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Non-split, alternating links bound unique Seifert surfaces in the 4-ball

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Abstract

We show that any two same-genus, oriented, boundary parallel surfaces bounded by a non-split, alternating link into the 4-ball are smoothly isotopic relative to the boundary. In other words, any same-genus Seifert surfaces for a non-split, alternating link become smoothly isotopic relative to the boundary once their interiors are pushed into the 4-ball. We conclude that a smooth surface in S^4 obtained by gluing two Seifert surfaces for a non-split alternating link is always smoothly unknotted.

1. Introduction

In this paper, we discuss isotopy of boundary parallel surfaces in the 4-ball.

Conventions 1.1. *All surfaces, knots, and links are oriented; all embeddings and isotopies are smooth. A Seifert surface will always denote an oriented, not necessarily connected surface embedded in S^3 that has no closed components. The boundary of such a surface S is a link L, and we say that S is a Seifert surface for L.*

Typically, we will obtain surfaces first as Seifert surfaces in S^3. When we say that two surfaces in S^3 are isotopic relative to the (rel.) boundary in B^4, we implicitly mean that the surfaces become isotopic rel. boundary after their interiors are pushed slightly into the interior of B^4. We prove the following main theorem.

Theorem 1.2. *Any two same-genus Seifert surfaces for a non-split, alternating link are smoothly isotopic rel. boundary in B^4.*

Corollary 1.3. *Up to isotopy rel. boundary in B^4, a non-split, alternating link L bounds a unique minimal genus Seifert surface S. Any other Seifert surface for L is isotopic rel. boundary in B^4 to a connected sum of S and some number of standard tori in S^4.*

Corollary 1.4. *Let S_1, S_2 be Seifert surfaces for a non-split, alternating link L in S^3. Viewing S^3 as an equator of S^4, push the interiors of S_1 and S_2 off S^3 toward opposite normal directions of S^3 to obtain surfaces S_1^+, S_2^- in S^4 with the same boundary but disjoint interior. The closed surface $S_1^+ \cup S_2^-$ bounds a handlebody smoothly embedded in S^4, i.e. it is smoothly unknotted.*

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There are certain cases where a knot or link is already known to have a unique minimal genus Seifert surface up to isotopy rel. boundary in B^4. For example, a fibered link L bounds a unique minimal genus Seifert surface up to isotopy rel. boundary in S^3. This condition is also known to hold for some other simple families of knots, see e.g. [Lyo74, Kob89]. Theorem 1.2 is the first theorem proving uniqueness of minimal genus Seifert surfaces for a family of knots up to isotopy rel. boundary in B^4 when uniqueness does not hold in S^3.

Many surfaces that are distinct in S^3 become isotopic rel. boundary in B^4. See [HKM+23, Section 2.1] for a detailed discussion of constructions in the literature. However, the first two authors along with Hayden, Park, and Sundberg constructed many examples of connected, minimal genus Seifert surfaces with common boundary that do not become isotopic rel. boundary in B^4 [HKM+23]. Even more examples of this behavior have since been produced by Aka, Feller, Miller and Wieser [AFMW24]. Thus, in general, we do not expect a knot or link to bound a unique minimal genus Seifert surface, even up to isotopy rel. boundary in B^4.

In Theorem 1.2, we focus on the setting of alternating links. A non-split, alternating link may bound multiple minimal genus Seifert surfaces up to isotopy rel. boundary in S^3. As an explicit example, Trotter showed the knot 9_4 bounds at least two distinct minimal genus Seifert surfaces [Tro75, Theorem 2.6]. Roberts further constructed a family of alternating knots $\{K_n\}_{n\in\mathbb{N}}$ such that K_n bounds at least 2^{2n-1} distinct minimal genus Seifert surfaces [Rob13]. However, we conclude from Theorem 1.2 that these Seifert surfaces are isotopic rel. boundary in B^4.

Remark 1.5. The adjective 'non-split' is essential in the proof of Theorem 1.2. Indeed, the unlink of three components is a trivial example of a split, alternating link, but it bounds an infinite family of Seifert surfaces of equivalent surface type that are not isotopic rel. boundary even in B^4 [HKM+23].

Thus, compressing a surface along splitting spheres and applying Theorem 1.2 to each non-split component does not prove the analogous theorem for non-split links. One failure of the argument is that when Σ is a Seifert surface of k components, then after pushing into B^4 the group $\pi_1(B^4 \setminus \Sigma)$ has k generators. There are many distinct homotopy classes of arcs with endpoints on Σ, complicating the behavior of stabilizations. In contrast, when Σ is a connected surface (as is the case in this paper), the group is cyclic and thus any two stabilizations are equivalent.

It would be interesting if one could adapt Theorem 1.2 to apply to connected Seifert surfaces for split alternating links using a direct argument. The analogue of Lemma 2.7 might prove difficult, due to its use of incompressibility of surfaces and cut-and-paste operations.

Corollary 1.4 is particularly interesting given the context of the following open problem.

Question 1.6. Let S_1, S_2 be Seifert surfaces for a knot K in S^3. Viewing S^3 as an equator of S^4, push the interiors of S_1 and S_2 off S^3 toward opposite normal directions of S^3 to obtain surfaces S_1^+, S_2^- in S^4 with the same boundary but disjoint interior. Is the closed surface $S_1^+ \cup S_2^-$ necessarily unknotted?

The answer to Question 1.6 is 'yes,' in the topological category of Conway and Powell [CP23]. (As a consequence, it was already known by [CP23] that the surface $S_1^+ \cup S_2^-$ in Corollary 1.4 bounds a locally flat handlebody in S^4.) In the smooth category, Corollary 1.4 answers Question 1.6 affirmatively in the case that K is alternating.

Question 1.6 is a well-known subquestion of the smooth unknotting conjecture, which posits that any smooth, oriented surface in S^4 whose complement has cyclic fundamental group is smoothly unknotted. Note that an affirmative answer to Question 1.6 would *not* imply that same-genus Seifert surfaces Σ_1, Σ_2 for a non-alternating knot K are isotopic rel. boundary once

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pushed into B^4, which is generally false. Even if Σ_1 and Σ_2 are not isotopic rel. boundary in B^4, their union as an embedded surface in S^4 may be smoothly unknotted, e.g. the examples of [HKM+23] and [AFMW24] have this property. See the discussion in [HKM+23, Example 2.1].

2. The Kakimizu complex

The proof of Theorem 1.2 hinges on understanding the space of incompressible Seifert surfaces for a non-split, alternating link. This space (for a general link) is called the *Kakimizu complex*. In the literature, it is common to restrict to minimal genus surfaces. For the sake of discussion, we give both definitions.

Definition 2.1 (Kakimizu [Kak92]). Let L be a non-split link in S^3. The *Kakimizu complexes of* L, denoted $\mathrm{MS}(L)$ and $\mathrm{IS}(L)$, are simplicial complexes constructed as follows.

- The vertices of $\mathrm{MS}(L)$ are in bijection with proper isotopy (rel. boundary) classes of minimal genus Seifert surfaces for L.
- The vertices of $\mathrm{IS}(L)$ are in bijection with proper isotopy (rel. boundary) classes of incompressible Seifert surfaces for L.
- In each of $\mathrm{MS}(L), \mathrm{IS}(L)$, for $n>0$, there is an n-simplex with vertices $v_0,\dots,v_n$ exactly when the isotopy classes corresponding to $v_0,\dots,v_1$ admit Seifert surface representatives with mutually disjoint interiors.

The notation '$\mathrm{MS}(L)$' and '$\mathrm{IS}(L)$' chosen by Kakimizu is meant to remind the reader that these simplicial complexes are associated to **m**inimal **s**urfaces or **i**ncompressible **s**urfaces bounded by the link L.

Theorem 2.2 (Kakimizu [Kak92]). *For any non-split link L, the complexes* $\mathrm{MS}(L)$ *and* $\mathrm{IS}(L)$ *are each connected.*

The methods of Kakimizu suggest the following strengthened version of Theorem 2.2.

Definition 2.3. Fix an integer k and a non-split link L in S^3. Let $\mathrm{IS}_k(L)$ be the simplicial subcomplex of $\mathrm{IS}(L)$ that contains an n-simplex with vertices $v_0,\dots,v_n$ exactly when incompressible Seifert surfaces representing $v_0,\dots,v_n$ each have Euler characteristic at least k.

Theorem 2.4. *Given a non-split link L, for each integer k the subcomplex* $\mathrm{IS}_k(L)$ *is connected.*

Note that a Seifert surface for L restricts to a properly embedded, oriented surface in $S^3\setminus\nu(L)$ that has no closed components and has boundary a longitude on each component of $\partial S^3\setminus\nu(L)$. For convenience, we will refer interchangeably to such properly embedded surfaces and Seifert surfaces for L.

We will make use of the following definition, used by Kakimizu in [Kak92].

Definition 2.5. Let S, S' be incompressible Seifert surfaces for a non-split link L. Let S_0 and S_0' respectively be lifts of S' to the infinite cyclic cover M of $S^3\setminus\nu(L)$. For each integer i, let S_i denote the translate of S_0 by the action of $i\in\mathbb{Z}$ and let E_i denote the closure of the region of M bounded by $-S_i\cup S_{i+1}$. Similarly, let S_j' denote the translate of S_0' by the action of $j\in\mathbb{Z}$ and let E_j' denote the closure of the region of M bounded by $-S_j'\cup S_{j+1}'$. Isotope S' (and hence equivariantly isotope $\{S_j'\}$) so as to minimize the number of integers i for which $E_0'\cap E_i\neq\emptyset$. Let a,b be integers with $E_0'\cap E_a, E_0'\cap E_b\neq\emptyset$, and $E_0'\cap E_i=\emptyset$ for $i>b$ and for $i<a$. The *covering distance* $d(S,S')$ is defined to be $b-a$.

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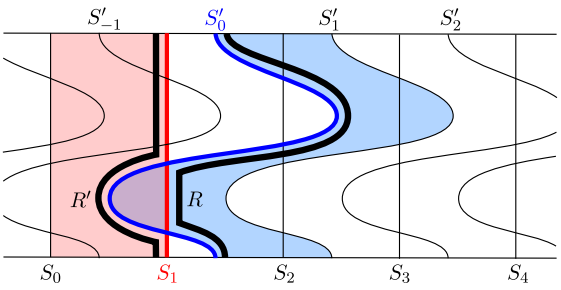


FIGURE 1 (colour online). A schematic of the infinite cyclic cover M of $E := S^3 \setminus \nu(L)$. We indicate lifts S_i of S and S'_j of S'. The region between S_i, S_{i+1} is E_i and the region between S'_j, S'_{j+1} is E'_j. In this schematic, $d(S, S') = 3$ as E'_0 intersects E_i for the values $i \in \{0, 1, 2, 3\}$. In the proof of Lemma 2.7, we obtain surfaces $R \subset E'_0$ and $R' \subset E_0$ whose union is isotopic to the cut-and-paste sum of S'_0 and S_1. It follows that one of R, R' has Euler characteristic bounded below by $\min\{\chi(S), \chi(S')\}$. Fully compressing that surface (in E'_0 or E_0 accordingly) yields an incompressible surface that projects to the desired Seifert surface S''.

Kakimizu [Kak92, Proposition 3.1(1)] proved that the covering distance of S and S' is equal to the minimum length of a path between vertices corresponding to S, S' in $\mathrm{IS}(L)$. Even without this fact (which we will not directly employ in this paper, so as to keep our arguments relatively self-contained), we might make the following observation.

Observation 2.6 (Kakimizu [Kak92]). Incompressible Seifert surfaces S and S' for L satisfy $d(S, S') = 0$ if and only if S, S' are properly isotopic rel. boundary: in Definition 2.5, if $d(S, S') = 0$ then up to isotopy we have $E'_0 \subset E_i$ for some i. Since the boundaries of E'_0 and E_i are lifts of S', S respectively, we must have $E'_0 = E_i$ and hence S, S' are isotopic rel. boundary in S^3.

We now prove Theorem 2.4 via the following lemma, which mirrors that of [Kak92, Theorem 2.1] and closely follows the same proof, taking care to consider the Euler characteristic of the produced surfaces.

LEMMA 2.7. *Let S, S' be non-isotopic, incompressible Seifert surfaces for a non-split link L. Let $n := d(S, S')$. Then there is an incompressible Seifert surface S'' for L such that S'' and S' do not intersect in their interiors, $d(S, S'') < n$, and $\chi(S'') \geqslant \min\{\chi(S), \chi(S')\}$.*

The following proof of Lemma 2.7 is illustrated in Figure 1.

Proof of Lemma 2.7. Let S_0 be a lift of S to M and let S_i denote the translate of S_i under the action of $i \in \mathbb{Z}$. Let E_i denote the closure of the region of M bounded by $-S_i \cup S_{i+1}$. Since $n = d(S, S')$, by definition of covering distance we may isotope S' and choose a lift S'_0 to M such that, for some integer a, S'_0 intersects E_a and E_{a+n-1} but does not intersect E_i for any $i < a$ or $i > a + n - 1$. Relabel the lifts S_i if necessary so that $a = 0$. Perturb S' if necessary so that intersections of S and S' in E are transverse and the interiors of S, S' intersect in closed curves. Since S, S' are each incompressible, each component of $S \cap S'$ is either essential in both S and S' or inessential in both S and S'. Remove any inessential intersections using a standard innermost circle surgery argument.

Similarly, let S'_j be the translate of S'_0 under the action of $j \in \mathbb{Z}$ and let E'_j denote the closure of the region of M bounded by $-S'_j \cup S'_{j+1}$. Then $E'_0 \cap E_0 \neq \emptyset$ and $E'_i \cap E_0 = \emptyset$ whenever $i > 0$. Let X be a regular neighborhood of $S'_0 \cup (E_0 \cap E'_0)$ in E'_0. Since $S'_0 \cap E_n = \emptyset$, we also have $X \cap E_n = \emptyset$. Let R be the closure of $\partial X \setminus \partial E'_0$.

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We simultaneously produce another surface, exchanging the roles of S and S'. That is, we note that $E_j \cap E'_0 = \emptyset$ whenever $j < 0$. Let X' be a regular neighborhood of $S_1 \cup (E_0 \cap E'_0)$ in E_0. Since $S_1 \cap E'_{-n} = \emptyset$, we also have $X' \cap E'_{-n} = \emptyset$. Let R' be the closure of $\partial X' \setminus \partial E_0$.

The union of R and R' is isotopic to the cut-and-paste sum of S_1 and S'_0. (That is, the union of R and R' is isotopic to the embedded surface obtained by resolving $S_1 \cup S'_0$ at intersections while respecting orientation. Note that if S_1, S'_0 are disjoint, as may be the case if $n = 1$, then R is a parallel copy of S'_0 while R' is a parallel copy of S_1.) We conclude $\chi(R) + \chi(R') = \chi(S) + \chi(S')$, so at least one of $\chi(R), \chi(R')$ is bounded below by $\min\{\chi(S), \chi(S')\}$. By potentially exchanging the roles of S and S' (and hence also R and R'), assume without loss of generality that $\chi(R) \geq \min\{\chi(S), \chi(S')\}$. Since S, S' intersect in essential curves, any closed components of R are positive genus. Delete any closed components; we still have $\chi(R) \geq \min\{\chi(S), \chi(S')\}$.

Since $R \subset X \subset E'_0$, R does not intersect any S'_i. Then $p(R) \cap S' = \emptyset$, where $p(R)$ is the projection of R to E (and hence a Seifert surface for L). Moreover, R does not intersect E_j with $j < 0$ or $j > n$. We already remarked that X does not intersect E_n, so R also does not intersect E_n. We conclude that if R is incompressible, then the Seifert surface $p(R)$ satisfies $d(S, p(R)) < n$, satisfying the claim (with $S'' := p(R)$).

Unfortunately, the surface R may fail to be incompressible. Since S, S' intersect in essential circles and R is a subset of the cut-and-paste sum of S_1 and S'_0, the surface R does not include any 2-sphere components. Since $\partial E'_0$ is incompressible in M, if a surface in the interior of E'_0 is incompressible in E'_0 then it is also incompressible in M. Suppose R is compressible in M and hence in E'_0. Let D be a compressing disk for R in E'_0. Since S_n is incompressible, again by a standard innermost circle argument we may assume D does not intersect S_n. Then compressing R along D yields a new surface in E'_0 that has no 2-sphere components and does not intersect E_n. Repeat until obtaining a surface that is incompressible in E'_0 and hence also incompressible in M. Delete any closed components and call the result $\tilde{R}$. Since $\tilde{R}$ is obtained from R by compression and deletion of positive-genus components, $\chi(\tilde{R}) \geq \chi(R) \geq \min\{\chi(S), \chi(S')\}$. Let S'' denote the incompressible Seifert surface $p(\tilde{R})$. Since $\tilde{R} \subset E'_0 \setminus E_n$, we find that the interiors of S'' and S' are disjoint and $d(S, S'') < n$ as desired. □

Theorem 2.4 follows easily from Lemma 2.7.

Proof of Theorem 2.4. Let v, v' be vertices in $\mathrm{IS}_k(L)$ for some k. We will construct a path in $\mathrm{IS}_k(L)$ from v' to v, implying that $\mathrm{IS}_k(L)$ is connected. Each of v, v' is represented by an incompressible Seifert surface S, S' (respectively) with $\chi(S), \chi(S') \geq k$. Let $n := d(S, S')$. By Lemma 2.7, there exists an incompressible Seifert surface F_1 (taking the place of S'') for L representing the vertex $v_1 \in \mathrm{IS}(L)$ such that $\chi(F_1) \geq \min\{\chi(S), \chi(S')\} \geq k$ (so in particular $v_1 \in \mathrm{IS}_k(L)$), the vertices v_1, v' are adjacent in $\mathrm{IS}(L)$, and $d(F_1, S) < n$.

If $d(F_1, S) = 0$, then by Observation 2.6 the surfaces F_1, S are isotopic rel. boundary and we conclude there is a path (v', v_1, v) in $\mathrm{IS}_k(L)$. Otherwise, repeat the above argument replacing S' with F_i to obtain a surface F_{i+1} representing a vertex $v_{i+1} \in \mathrm{IS}(L)$ such that $\chi(F_{i+1}) \geq \min\{\chi(S), \chi(F_i)\} \geq \min\{\chi(S), \chi(S')\} \geq k$ (so in particular $v_{i+1} \in \mathrm{IS}_k(L)$), the vertices v_{i+1}, v_i are adjacent in $\mathrm{IS}(L)$, and $d(F_{i+1}, S) < d(F_i, S)$. Repeat until obtaining a surface F_a with $d(F_a, S) = 0$ and hence (again by Observation 2.6) the surface F_a is isotopic rel. boundary to S. We conclude there is a path $(v', v_1, v_2, \ldots, v_a, v)$ in $\mathrm{IS}_k(L)$. □

The following corollary follows from Kakimizu [Kak92, Proposition 3.1(1)], but we prove it here for completeness.

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COROLLARY 2.8. *Let S, S' be incompressible Seifert surfaces for a link L. Then $d(S, S') = 1$ if and only if S and S' represent adjacent vertices in* $\mathrm{IS}(L)$.

Proof. If S, S' represent adjacent vertices, then up to isotopy rel. boundary the interiors of S and S' are disjoint and hence $d(S, S') = 1$.

On the other hand, if $d(S, S') = 1$, then it follows from Lemma 2.7 that there is a third Seifert surface S'' such that S'' is disjoint from S' in its interior and $d(S, S'') < 1$. Then $d(S, S'') = 0$ and Observation 2.6 shows that S, S'' are isotopic rel. boundary in S^3. We conclude that, up to isotopy rel. boundary in S^3, the surfaces S, S' have disjoint interior and thus represent adjacent vertices in $\mathrm{IS}(L)$. □

Remark 2.9. In [PS12], Przytycki and Schultens give a similar but inequivalent definition of $\mathrm{MS}(L)$ (in their notation, $\mathrm{MS}(E(L))$) that differs from $\mathrm{MS}(L)$ when L admits minimal genus, disconnected surfaces. They show that $\mathrm{MS}(E(L))$ is contractible. If L is non-split and alternating, then every oriented surface bounded by L is connected: consider the classic exercise [Lic97, Proposition 4.8] showing that an incompressible positive-genus surface F in the complement of L admits a compressing disk intersecting L in a single point. This would be violated by $F := \partial\nu(S_1)$ if $S_1 \sqcup S_2$ is a disconnected surface bounded by L (possibly after compressing F until reaching an incompressible splitting surface for L), since every simple closed curve of F links L zero times. (Alternatively, one could cite [AK13, Corollary 5.2].) Therefore, in the settings considered in this paper, we can ignore this distinction (and hence $\mathrm{MS}(L)$ is contractible, although we do not require more than connectivity). We similarly need not distinguish between 'minimal genus' and 'maximal Euler characteristic', since all surfaces are connected. It remains open whether $\mathrm{IS}(L)$ is contractible (see [PS12, § 1] for discussion).

In the case that L is an alternating link, there is even more known about surfaces bounded by or containing L. It is a well-known theorem of [Cro59, Mur58] that Seifert's algorithm applied to an alternating diagram yields a minimal genus Seifert surface. However, a minimal genus Seifert surface for an alternating link need *not* arise from Seifert's algorithm applied to an alternating diagram [HS97, Theorem 1.2][1], and in general it is not clear what, if any, relationship is held between distinct minimal genus Seifert surfaces for an alternating link.

DEFINITION 2.10. Given a closed surface F in S^3 containing a link L, we say that F is *essential with respect to* L when $F \cap (S^3 \setminus \nu(L))$ is essential in $S^3 \setminus \nu(L)$. When L is clear, we may simply write that F is essential.

We will make use of the following key lemma proved by Kindred [Kin18].

LEMMA 2.11 (Kindred [Kin18] (Crossing Tube Lemma)). *Given a nontrivial, reduced alternating diagram $\mathcal{D}$ of a non-split link L and a closed, essential surface F containing L, there exists an isotopy of S^3 fixing L after which F has a standard tube above some crossing c of $\mathcal{D}$ as in Figure 2. That is, F admits a compressing disk lying above a small neighborhood of c and that intersects L in the two points of L above c.*

[1]Thanks to Thomas Kindred for pointing out the following subtlety to the authors: while [HS97] produces Seifert surfaces not arising from Seifert's algorithm applied to any *alternating* diagram, it is unknown whether an alternating link can admit a minimal genus Seifert surface not arising from Seifert's algorithm applied to *some* (not necessarily alternating) diagram.

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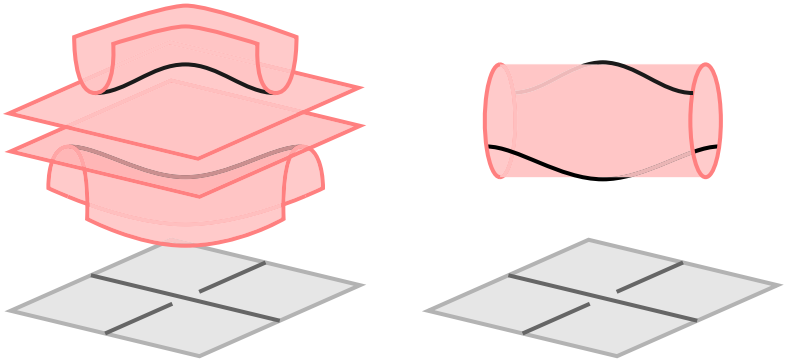


FIGURE 2 (colour online). Left: a generic essential surface F containing a link L. We feature the portion of F lying above a neighborhood of some crossing in the diagram $\mathcal{D}$ (shown at the bottom of the figure). This subset of F includes two disks that contain two arcs of L and potentially other disks that do not meet L. Right: above this crossing, the surface F includes a standard tube containing the two strands that project to the crossing.

3. Proof of Theorem 1.2

We will reduce to the setting of incompressible surfaces using the following lemma, which is based on a commonly used fact in the study of surfaces in 4-manifolds. See [Boy88, Corollary 4] for a general statement and proof. Here, when we say a Seifert surface S_i is obtained by compressing another Seifert surface Σ_i, we mean that we compress Σ_i and delete any resulting closed components.

LEMMA 3.1. *Let Σ be a connected Seifert surface for a link L. Suppose that S is a connected Seifert surface obtained by compressing Σ along a disk (possibly deleting a resulting closed component). Push the interiors of both surfaces into B^4 and let T denote the standard torus in S^4. Then Σ is isotopic rel. boundary to $S\#(g(\Sigma)-g(S))T$.*

Proof. If the compression takes place along a non-separating curve, then Σ is obtained from S by attaching one tube, i.e. surgery along a framed arc γ. Because S is boundary parallel in B^4, we have $\pi_1(B^4 \setminus S) \cong \mathbb{Z}$ with generator represented by a meridian of S. Therefore, any two arcs with endpoints on S and interior in $B^4 \setminus S$ are isotopic keeping ends on S and interior disjoint from S. In ambient dimension four, the normal bundle $N(\gamma)$ of the arc γ rel. boundary admits framings in bijection with $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Such a framing corresponds with a choice of how to attach a tube to S along γ: we choose a trivialization $\langle v_1, v_2, v_3\rangle$ of $N(\gamma)$ with $\langle v_1, v_2\rangle$ tangent to S at the ends of γ and then surger S along the three-dimensional 1-handle $\gamma \times \langle v_1, v_2\rangle$. Exactly one of the two potential framings of $N(\gamma)$ yields a 1-handle over which the orientation of S extends, while the other framing yields a 1-handle over which the orientation does not extend. Since Σ is oriented, the framing of γ is uniquely determined. We conclude that there is a unique way of attaching an oriented tube to S in B^4, one of which would yield $S\#T$, and hence that Σ is isotopic rel. boundary to $S\#T$.

On the other hand, suppose that S is obtained by compressing Σ along a separating curve and then deleting a closed component S' disjoint from S. Recall that a closed surface in S^3 always bounds a handlebody in B^4 (see e.g. the discussion in the beginning of [Hat07, § 1.2]: every positive-genus surface in S^3 is compressible and every smooth S^2 in S^3 bounds a 3-ball). Let V be a handlebody bounded by S' into B^4. Let γ be a framed arc connecting S, S' such that surgery along γ yields Σ. Push S', V, γ into the interior of B^4. Since γ is one-dimensional and disjoint in its interior from V and $\pi_1(B^4 \setminus V) = 1$, as the nonseparating case we conclude that

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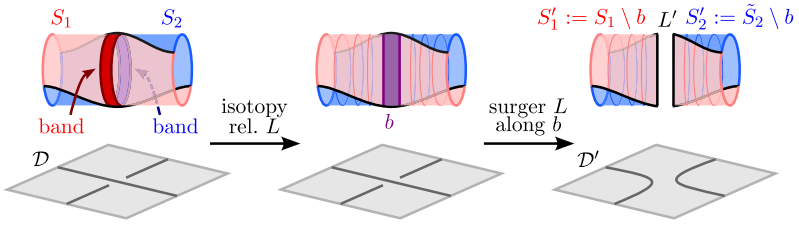


FIGURE 3 (colour online). Left: incompressible surfaces S_1, S_2 bounded by L. By Lemma 2.11, the essential surface $S_1 \cup S_2$ includes a standard tube over some crossing c in a reduced, alternating diagram $\mathcal{D}$ for L. Center: we isotope S_1, S_2 near the standard tube so that they agree in a band b. Elsewhere, we add $g(S_1) - g(S_2)$ tubes to S_2 to obtain a surface $\tilde{S}_2$ of the same genus as S_1. Right: deleting the band b from $S_1, \tilde{S}_2$ yields surfaces S_1', S_2' bounded by a link L'.

γ is isotopic to any other arc connecting S to $S' = gT$ that is disjoint from V and also that the framing on γ is determined by the orientation of Σ. Therefore, Σ is isotopic rel. boundary in B^4 to $S\#(g(S')T)$. □

We will make use of the following subcase of Theorem 1.2 in the proof of the main theorem.

LEMMA 3.2. *Let Σ_1, Σ_2 be same-genus Seifert surfaces for a non-split, alternating link. Suppose that Σ_1, Σ_2 can be fully compressed to incompressible Seifert surfaces S_1, S_2 (respectively) with $d(S_1, S_2) \leq 1$. Then Σ_1 and Σ_2 are isotopic rel. boundary in B^4.*

In the proof of Lemma 3.2, we use the following easy fact.

PROPOSITION 3.3. *If Σ_1, Σ_2 are Seifert surfaces for a non-split alternating link L and Σ_1, Σ_2 have disjoint interior, then Σ_1, Σ_2 can be compressed to incompressible Seifert surfaces S_1, S_2 (respectively) that also have disjoint interior.*

Proof. Suppose D is a compressing disk for Σ_1. Let C be a curve of intersection in $D \cap \Sigma_2$ that is innermost on D, so C bounds a disk $D' \subset D$ whose interior is disjoint from Σ_2. Compress Σ_2 along D' and delete any closed components formed. Repeat until the interior of D is disjoint from Σ_2, and finally perform the compression along Σ_1. Repeat until obtaining an incompressible Seifert surface S_1. Now any compressing disk for Σ_2 intersects S_1 in inessential curves that can similarly be removed by an innermost circle argument, so compress Σ_2 along disks disjoint from S_1 to obtain an incompressible surface S_2 whose interior is disjoint from S_1. □

Proof of Lemma 3.2. If $d(S_1, S_2) = 0$, then by Observation 2.6 the surfaces S_1 and S_2 are isotopic rel. boundary. Then up to isotopy rel. boundary in S^3, the surfaces Σ_1, Σ_2 are each obtained from S_1 by surgery along n arcs for some n. By Lemma 3.1, Σ_1 and Σ_2 are isotopic rel. boundary in B^4.

Now suppose $d(S_1, S_2) \leq 1$. We will prove that Σ_1, Σ_2 are isotopic rel. boundary in B^4 by inducting on the genus g of Σ_1, Σ_2 (although for convenience, we will phrase this as a condition on the Euler characteristic, inducting on the Euler characteristic decreasing from the maximum possible value of 1).

If $\chi(\Sigma_i) = 1$, then it is a well-known consequence of the three-dimensional Schoenflies theorem (see e.g. the discussion in [Hat07, Section 1.1]) that the disks Σ_1, Σ_2 are isotopic rel. boundary in S^3 and hence in B^4.

Now suppose that Lemma 3.2 holds for Seifert surfaces of Euler characteristic strictly greater than $\chi(\Sigma_i)$. If $d(S_1, S_2) = 0$ then Σ_1 and Σ_2 are isotopic rel. boundary in B^4, so assume that $d(S_1, S_2) = 1$. By Corollary 2.8, we may apply an isotopy rel. boundary in S^3 to S_1, S_2 so

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that they have disjoint interior. Then $S_1 \cup S_2$ is an embedded surface F that is essential with respect to L.

Choose a reduced, alternating diagram $\mathcal{D}$ for L. Lemma 2.11 implies that F includes a standard tube over some crossing c of $\mathcal{D}$ as in Figure 3 (left). Fixing L, isotope S_1 and S_2 so that their interiors agree precisely at a band b above c as in Figure 3 (center). Let $\mathcal{D}'$ be the diagram obtained by smoothing $\mathcal{D}$ at c, respecting orientations, and let L' be the link described by $\mathcal{D}'$. Since $\mathcal{D}$ is alternating, non-split, and reduced, the smoothed diagram $\mathcal{D}'$ is alternating and non-split.

Without loss of generality, assume $g(S_1) \geq g(S_2)$. Attach tubes to S_2 (away from b) to obtain a surface $\tilde{S}_2$ of the same genus as S_1. By Lemma 3.1, $\tilde{S}_2$ is isotopic rel. boundary in B^4 to $S_2 \# (g(S_1) - g(S_2))T$, where T is a standard torus in S^4.

Consider the surfaces $S_1' := S_1 \setminus b$, $S_2' := \tilde{S}_2 \setminus b$ bounded by the link L'. We have $\chi(S_1') = \chi(S_2') = \chi(S_1) + 1 > \chi(\Sigma_i)$. By Proposition 3.3, S_1', S_2' compress to surfaces with disjoint interior and hence covering distance at most 1. Then by inductive hypothesis, we find that S_1' and S_2' are isotopic rel. boundary in B^4. Since $\Sigma_1 = S_1' \cup b$ and $\tilde{S}_2 = S_2' \cup b$, we find also that S_1 and $\tilde{S}_2$ are isotopic rel. boundary in B^4.

By Lemma 3.1, the surface Σ_1 is isotopic rel. boundary in B^4 to

$$S_1 \# (g(\Sigma_1) - g(S_1))T.$$

Similarly, Σ_2 is isotopic rel. boundary in B^4 to

$$\begin{aligned} S_2 \# (g(\Sigma_2) - g(S_2))T &= (S_2 \# (g(S_1) - g(S_2))T) \# (g(\Sigma_1) - g(S_1))T \\ &= \tilde{S}_2 \# (g(\Sigma_1) - g(S_1))T. \end{aligned}$$

Since S_1 and $\tilde{S}_2$ are isotopic rel. boundary in B^4, we conclude that Σ_1 and Σ_2 are isotopic rel. boundary in B^4 as claimed. □

Remark 3.4. In the proof of Lemma 3.2, it is essential that we are concluding isotopy in B^4 rel. boundary rather than in S^3. Even if S_1', S_2' were isotopic rel. boundary in S^3, an isotopy from S_1' to S_2' might pass S_1' through the band b and hence not extend to an isotopy from S_1 to $\tilde{S}_2$. Thus, the induction fails to prove that same-genus Seifert surfaces for non-split, alternating links are unique up to isotopy rel. boundary in S^3, even when we restrict to minimal genus Seifert surfaces. This is consistent with the fact that minimal genus Seifert surfaces for non-split, alternating links are generally *not* unique up to isotopy in S^3, as discussed in Section 1.

We can now prove Theorem 1.2.

Theorem 1.2. *Let Σ_1, Σ_2 be same-genus Seifert surfaces for a non-split, alternating link. Then Σ_1 and Σ_2 are isotopic rel. boundary in B^4.*

Proof. Let $\chi := \chi(\Sigma_i)$. Let S_i be an incompressible surface obtained from Σ_i by k_i compressions. Assume $k_1 \leq k_2$, so $\chi \leq \chi(S_1) \leq \chi(S_2)$.

By Theorem 2.4, there exist incompressible Seifert surfaces $S_1 = F_0, F_1, F_2, \ldots, F_n, F_{n+1} = S_2$ such that $\chi(F_j) \geq \chi(S_2) \geq \chi$ and the surfaces F_j, F_{j+1} have disjoint interior for each $j \in \{0, \ldots, n\}$. For each $j = 1, \ldots, n$, let Σ_j' be a Seifert surface obtained from F_j by attaching tubes to F_j such that $\chi(\Sigma_j') = \chi$ for all j. Let $\Sigma_0' := \Sigma_1, \Sigma_{n+1}' := \Sigma_2$.

Now for each $j = 0, \ldots, n$, the surfaces Σ_j', Σ_{j+1}' are same-genus Seifert surfaces for L that compress to incompressible Seifert surfaces F_j, F_{j+1}. Note that $d(F_j, F_{j+1}) \leq 1$ since the interiors of F_j, F_{j+1} are disjoint. Then by Lemma 3.2, Σ_j', Σ_{j+1}' are isotopic rel. boundary in B^4. Since $\Sigma_1 = \Sigma_0', \Sigma_2 = \Sigma_{n+1}'$, we conclude that Σ_1, Σ_2 are isotopic rel. boundary in B^4. □

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Corollaries 1.3 and 1.4 now follow easily.

COROLLARY 1.3. *Up to isotopy rel. boundary in the 4-ball, a non-split, alternating link L bounds a unique minimal genus Seifert surface S. Any other Seifert surface for L is isotopic rel. boundary in B^4 to a connected sum of S and some number of standard tori in S^4.*

Proof. Let S be a minimal genus Seifert surface for L. By Theorem 1.2, any other minimal genus Seifert surface for L is isotopic to S rel. boundary in B^4. Let Σ be a Seifert surface for L, and let S' be a genus-$g(\Sigma)$ Seifert surface obtained by attaching n tubes to S. By Theorem 1.2, Σ is isotopic rel. boundary in B^4 to S', which by Lemma 3.1 is isotopic rel. boundary in B^4 to $S\#_n T$. □

COROLLARY 1.4. *Let S_1, S_2 be Seifert surfaces for a non-split, alternating link L in S^3. Viewing S^3 as an equator of S^4, push the interiors of S_1 and S_2 off S^3 toward opposite normal directions of S^3 to obtain surface S_1^+, S_2^- in S^4 with the same boundary but disjoint interior. The closed surface $S_1^+ \cup S_2^-$ bounds a handlebody smoothly embedded in S^4, i.e. is smoothly unknotted.*

Proof. Let S be a minimal genus Seifert surface for L. Obtain a closed surface F in S^4 by gluing two copies of S, with the interiors of each copy pushed off S^3 toward opposite sides. Then F is isotopic to the boundary of the handlebody $S \times I$.

By Theorem 1.2, $S_1^+ \cup S_2^-$ is isotopic to $F\#_n T$, where $n = (g(S_1) - g(S)) + (g(S_2) - g(S))$. We conclude $S_1^+ \cup S_2^-$ also bounds a handlebody. □

4. Open questions

We end with some natural open questions for other families of links. We first suggest one consider almost-alternating links, i.e. links that admit a diagram which becomes alternating after applying one crossing change (note the distinction between being almost alternating vs. admitting a crossing change to some alternating link).

Questions 4.1.

(a) If K is almost alternating, must any two same-genus Seifert surfaces for K be smoothly isotopic in B^4?

(b) If K is almost alternating, must the union of any two Seifert surfaces for K yielding a closed surface in S^4 (as in Question 1.6) be smoothly unknotted?

An affirmative answer to (a) implies an affirmative answer to (b).

Arguably, it would be more interesting if the answer to Question 4.1(a) is 'no,' as this would provide a sharpness result for Theorem 1.2 is sharp.

Problem 4.2. Find same-genus Seifert surfaces for an almost-alternating knot K that are not isotopic in B^4. Even better: arrange for these surfaces to be minimal genus.

Theorem 1.2 is the first theorem proving uniqueness of minimal genus Seifert surfaces for a family of knots up to isotopy rel. boundary in B^4 without having uniqueness up to isotopy in S^3.

Question 4.3. How many minimal genus Seifert surfaces does a given hyperbolic knot bound up to isotopy rel. boundary in B^4?

Hyperbolic knots may bound more than one minimal genus Seifert surface up to isotopy rel. boundary in B^4 [HKM+23]. However, a hyperbolic knot bounds a finite number of minimal genus

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Seifert surfaces in S^3 (and in fact, a finite number of Seifert surfaces of any fixed genus) according to constraints from normal surface theory [SS64] (see also [JPW14, Theorem 4]). Perhaps a similar analysis could be used to give a nontrivial explicit upper bound on the number of Seifert surfaces for a hyperbolic knot up to isotopy rel. boundary in B^4.

Conflicts of interest

None.

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