perpendicular magnetic field is considered within the framework of kinetic theory. It is shown that the availability of the weak magnetic field, as a small parameter of the problem, allows one to determine a unique charged particle distribution function. As a result,

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the structure of the nonlinear wave becomes quite definitive in contrast to the uncertainty characteristic of the broad class of BGK waves. As this takes place, a nonlinear dispersion relation taking into account the specific structure of the distribution function in the resonant region can be established similarly to the work by Krasovsky (1992, 1995).

The plan of the paper is as follows. In Sec. 2 the solution of Vlasov kinetic equation is found in the weak magnetic field limit. In Sec. 3 the charged particle density is calculated as the sum of the contributions of resonant and non-resonant particles. The solution of Poisson equation and the derivation of the nonlinear dispersion law are presented in Sec. 4. Finally, Sec. 5 summarizes the main results and conclusions.

2. Electron distribution function

Let us consider a periodic electrostatic wave propagating in a collisionless plasma along the x-axis perpendicularly to a weak magnetic field B_0 directed along the z-axis. For brevity, we will use the following units of measurement to describe the physical variables and parameters

$$[t] = [\omega]^{-1} = \omega_{\mathrm{p}}^{-1}, \quad [x] = [\lambda] = [k]^{-1} = v_{\mathrm{Te}}/\omega_{\mathrm{p}}, \quad [v] = [u] = v_{\mathrm{Te}},$$

$$[\phi] = mv_{\mathrm{Te}}^2/e, \quad [n] = n_0, \quad h \equiv \omega_{\mathrm{H}}/\omega_{\mathrm{p}} \ll 1, \tag{1}$$

where $\omega_{\mathrm{p}} = (4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency, $\omega_{\mathrm{H}} = eB_0/mc$ is the electron gyrofrequency, v_{Te} is the electron thermal velocity, n_0 is the undisturbed plasma density, ϕ is the electrostatic potential of the wave, λ, k, u and $\omega = ku$ are the wavelength, wavenumber, velocity and frequency of the wave, respectively. In the units (1) the finite amplitude wave is described by the Vlasov kinetic equation

$$\frac{\partial f}{\partial t} + V_x \frac{\partial f}{\partial x} + \left(\frac{\partial \phi}{\partial x} - hV_y\right)\frac{\partial f}{\partial V_x} + hV_x \frac{\partial f}{\partial V_y} = 0, \tag{2}$$

where f is the electron distribution function integrated over the velocity V_z, and by the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = \tilde{n}_{\mathrm{e}} = n_{\mathrm{e}} - 1, \quad n_{\mathrm{e}} = \iint dV_x\, dV_y f, \tag{3}$$

where n_{e} is the electron density and the ions are assumed to be immobile, so that the ion density is constant $n_{\mathrm{i}} = 1$.

It should be particularly emphasized that there is an essential qualitative difference in the dynamics of the trapped and passing (untrapped) electrons and, respectively, in their contribution to the damping of the wave. As is shown by Sagdeev and Shapiro (1973), the trapped particles oscillating in the potential energy wells are accelerated in the y-direction perpendicularly to the magnetic field. As a result of the energy exchange with these particles, the wave experiences a peculiar nonlinear damping. Hence the steady finite amplitude waves cannot exist in the presence of the trapped electrons. However, as time goes on, these particles escape the potential wells because, starting at a certain time, the growing projection of the Lorentz force on the x-direction cannot be balanced by the electrostatic force. Consequently, phase density voids must arise in the trapping region of phase space. Therefore, the steady wave may be formed at the final stage of the wave–particle interaction after the electron detrapping. On this basis, the following treatment of

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the stationary waves implies the absence of the trapped electrons. Thus the electron density in the Poisson equation (3) is entirely determined by the contribution of the untrapped electrons.

Let us look for periodic solutions of the equations (2) and (3) of the type of a stationary wave propagating with the constant velocity u, i.e. $f = f(x-ut, V_x, V_y) = f(\xi, V_x, V_y)$, $\phi = \phi(x-ut) = \phi(\xi)$. In the frame of reference moving with the wave $\xi = x - ut$, $v = V_x - u$ the kinetic equation (2) takes the time-independent form

$$v\frac{\partial f}{\partial \xi} + \left(\frac{\partial \phi}{\partial \xi} - hV_y\right)\frac{\partial f}{\partial v} + h(u+v)\frac{\partial f}{\partial V_y} = 0. \tag{4}$$

Going over to the total electron energy $W = v^2/2 - \phi(\xi)$ as to a new independent variable, similarly to the well-known BGK technique, we get

$$\frac{\partial f}{\partial \xi} = h\left[V_y\frac{\partial f}{\partial W} - \left(\frac{u}{v_\pm} + 1\right)\frac{\partial f}{\partial V_y}\right], \tag{5}$$

where $v_\pm \equiv \pm\sqrt{2[W+\phi(\xi)]}$ and the $\pm$ signs correspond to the particles moving respectively faster or slower than the wave in the longitudinal direction. When $h = 0$, the distribution function is obviously an arbitrary function of the variables W and V_y. Hence, at small but finite h, it is natural to look for the solution of (5) by expanding the distribution function in $h \ll 1$

$$f = F_0(W, V_y) + hF_1(W, V_y, \xi) + \cdots. \tag{6}$$

Substitution (6) into (5) yields to the first order in h

$$\frac{\partial F_1}{\partial \xi} = V_y\frac{\partial F_0}{\partial W} - \left(\frac{u}{v_\pm} + 1\right)\frac{\partial F_0}{\partial V_y}. \tag{7}$$

Averaging (7) over the wavelength and taking into account the periodicity of the distribution function, $\langle \partial F_1/\partial \xi \rangle = 0$, we find the equation relative to F_0

$$V_y\frac{\partial F_0}{\partial W} - \frac{\partial F_0}{\partial V_y}\left[1 + u\left\langle\frac{1}{v_\pm(\xi, W)}\right\rangle\right] = 0, \tag{8}$$

where the angle brackets denote the average value

$$\langle \dots \rangle \equiv \int_0^\lambda \frac{d\xi}{\lambda}(\dots). \tag{9}$$

Solution of (8) is an arbitrary function of the variable

$$U_\pm = \frac{u^2 + V_y^2}{2} + W + u\langle v_\pm \rangle + \frac{c_\pm}{2}, \tag{10}$$

where $c_\pm = c_\pm\{\phi\}$ are the integration constants representing, generally speaking, some functionals of the waveform $\phi(\xi)$.

An explicit form $F_0(U_\pm)$ must be consistent with some undisturbed distribution in the absence of the wave, $\phi \to 0$. For definiteness, consider, for example, Maxwellian distribution $f_0 = (1/\pi)\exp[-(V_x^2 + V_y^2)]$. Then, the electron distribution function to the leading order in h is given by

$$f = F_0 = (1/\pi)\exp(-2U_\pm). \tag{11}$$

'Sewing together' of the distribution functions $F_0(U_+) = F_0(U_-)$ for positive and negative velocities $v = v_\pm$ at the separatrix, $W = W_s = -\phi_{\min}$, dividing the

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phase space regions corresponding to the passing and trapped particles yields the interconnection between c_+ and c_-

$$c_- - c_+ = 4u\langle\sqrt{2[W_s + \phi(\xi)]}\rangle, \tag{12}$$

so that (10) may be rewritten in the form

$$U_\pm = \frac{u^2 + V_y^2}{2} + W \pm u\langle\sqrt{2[W + \phi(\xi)]} - \sqrt{2[W_s + \phi(\xi)]}\rangle + \frac{c}{2}. \tag{13}$$

The unknown functional $c = c\{\phi\}$ may be determined on the basis of the constancy of the average electron density (see Sec. 3).

Physically, the distribution function (11) approximated by the first term of the expansion (6), i.e. by the zeroth spatial harmonic of the distribution function in the variables V_y, W and ξ, is in essence a distribution average over phases of particles relative to the wave. This approximation is well justified in the limit $h \to 0$, when particles undergo rapid phase mixing in the process of their comparatively slow Larmor gyration. The distribution function defined by (11) and (13) may also be treated as an analog of the 'ergodic' distribution arising as a result of the damping of a finite amplitude wave (O'Neil 1965). If necessary, calculations of the higher-order terms in (6) may also be carried out. In particular, the correction of order h^2, which is typical of upper hybrid plasma oscillations, may be found. However, it is of little importance for our purposes.

Thus the presence of the small perturbation proportional to $h \ll 1$ in the right-hand side of the kinetic equation (5) makes it possible to select the unique solution from the broad class of the BGK modes. A situation similar to some degree arises in the study of the damping of a finite amplitude wave under the action of rare collisions (Zakharov and Karpman 1962), when the collision frequency plays the role of the small parameter. Another way to find the realizable solutions of the BGK type is to solve an initial value problem of the time (space) evolution of the finite amplitude wave (O'Neil 1965; Krasovsky 1995).

3. Electron density

Knowing the distribution function (11), one can calculate the electron density n_e entering the Poisson equation (3). In the course of the calculation the unknown constant c in (13) can also be determined with the aid of the plasma neutrality condition $\langle n_e\rangle = n_i = 1$. Going to the integration variable W in (3), we obtain

$$n_e = \int_{-\infty}^{\infty} dV_y \int_{W_s}^{\infty} \frac{dW(f_+ + f_-)}{\sqrt{2(W + \phi)}}, \tag{14}$$

where $f_\pm = (1/\pi)\exp(-2U_\pm)$ according to (11), and the lower limit of integration $W_s = -\phi_{\min}$ excludes the phase space region corresponding to the trapped electrons since their distribution function is assumed to be zero $f_T = 0$.

The electrostatic potential of the wave may be conveniently represented in the form

$$\phi = \phi_{\max} - (\phi_{\max} - \phi_{\min})\alpha(\xi) = 2A(\langle\alpha\rangle - \alpha), \tag{15}$$

where $A = (\phi_{\max} - \phi_{\min})/2$ is the wave amplitude and $\alpha = \alpha(\xi)$, $(0 \leqslant \alpha \leqslant 1)$, is the function describing the shape of the wave. In accordance with (15), $\langle\phi\rangle \equiv 0$, $\phi_{\max} = 2A\langle\alpha\rangle$ and $\phi_{\min} = 2A(\langle\alpha\rangle - 1)$.

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Excluding strongly nonlinear waves from consideration, hereafter we assume that $A \ll 1$. In doing so, the calculation of the integral (14) can be simplified by expanding in powers of $A^{1/2}$. With the use of the new variable $w = W - W_s$ in (13) and (14) the distribution function may be rewritten in the form

$$f_\pm = (1/\sqrt{\pi}) g_\pm(w) \exp[-(u^2 + V_y^2 + \langle 2\delta \rangle + c)], \tag{16}$$

where

$$g_\pm = (1/\sqrt{\pi}) \exp[-(2w \pm 2u\langle \sqrt{2(w+\delta)} - \sqrt{2\delta} \rangle)], \tag{17}$$

and $\delta \equiv 2A(1-\alpha) \ll 1$ is small. Then, upon substitution (16) and integration over V_y, the equation (14) yields

$$n_e = \exp[-(u^2 + \langle 2\delta \rangle + c)] \int_0^\infty \frac{dw(g_+ + g_-)}{\sqrt{2(w+\delta)}}. \tag{18}$$

The integral in (18) may be calculated by parts to give

$$n_e \exp[(u^2 + \langle 2\delta \rangle + c)] = 2I - 2(2\delta/\pi)^{1/2}, \tag{19}$$

where

$$I = I^+ + I^-, \tag{20}$$

and

$$I^\pm = \int_0^\infty dw g_\pm(w) \sqrt{2(w+\delta)} \left(1 \pm \left\langle \frac{u}{\sqrt{2(w+\delta)}} \right\rangle\right). \tag{21}$$

The second term in the right-hand side of (19) is associated with the void or 'electron hole' in the phase space region corresponding to the trapped particles. The contribution of the electron hole to the electron density is negative and proportional to the typical width of the hole in velocity space $\Delta v \sim \delta^{1/2} \sim A^{1/2}$.

To calculate the first term in the right-hand side of (19), it is convenient to break up the range of integration into two parts

$$I = I_R + I_N, \tag{22}$$

where $I_R = I_R^+ + I_R^-$, $I_N = I_N^+ + I_N^-$, and

$$I_N^\pm = \int_{A^{1/2}}^\infty dw g_\pm(w) \sqrt{2(w+\delta)} \left(1 \pm \left\langle \frac{u}{\sqrt{2(w+\delta)}} \right\rangle\right), \tag{23}$$

$$I_R^\pm = \int_0^{A^{1/2}} dw g_\pm(w) \sqrt{2(w+\delta)} \left(1 \pm \left\langle \frac{u}{\sqrt{2(w+\delta)}} \right\rangle\right). \tag{24}$$

Then, in the 'non-resonant' part $I_N^\pm$ the square roots $\sqrt{2(w+\delta)}$ in (17) and (23) can be expanded since $w > A^{1/2} \gg \delta \sim A$ at $A \ll 1$ everywhere in the range of integration, whereas in the 'resonant' part $I_R^\pm$ the distributions $g_\pm$ can be expanded near $w = 0$ taking into account the narrowness of the range of integration in the limit $A \to 0$. A similar procedure has already been demonstrated previously (Krasovsky 1995). Because of this, omitting the simple but somewhat lengthy mathematical treatment, we here write out only the final result. With the proviso that the wave velocity is well above the electron thermal velocity $u \gg 1$, the expansion of the integrals (23) and (24) up to the terms of the order of A leads to the following

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expressions

$$I_N = (4/3)(2/\pi)^{1/2}A^{3/4}(2u^2-1) + \exp(u^2)[1/2 - \langle\sqrt{2\delta}\rangle u\Phi(u) - \langle\delta\rangle(2u-1)$$
$$+ 2u^2\langle\sqrt{\delta}\,\rangle^2 + (\langle\delta\rangle - \delta)(1/2u^2)(1+3/2u^2)], \tag{25}$$

$$I_R = -(4/3)(2/\pi)^{1/2}A^{3/4}(2u^2-1) + (8/\sqrt{\pi})u^2A\sqrt{1-\alpha}, \tag{26}$$

where

$$\Phi(u) \equiv 1 - (2/\sqrt{\pi})\int_u^\infty dV\exp(-V^2).$$

The terms proportional to $A^{3/4}$ are connected with the artificially introduced boundary $w = A^{1/2}$ and cancel one another in the sum (22). Substituting (25) and (26) into (22), we find with the help of (19)

$$n_e\exp(c + \langle 2\delta\rangle) = 1 - 2\sqrt{2\delta}f_0(u)(1-4u^2A^{1/2}) - 2\langle\sqrt{2\delta}\rangle u\Phi(u) - 2\langle\delta\rangle(2u-1)$$
$$+ 4u^2\langle\sqrt{\delta}\rangle^2 + (\langle\delta\rangle - \delta)P(u), \tag{27}$$

where $f_0(u) = (1/\sqrt{\pi})\exp(-u^2)$ is the value of the undisturbed distribution function integrated over V_y in the resonance $V_x = u$ and

$$P(u) \simeq \frac{1}{u^2}\left(1 + \frac{3}{2u^2}\right)$$

is the leading order terms of the expansion in powers of $1/u^2 \ll 1$ of the principal value Vlasov integral well known in linear plasma wave theory

$$P(u) \equiv P.V.\int_{-\infty}^{\infty}\frac{dV_x}{V_x - u}\left(\frac{\partial f_0}{\partial V_x}\right). \tag{28}$$

Averaging (27) over the wavelength according to (9) and taking into account the plasma neutrality $\langle n_e\rangle = 1$, we obtain the equation for the functional c

$$\exp(c + 2\langle\delta\rangle) = 1 - 2\langle\sqrt{2\delta}\rangle[f_0(u)(1-4u^2A^{1/2}) + u\Phi(u)] + 2\langle\delta\rangle + 4u^2\langle\sqrt{\delta}\rangle^2. \tag{29}$$

Expansion of the left-hand side leads to the approximate expression

$$c \simeq -2u\langle\sqrt{2\delta}\rangle = -4uA^{1/2}\langle\sqrt{1-\alpha(\xi)}\rangle, \tag{30}$$

where small corrections proportional to the products $Au^2f_0(u)$, $A^{1/2}f_0(u)$ and $A^{1/2}[1-\Phi(u)] \ll 1$ have been omitted.

Now that we have found c, the electron distribution function becomes completely determined by (11) and by the expressions following from (13)

$$2U_+ = u^2 + V_y^2 + 2w + 2\langle\delta + u\sqrt{2(w+\delta)} - 2u\sqrt{2\delta}\rangle, \tag{31}$$

$$2U_- = u^2 + V_y^2 + 2w + 2\langle\delta - u\sqrt{2(w+\delta)}\rangle, \quad \delta = 2A(1-\alpha), \tag{32}$$

provided the shape of the wave $\alpha(\xi)$ is known. The waveform $\alpha(\xi)$ will be found in Sec. 4 with the aid of the Poisson equation (3).

It should be noted that the expression (30) takes into account only the resonant electron contribution proportional to the typical width of the resonant region in velocity space $\Delta v \sim A^{1/2}$. Generally, the non-resonant plasma component contributes to the value c along with the resonant electrons. However, the non-resonant contribution is very small, as may be shown $c = -\langle\phi^2\rangle P(u) \sim A^2$, and here we will

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not take into account this weakly nonlinear effect associated with the background plasma electrons.

To solve the Poisson equation (3), we need to calculate the electron density perturbation $\tilde{n}_e = n_e - 1$. This can be done by subtracting (29) from (27), using (30) and substituting $\delta = 2A(1-\alpha)$ into the resulting expression. In doing so, we find

$$\tilde{n}_e = 2AP(u)(\alpha - \langle\alpha\rangle) + 4A^{1/2} f_0(u)(\langle\sqrt{1-\alpha}\rangle - \sqrt{1-\alpha}), \tag{33}$$

where the negligible term proportional to the small product of $Af_0(u)$ has been dropped out. The density perturbation (33) consists of the linear contribution of the non-resonant particles and contribution of the electron phase space hole in the resonant region due to the absence of the trapped electrons. The deficit of particles in the resonant region is proportional to the value of the distribution function in the resonance $f_0(u)$ and to the width of the trapping region $\Delta v \sim A^{1/2}$.

4. Nonlinear dispersion relation

Now let us consider spatial dependence of the wave electrostatic potential. Substituting (15) and (33) into (3), we come to the Poisson equation in the form

$$\frac{d^2\alpha}{d\psi^2} + p(\alpha - \langle\alpha\rangle) + q(\langle\sqrt{1-\alpha}\rangle - \sqrt{1-\alpha}) = 0, \tag{34}$$

where $\psi = k\xi = k(x-ut)$ is the phase of the wave, $p \equiv P(u)/k^2 = 1 - \epsilon_L(u,k) \simeq (ku)^{-2}(1+3/2u^2)$, $\epsilon_L(u,k)$ is the linear plasma dielectric constant, λ and $k = 2\pi/\lambda$ are the wavelength and wavenumber respectively and $q \equiv 2f_0(u)/k^2A^{1/2}$. Owing to the periodicity of the wave $\alpha(\psi + 2\pi) = \alpha(\psi)$ and the evenness of the wave profile $\alpha(-\psi) = \alpha(\psi)$ it will suffice to solve (34) at $0 \leqslant \psi \leqslant \pi$.

Multiplication by $d\alpha/d\psi$ and integration of (34) leads to

$$\frac{1}{2}\left(\frac{d\alpha}{d\psi}\right)^2 + p\alpha\left(\frac{\alpha}{2} - \langle\alpha\rangle\right) + \frac{2}{3}q(1-\alpha)^{3/2} + q\alpha\langle\sqrt{1-\alpha}\rangle = C, \tag{35}$$

where C is the integration constant. The boundary conditions at the point of the electrostatic potential maximum (15) $\psi = 0$, $\alpha = 0$, $d\alpha/d\psi = 0$ and at the minimum $\psi = \pi$, $\alpha = 1$, $d\alpha/d\psi = 0$ yield the following interrelations

$$C = 2q/3, \quad q\langle\sqrt{1-\alpha}\rangle = 2q/3 - p(1/2 - \langle\alpha\rangle), \tag{36}$$

so that (35) takes the form

$$\frac{1}{p}\left(\frac{d\alpha}{d\psi}\right)^2 = (1-\alpha)[\alpha + \epsilon(1-\sqrt{1-\alpha})], \tag{37}$$

where $\epsilon \equiv 4q/3p = 8f_0(u)/3A^{1/2}P(u)$. This equation can be integrated to give

$$\begin{aligned} p^{1/2}\psi &= \int_0^\alpha \frac{dt}{\sqrt{(1-t)[1+\epsilon(1-\sqrt{1-t})]}} \\ &= 4\int_0^{\sqrt{1-\sqrt{1-\alpha}}} \frac{d\theta}{\sqrt{2+\epsilon-\theta^2}} \\ &= 4\arcsin\sqrt{(1-\sqrt{1-\alpha})/(2+\epsilon)}, \end{aligned} \tag{38}$$

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where the integration variable t has been changed for $\theta = \sqrt{1-\sqrt{1-t}}$. Substituting $\psi = \pi$, $\alpha = 1$ into (38), we arrive at the interconnection between p and ϵ

$$(\pi/4)p^{1/2} = \arcsin(1/\sqrt{2+\epsilon}), \tag{39}$$

which may be treated as a nonlinear dispersion law. According to (38) and (39) the function describing the shape of the wave is given by

$$\alpha = (2+\epsilon)\sin^2\varphi[2-(2+\epsilon)\sin^2\varphi], \tag{40}$$

$$\varphi \equiv (\psi/\pi)\arcsin(1/\sqrt{2+\epsilon}). \tag{41}$$

The average values $\langle\alpha\rangle$ and $\langle\sqrt{1-\alpha}\rangle$ entering the expression (36) can be calculated with the help of (38)

$$\langle\alpha\rangle = (1/8)[(2+\epsilon)(2-3\epsilon) + 3\epsilon\sqrt{1+\epsilon}/\arcsin(1/\sqrt{2+\epsilon})], \tag{42}$$

$$\langle\sqrt{1-\alpha}\rangle = (1/2)[\sqrt{1+\epsilon}/\arcsin(1/\sqrt{2+\epsilon}) - \epsilon]. \tag{43}$$

Finally, the values $\phi_{\max}$, $\phi_{\min}$ and c may be determined with the aid of (15) and (30). Thus we have completely described the structure of the wave.

In the limit $\epsilon \to 0$, the influence of the electron holes on the structure of the wave is negligible. In that case we arrive at the usual result of linear plasma wave theory, i.e. the wave is harmonic $\alpha = (1/2)(1-\cos\psi)$, $\phi = A\cos[k(x-ut)]$, and in the units (1) the frequency $\omega = ku$ satisfies the linear dispersion relation

$$\epsilon_{\mathrm{L}}(\omega, k) = 1 - P(\omega/k)/k^2 \simeq 1 - (1/\omega^2)(1 + 3k^2/2) = 0. \tag{44}$$

According to (15), (39) and (40), for small but finite $\epsilon \ll 1$ the wave becomes slightly anharmonic

$$\phi \simeq A\cos\psi + \epsilon A[3/\pi + \cos\psi - 2\cos(\psi/2) + (\psi/\pi)\sin\psi], \quad (0 \leqslant \psi \leqslant 2\pi), \tag{45}$$

where $\epsilon \simeq 8f_0(u)/3k^2A^{1/2}$, and the wave velocity obeys the nonlinear dispersion relation $1 - P(u)/k^2 - 2\epsilon/\pi = 0$ or

$$\epsilon_{\mathrm{NL}}(u, k, A) = 1 - \frac{1}{(ku)^2}\left(1 + \frac{3}{2u^2}\right) - \left(\frac{16}{3\pi}\right)\frac{f_0(u)}{k^2A^{1/2}} = 0. \tag{46}$$

It follows from (46) that the dimensionless frequency of the wave $\omega = ku$ is determined by the equation

$$\omega^2 \simeq 1 + \frac{3}{2}k^2 + \left(\frac{16}{3\pi}\right)\frac{f_0(1/k)}{k^2A^{1/2}}, \tag{47}$$

where the second term is the well-known thermal correction and the third term is associated with the electron phase density holes. Thus the deficit of the trapped electrons manifests itself in the correction to the wave frequency $\Delta\omega \simeq (8/3\pi)f_0/k^2A^{1/2}$.

It is particularly remarkable that the nonlinear frequency shift becomes significant in the small amplitude limit. The reason is that the resonant particles moving with comparatively small velocities in the wave frame have a small inertia, resulting in a greater contribution to screening process than the contribution of nonresonant particles. The contribution of the resonant particles to the plasma density perturbation is proportional to the width of the resonant region, $n_{\mathrm{R}} \sim \Delta v \sim A^{1/2}$, while the linear plasma response to the electrostatic perturbation is proportional

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to the first power of the amplitude $n_N \sim A \ll 1$ [see (33)]. Therefore, the resonant region of phase space may contribute significantly to the electrostatic structure of the wave in spite of the small particle density in this region. Because of this, the dispersion properties of the steady wave are very sensitive to a specific behaviour of the distribution function near the resonance, in particular, to the presence of the voids in the trapping region of phase space. The essential contribution of the trapped particles to the charge density perturbation and the effect of their deficit on the screening also explain the possibility of the existence of localized one-dimensional electrostatic perturbations of the type of BGK solitons (see e.g. Krasovsky et al. 2003 and references therein).

5. Conclusions

The above analysis clarifies the structure of finite amplitude electrostatic plasma waves in a weak transverse magnetic field. Such stationary waves exist in the absence of trapped particles, which undergo acceleration along the wave front and subsequently escape the potential energy wells. These waves are close in their physical structure to the well-known BGK waves of a particular form and may be described by a similar set of equations. However, the weak magnetic field, as a small disturbance introduced into the physical system, provides the uniqueness of the solution in contrast to the well-known BGK analysis. A similar selection of physically realizable solutions from the broad class of the BGK modes has been demonstrated previously by Zakharov and Karpman (1962) and Krasovsky (1995) who studied the finite amplitude waves under quite different physical conditions.

The theory developed is based on a closed consistent description of the steady wave propagating across the magnetic field. As a result of the analysis performed, the physical structure and properties of the wave have been studied in detail. We have determined the electron distribution function perturbed under the action of the wave, the perturbation of the electron density as a sum of the resonant and non-resonant contributions, the spatial shape of the wave potential and, finally, the nonlinear dispersion relation taking into account the frequency shift due to the specific hole-like structure of the distribution function in the resonant region. A similar nonlinear frequency shift due to the lack of the particles trapped by the wave may also arise in a weakly inhomogeneous plasma (Krasovsky 1995). It is in inverse proportion to the square root of the wave amplitude, suggesting that the specific structure of the distribution function in the resonant region may strongly modify the dispersion properties of the nonlinear waves. This might be a subject for experimental check in laboratory as well as in space plasmas.

Acknowledgements

The author thanks R. Z. Sagdeev, L. M. Zelenyi and A. I. Neishtadt for the discussion of the work and helpful remarks.

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