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Decay of correlations and mixing properties in a dynamical system with zero K–S entropy

M. COURBAGE†‡ and D. HAMDAN†

† *Laboratoire de Probabilités, 4, Place Jussieu, Tour 56, 3ème Etage 75252, Paris Cedex 05, France*

‡ *Université Paris VII, L.P.T.M., Tour Centrale, 2, Place Jussieu 75251, Paris Cedex 05, France*

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Abstract. We consider the ergodic skew product of translations on the torus $\mathbb{T}^2$. We characterize a class of smooth functions with exponential correlation decay rate and a class of subsets with an exponential rate of mixing. We show that the class of functions with sub-exponential correlation decay rate is generic. We also characterize another class of subsets with mixing rate of the order of $1/t$.

0. *Introduction*

An important statistical property of a dynamical system, given by a transformation T, preserving a probability measure μ, is the exponential decay of correlations

$$\hat{\mu}_{f,g}(n) = \int \overline{f(x)} g(T^n x)\, d\mu - \int \overline{f(x)}\, d\mu \int g(x)\, d\mu,$$

where f and g belong to a certain class of functions. Such a property has been proved for Axiom A systems and other dynamical systems with positive Kolmogorov–Sinaï entropy [**3, 4, 7, 10**]. It has also been proved that the rate of mixing of these systems, $\mu(E \cap T^{-n}F) - \mu(E)\mu(F)$, is an exponential rate. In general, the exponential decay of the autocorrelation coefficients of a function, f, orthogonal to 1, implies that the spectral measure μ_f on $[0, 2\pi[$, whose Fourier coefficients are

$$\hat{\mu}_f(n) = \int e^{inx}\, d\mu_f(x) = \int \overline{f(x)} f(T^n x)\, d\mu(x),$$

is equivalent to the Lebesgue measure. In fact, in this case the Fourier coefficients $\hat{\mu}_f(n)$ form a summable series; this implies that μ_f then has a density $g(x)$ with respect to the Lebesgue measure:

$$g(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \hat{\mu}_f(n) e^{-inx}$$

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which is analytic in $[0, 2\pi[$. Thus, $g(x)$ has at most a finite number of zeros, and μ_f is equivalent to the Lebesgue measure. In particular, for a subset E, the autocorrelation coefficients $\hat{\mu}_f(n)$ of $f = 1_E - \mu(E)$ are just $\mu(E \cap T^{-n}E) - (\mu(E))^2$. Thus, if the rate of self-mixing of the subset E is exponential, then μ_f is equivalent to the Lebesgue measure.

The converse question of the properties of functions realizing a Lebesgue spectral measure of a dynamical system has been investigated by several authors, among them, Alexeyev [**2**]. In [**6**], we considered partitions $\mathcal{P} = \{P_0, P_1\}$ of the phase space having the Chapman–Kolmogorov property, which is equivalent to the following:

$$\mu(P_i \cap T^{-n}P_j) - \mu(P_i)\mu(P_j) = \alpha_{i,j}\lambda^{|n|}$$

for any $n \in \mathbb{Z}$, $n \neq 0$, where $|\lambda| < 1$. An important question is the relation between the existence of these partitions to the K–S entropy of (T, μ). When $\lambda = 0$, we say that $\mathcal{P}$ is pairwise independent. Contrasting with the independence of $\mathcal{P}$ (which implies a positive K–S entropy), the pairwise independence is compatible with zero entropy [**6**].

In this paper, we investigate these issues in a family of maps on the torus $\mathbb{T}^2$, with zero entropy, having countable Lebesgue components in their spectrum. These maps are given by

$$T(x, y) = (x + \alpha \pmod{2\pi}, px + y \pmod{2\pi})$$

with the invariant Lebesgue measure μ on $\mathbb{T}^2$, $\alpha \in [0, 2\pi[$ and $p \in \mathbb{Z} \setminus \{0\}$. This is an example of compound skew translations on the product of two compact groups (see [**5**]). For irrational α, T is ergodic, and according to Furstenberg's result [**5**], it is uniquely ergodic. It comes from a theorem of Abramov and Roklin [**1**] that the entropy of (T, μ) is zero. This implies that the topological entropy of T is also zero.

The spectral analysis of the operator U_T on L^2_μ, associated to T by $U_T f = f \circ T$, shows that the spectrum of U_T has countable Lebesgue components and a purely discrete part. If we denote by $\mathcal{H}_{\text{a.c.}}$ and $\mathcal{H}_{\text{d}}$ the corresponding invariant subspaces of L^2_μ, then for any $f, g \in \mathcal{H}_{\text{a.c.}}$, $\langle f, U^t g\rangle$ decays to zero as $|t| \to \infty$.

In §1 we characterize a class of smooth functions with exponential correlation decay rate. This class is dense in $\mathcal{H}_{\text{a.c.}}$, nevertheless, the class of functions with sub-exponential correlation decay rate is generic, that is, it is a G_δ-dense set with respect to the uniform topology.

In §2 we characterize a class of subsets $E \subset \mathbb{T}^2$ with an exponential rate of mixing; in particular, subsets which are pairwise independent. On the other hand, we show that for the subsets which are unions of squares of side $2\pi/n$ and which have x-section of constant length, the rate of mixing is $O(1/t)$, and if, moreover, this rate is $o(1/t)$ then $\{T^n E\}$ are pairwise independent. Finally, we display a class of subsets for which the rate of mixing is oscillating around $1/t$.

As an application, the partitions constructed here give examples of stationary stochastic processes $\{x_n\}$ of zero entropy with pairwise independent variables.

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1. *Decay of correlations*

On the two-dimensional torus $\mathbb{T}^2$, where $\mathbb{T}$ denotes the circle $[0, 2\pi[$, we consider the transformation $T_{\alpha,p}$, $\alpha \in \mathbb{T}$, $p \in \mathbb{Z}$, $p \neq 0$ defined by

$$T_{\alpha,p}(x, y) = (x + \alpha, px + y). \tag{1}$$

In what follows, we denote $T_{\alpha,p}$ simply by T, unless needed for particular cases. As the normalized Lebesgue measure μ on $\mathbb{T}^2$ is invariant under T, we consider the unitary operator on L^2_μ associated with T by

$$Uf = f \circ T.$$

In considering a basis of $L^2_\mu(\mathbb{T}^2)$ of the form $\chi_{n,m}(x, y) = a_{n,m}e^{i(nx+my)}$, it is easy to see that the subspace of functions which only depend on the variable x, is an invariant subspace under U, which we denote $\mathcal{H}_{\mathrm{d}}$ because U has a discrete spectrum on $\mathcal{H}_{\mathrm{d}}$. We denote the orthogonal of $\mathcal{H}_{\mathrm{d}}$ by $\mathcal{H}_{\mathrm{a.c.}}$ because $U|_{\mathcal{H}_{\mathrm{a.c.}}}$ has a countable Lebesgue spectrum. The orthogonal projection onto H_{d} is given by

$$P_{\mathrm{d}} f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x, y)\, dy. \tag{2}$$

We denote by $P_{\mathrm{a.c.}}$ the orthogonal projection onto $\mathcal{H}_{\mathrm{a.c.}}$, $P_{\mathrm{a.c.}} = 1 - P_{\mathrm{d}}$. Thus, $f \in \mathcal{H}_{\mathrm{a.c.}}$ if and only if $\int_0^{2\pi} f(x, y)\, dy = 0$, a.e. It follows from the Riemann–Lebesgue lemma that, for any $f \in \mathcal{H}_{\mathrm{a.c.}}$, the auto-correlation sequence

$$\hat{\mu}_f(t) = \int_{\mathbb{T}^2} \overline{f(x, y)} f(T^t(x, y))\, dx\, dy \equiv \langle f, U^t f \rangle \tag{3}$$

converges to zero as $|t| \to \infty$.

We consider a class of functions $\mathcal{F}$ on $\mathbb{T}^2$ such that $f \in \mathcal{F}$ if it verifies:

(1) f is continuous on $\mathbb{T}^2$;

(2) f has a continuous extension $\tilde{f}(z, y)$ on the subset $A(0, r, R) \times \mathbb{T}$, where $A(0, r, R)$ is the annulus of $\mathbb{C}$:

$$A(0, r, R) = \{z \in \mathbb{C} : 0 < r < |z| < R\}$$

for some r and R such that $r < 1 < R$;

(3) for almost every $y \in \mathbb{T}$, the function $z \to \tilde{f}(z, y)$ is analytic in $A(0, r, R)$.

We denote by $\mathcal{F}_{r,R}$ the set of functions verifying (1), (2) and (3) for fixed r and R.

THEOREM 1. *For $f, g \in \mathcal{F} \cap \mathcal{H}_{\mathrm{a.c.}}$ there exist $\lambda \in]0, 1[$ and $C > 0$ such that for every $t \in \mathbb{Z}$,*

$$|\langle f, U^t g \rangle| \leq C\lambda^{|t|}.$$

Proof. In writing $f(x, y) = \sum_{m\in\mathbb{Z}} e^{imy} f_m(x)$ with

$$f_m(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-imy} f(x, y)\, dy,$$

we have the following expression of the scalar product:

$$\langle f, g \rangle = \sum_{m\in\mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} \overline{f_m(x)} g_m(x)\, dx.$$

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Using the formula,

$$T^t(x, y) = \left(x + t\alpha, tpx + t(t-1)\frac{p\alpha}{2} + y\right)$$

we obtain

$$U^t f(x, y) = \sum_{m\in\mathbb{Z}} e^{imy} e^{imtpx} e^{imt(t-1)p\alpha/2} f_m(x + t\alpha)$$

and

$$\langle f, U_t g\rangle = \sum_{m\in\mathbb{Z}} e^{imt(t-1)p\alpha/2} \frac{1}{2\pi} \int_0^{2\pi} \overline{f_m(x)} g_m(x + t\alpha) e^{imtpx}\, dx. \tag{4}$$

Now, we shall use the following lemma.

LEMMA 1. *Let $\varphi \in L^2(\mathbb{T})$. Then:*

(i) *if φ has analytic continuation $\tilde{\varphi}$ in $A(0, r, R)$, then its Fourier coefficients verify*

$$|a_n| < C\rho^{|n|}$$

for any ρ such that $\max(r, 1/R) < \rho < 1$ and for any $n \in \mathbb{Z}$, with

$$C = \max\left\{\frac{1}{2\pi}\int_0^{2\pi} |\tilde{\varphi}(\rho e^{i\theta})|\, d\theta, \frac{1}{2\pi}\int_0^{2\pi} \left|\tilde{\varphi}\left(\frac{1}{\rho}e^{i\theta}\right)\right| d\theta\right\};$$

(ii) *if the Fourier coefficients a_n of φ verify $\forall n \in \mathbb{Z}$ that*

$$|a_n| < C\lambda^{|n|},$$

where $\lambda \in]0, 1[$ and $C > 0$, then φ has analytic continuation in $A(0, \lambda, 1/\lambda)$;

(iii) *if φ has analytic continuation $\tilde{\varphi}$ in $A(0, r, R)$, then $\bar{\varphi}$, defined by $\bar{\varphi}(x) = \overline{\varphi(x)}$, also has analytic continuation $\tilde{\bar{\varphi}}$ in $A(0, \rho, 1/\rho)$ and*

$$\tilde{\bar{\varphi}}(\rho e^{i\theta}) = \bar{\tilde{\varphi}}\left(\frac{1}{\rho}e^{i\theta}\right)$$

for any ρ such that $\max(r, 1/R) < \rho < 1$.

Proof. (i) The Fourier coefficients a_n of φ are also the coefficients of the Laurent series, which are given by

$$a_n = \frac{1}{2i\pi}\int_\Gamma \frac{\tilde{\varphi}(z)}{z^{n+1}}\, dz,$$

where Γ is a circle centered at the origin contained in $A(0, r, R)$. By taking a circle of radius $1/\rho$ for $n \geq 0$ and a circle of radius ρ for $n < 0$, we obtain the inequality.

(ii) The results come directly from the fact that $\tilde{\varphi}(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is a uniformly convergent series in $A(0, \lambda, 1/\lambda)$.

(iii) The assertion follows from the fact that $\tilde{\bar{\varphi}}(z) = \sum_{n\in\mathbb{Z}} \bar{a}_{-n} z^n$ and from (i) and (ii). This achieves the proof of Lemma 1. □

We now continue the proof of Theorem 1. As f is continuous on $\mathbb{T}^2$ and analytic as a function of x for almost all y, then, by Fubini's theorem and Morera's theorem, f_m has analytic continuation

$$\tilde{f}_m(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-imy} \tilde{f}(z, y)\, dy$$

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in $A(0, r, R)$, and the same is true for g. Denote

$$\varphi_{m,t}(x) = \bar{f}_m(x) g_m(x + t\alpha).$$

It follows from (i) and (iii) of Lemma 1 that $\varphi_{m,t}$ has analytic continuation $\tilde{g}_m(ze^{it\alpha})\tilde{\bar{f}}_m(z)$ in $A(0, \rho, 1/\rho)$, $\rho \in]\max(r, 1/R), 1[$. We now use (i) of Lemma 1 to obtain a bound of the Fourier coefficients $a_n^{(m,t)}$ of $\varphi_{m,t}$:

$$|a_n^{(m,t)}| < C_{m,t}\rho^{|n|}, \tag{5}$$

$$C_{m,t} = \max\left\{\frac{1}{2\pi}\int_0^{2\pi} |\tilde{\varphi}_{m,t}(\rho e^{i\theta})|\, d\theta, \frac{1}{2\pi}\int_0^{2\pi}\left|\tilde{\varphi}_{m,t}\left(\frac{1}{\rho}e^{i\theta}\right)\right| d\theta\right\}.$$

Using the expression of $\tilde{g}_m$ and $\tilde{f}_m$, we obtain

$$\begin{aligned} |\tilde{\varphi}_{m,t}(\rho e^{i\theta})| &\leq \frac{1}{(2\pi)^2}\int_0^{2\pi} |\tilde{g}(\rho e^{i(t\alpha+\theta)}, y)|\, dy \cdot \int_0^{2\pi}\left|\tilde{f}\left(\frac{1}{\rho}e^{i\theta}, u\right)\right| du \\ &\leq \|\tilde{g}\|_{C(S(0,\rho)\times\mathbb{T})}\|\tilde{f}\|_{C(S(0,1/\rho)\times\mathbb{T})}, \end{aligned}$$

where $S(0, \rho)$ is the circle of radius ρ centered at the origin and $\|\cdot\|_{C(X)}$ denotes the supremum norm of the space of continuous functions on X. This, together with a similar estimation of $\tilde{\varphi}_{m,t}(1/\rho e^{i\theta})$, show that there is a constant A such that

$$|C_{m,t}| \leq A \tag{6}$$

independently of m and t. Now, $f, g \in \mathcal{H}_{\text{a.c.}}$ if and only if $f_0 = g_0 = 0$. Therefore, we write (4) as

$$\langle f, U_t g\rangle = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} e^{imt(t-1)}\frac{p\alpha}{2} a_{-mtp}^{(m,t)},$$

which yields, for $t \neq 0$,

$$|\langle f, U_t g\rangle| \leq 2A\sum_{m=1}^{\infty}(\rho^{|tp|})^m \leq \frac{2A}{1-\rho}(\rho^{|p|})^{|t|}.$$

□

For any r and R, the class $\mathcal{F}_{r,R}$ contains the linear manifold $\mathcal{G}$ generated by functions of the form $(x, y) \in \mathbb{T}^2 \to P(e^{ix})h(y)$, where h is a continuous function on $\mathbb{T}$ and P is a trigonometric polynomial. Theorem 1 implies that, for any function f of this manifold, that is, for any $f(x, y) = P(e^{ix})h(y)$ such that

$$\int_0^{2\pi} h(y)\, dy = 0,$$

f has a correlation decay with an arbitrary exponential rate. In fact, it can be seen from (4) that these correlations are identically zero for sufficiently great $|t|$. $\mathcal{G}$ is a dense set of functions in $C(\mathbb{T}^2) \cap \mathcal{H}_{\text{a.c.}}$ with respect to the uniform topology. More generally, let $\mathcal{D}_\lambda$ be

$$\mathcal{D}_\lambda = \bigcup_{C>0}\{f \in \mathcal{H}_{\text{a.c.}} : |\hat{\mu}_f(t)| \leq C\lambda^{|t|}, \forall t \in \mathbb{Z}\}.$$

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Then $\mathcal{G} \subset \bigcap_{0<\lambda<1} \mathcal{D}_\lambda$, therefore, $\bigcap_{0<\lambda<1} \mathcal{D}_\lambda$ is dense in $C(\mathbb{T}^2) \cap \mathcal{H}_{\text{a.c.}}$ in the uniform topology.

However, this exponential decay is far from being abundant. More precisely, let $\mathcal{E}$ be the set of functions in $\mathcal{H}_{\text{a.c.}}$ with exponentially decaying correlations:

$$\mathcal{E} = \bigcup_{\substack{0\le\lambda<1 \\ C>0}} \{f \in \mathcal{H}_{\text{a.c.}} : \exists T > 0, |\hat{\mu}_f(t)| \le C\lambda^{|t|}, \text{ for } |t| > T\}.$$

Thus, a function $f \in \mathcal{H}_{\text{a.c.}} \setminus \mathcal{E}$ has decaying correlations, with a rate slower than any exponential rate. That is for any $C > 0$, and $\lambda \in]0, 1[$, there is an infinite sequence t_k, such that

$$|\hat{\mu}_f(t_k)| > C\lambda^{|t_k|},$$

that is, for any $\lambda \in]0, 1[$,

$$\overline{\lim_t} \left| \frac{\hat{\mu}_f(t)}{\lambda^{|t|}} \right| = +\infty.$$

The following theorem shows that the set of all f in $\mathcal{H}_{\text{a.c.}}$ with a correlation decay rate slower than any exponential rate is generic.

THEOREM 2.
(1) *The set $C(\mathbb{T}^2) \setminus \mathcal{E}$ is a G_δ-dense set in $C(\mathbb{T}^2)$ in the uniform topology.*
(2) *The set $(H_{\text{a.c.}} \setminus \mathcal{E}) \cap C(\mathbb{T}^2)$ is a G_δ-dense set in $\mathcal{H}_{\text{a.c.}} \cap C(\mathbb{T}^2)$ in the uniform topology.*

Proof. To see that $C(\mathbb{T}^2) \setminus \mathcal{E}$ is a G_δ-set, notice that the projection $f \to P_{\text{d}} f$ is continuous on $C(\mathbb{T}^2)$; hence, $P_{\text{a.c.}}$ is also continuous. Thus, $H_{\text{a.c.}} \cap C(\mathbb{T}^2)$ is closed in $C(\mathbb{T}^2)$. From the definition of $\mathcal{E}$, we have that $\mathcal{E} \cap C(\mathbb{T}^2)$ is a countable union of closed sets. It follows that $C(\mathbb{T}^2) \setminus \mathcal{E}$ and $(H_{\text{a.c.}} \setminus \mathcal{E}) \cap C(\mathbb{T}^2)$ are G_δ-sets.

(1) It is enough to show that for any $f \in \mathcal{E}$, and any $\varepsilon > 0$, there is $g \in C(\mathbb{T}^2) \setminus \mathcal{E}$ such that $\|f - g\|_\infty \le \varepsilon$. Take $h \in C(\mathbb{T}^2) \cap \mathcal{H}_{\text{d}}$, e.g. $h(x, y) = h(x) = \varepsilon e^{ix}$; then $g = f + h$ verifies the requirement.

(2) Let $\mathcal{G}$ be the set of all $f \in C(\mathbb{T}^2) \cap H_{\text{a.c.}}$ of the form $f(x, y) = \sum_{j=1}^n P_j(e^{ix}) h_j(y)$, where n runs over the positive integers, P_j runs over the complex polynomials and h_j runs over the continuous functions on $\mathbb{T}$. By Theorem 1, $\mathcal{G}$ is contained in $\mathcal{E}$. On the other hand, $\mathcal{G}$ is dense in $C(\mathbb{T}^2) \cap H_{\text{a.c.}}$ with respect to the uniform topology, since $P_{\text{a.c.}}$ is continuous on $C(\mathbb{T}^2)$. Then, it suffices to approximate, in the $C(\mathbb{T}^2)$ norm, every f in $\mathcal{G}$ by functions in $C(\mathbb{T}^2) \cap (H_{\text{a.c.}} \setminus \mathcal{E})$. So let us fix some $f \in \mathcal{G}$ of the above form, and denote $P_j(z) = \sum_{k \in K_j} a_{j,k} z^k$, K_j being a finite set of natural numbers. We will show that in 'perturbing' f by some $\varphi \in C(\mathbb{T}^2)$, the result does not belong to $\mathcal{E}$. Many such φ can be constructed. Here we shall consider a function φ given by

$$\varphi(x, y) = h(x)e^{iy}, \quad h(x) = \sum_{n \ne 0} \frac{e^{inx}}{n^2}.$$

We will show that, for any real number $\delta \ne 0$, $g = f + \delta\varphi$ does not belong to $\mathcal{E}$. The equality

$$\langle g, U^t g \rangle = \langle f, U^t f \rangle + \delta \langle f, U^t \varphi \rangle + \delta \langle \varphi, U^t f \rangle + \delta^2 \langle \varphi, U^t \varphi \rangle$$

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shows that $g \in \mathcal{E}$ if and only if for some $\gamma > 0$,

$$\sup_{t\in\mathbb{Z}} |e^{\gamma|t|}[\langle f, U^t\varphi\rangle + \langle \varphi, U^t f\rangle + \delta\langle \varphi, U^t\varphi\rangle]|$$

is finite. In particular, if $g \in \mathcal{E}$, we have

$$\lim_{|t|\to\infty} t^m(\langle f, U^t\varphi\rangle + \langle \varphi, U^t f\rangle + \delta\langle \varphi, U^t\varphi\rangle) = 0 \tag{6*}$$

for any $m \in \mathbb{N}$.

We shall show that this fails to be true.

By direct computations using (4), we obtain

$$\langle f, U^t\varphi\rangle = e^{-it(t+1)p\alpha/2}\sum_{j=1}^{n}\bar{\hat{h}}_j\sum_{k\in K_j}\bar{a}_{j,k}\frac{e^{ikt\alpha}}{(tp-k)^2}, \tag{6a}$$

where

$$\hat{h}_j = \frac{1}{2\pi}\int_0^{2\pi} e^{-iy}h_j(y)\,dy.$$

As $\langle \varphi, U^t f\rangle = \overline{\langle f, U^{-t}\varphi\rangle}$, we obtain

$$\langle \varphi, U^t f\rangle = e^{it(t-1)p\alpha/2}\sum_{j=1}^{n}\hat{h}_j\sum_{k\in K_j}a_{j,k}\frac{e^{ikt\alpha}}{(tp+k)^2}. \tag{6b}$$

Similarly, we have

$$\begin{aligned}\langle \varphi, U^t\varphi\rangle &= e^{it(t-1)p\alpha/2}\frac{1}{2\pi}\int_0^{2\pi}\overline{h(x)}h(x+t\alpha)e^{itpx}\,dx\\ &= e^{it(t-1)p\alpha/2}\theta_{tp}(t\alpha),\end{aligned} \tag{6c}$$

where

$$\theta_k(x) = \sum_{\substack{n=-\infty\\ n\neq 0\\ n\neq -k}}^{\infty}\frac{e^{inx}}{n^2(n+k)^2}.$$

This series is computed by using the identity

$$\frac{1}{n^2(n+k)^2} = \frac{1}{k^2}\left[\frac{1}{n^2} + \frac{1}{(n+k)^2} - \frac{2}{kn} + \frac{2}{k(n+k)}\right].$$

Hence, we obtain

$$\theta_k(x) = \frac{1}{k^2}\left[(1+e^{-ikx})h(x) - \frac{1}{k^2}(1+e^{-ikx}) - 2\mathcal{E}_k(x)\right], \tag{6d}$$

where

$$\mathcal{E}_k(x) = \sum_{\substack{n=-\infty\\ n\neq 0\\ n\neq -k}}^{+\infty}\frac{e^{inx}}{n(n+k)}.$$

It is easy to see that $\sup_{x\in\mathbb{T}}|\mathcal{E}_k(x)|$ converges to zero when $k\to\infty$, and $\mathcal{E}_k(0) = 2/k^2$, which implies that $\theta_k(0)$ is a rational function of k with a unique pole, at the origin, of order 4.

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Now we distinguish two cases:

(1) $\alpha/2\pi$ is a rational number. Denote $\alpha/2 = r/q \cdot 2\pi$ and take the infinite subsequence $t = uq$, $u \in \mathbb{Z}$. If $g \in \mathcal{E}$, then (6$*$), (6a), (6b) and (6c) imply that

$$\lim_{|u|\to\infty} u^m (R(u) + \delta\theta_{uqp}(0)) = 0$$

for any $m > 0$, where $R(u)$ is a rational function of u, with poles of order 2. But, as we noticed above, $\theta_{uqp}(0)$ is a rational function of u, with poles of order 4. This implies that $\delta = 0$, a contradiction.

(2) $\alpha/2\pi$ is an irrational number. Then, the sequence

$$(a(t), b(t)) \equiv \left(t^2\frac{\alpha}{2}, t\frac{\alpha}{2}\right)$$

is uniformly distributed in $\mathbb{T}^2$. In fact, according to Weyl criterion [**8**, Theorem 6.2, p. 48], the sequence $(a(t), b(t))$ is uniformly distributed in $\mathbb{T}^2$ if and only if, for any couple of integers $(n, m) \neq (0, 0)$,

$$\frac{1}{N}\sum_{t=0}^{N-1} e^{i(na(t)+mb(t))} \to 0, \quad \text{as } N \to \infty.$$

But, $na(t) + mb(t)$ is a polynomial with at least one irrational coefficient. Then, it follows from another theorem of Weyl [**8**, Theorem 3.2, p. 27], that this sequence is uniformly distributed in $\mathbb{T}$. This shows that $(t^2(\alpha/2), t(\alpha/2))$ is uniformly distributed and, therefore, dense in $\mathbb{T}^2$. Thus, for any $(x, y) \in \mathbb{T}^2$, let (t_ℓ) be a sequence of positive integers such that $(t_\ell^2(\alpha/2), t_\ell(\alpha/2))$ converges to (x, y). It follows from (6a), (6b), (6c) and (6d) that

$$\begin{aligned}\lim_{\ell\to\infty} t_\ell^2[\langle f, U^{t_\ell}\varphi\rangle + \langle\varphi, U^{t_\ell} f\rangle + \delta\langle\varphi, U^{t_\ell}\varphi\rangle] \\ = \frac{1}{p^2}\Bigg[e^{-ip(x+y)}\sum_{j=1}^{n}\bar{\hat{h}}_j\overline{P_j(e^{-2iy})} + e^{ip(x-y)}\sum_{j=1}^{n}\hat{h}_j P_j(e^{2iy}) \\ +\delta e^{ip(x-y)}(1 + e^{-2ipx})h(2y)\Bigg] = 0.\end{aligned}$$

This shows that $h(y)$ is a rational function of e^{iy}, which is, clearly, not true. □

Remark. It follows from (6c) and (6d) that $\langle\varphi, U^t\varphi\rangle$ is $O(1/t^2)$ and not $o(1/t^2)$. This method allows the construction of examples of functions of arbitrary rate of decay of correlations. More precisely, let $\mathcal{C}_\mathcal{K}$, $k \geq 2$, be the subset of $C(\mathbb{T}^2)$ such that

$$\frac{\partial^l}{\partial x^l}(f(x, y) \in C(\mathbb{T}^2), \quad \text{for all } l = 1, \ldots, k.$$

Then, for all f and g in $\mathcal{C}_\mathcal{K}$,

$$|\langle f, U^t g\rangle| \leq C/|t|^k.$$

In conclusion, we obtain a partition of $C(\mathbb{T}^2)$ into two dense sets, the first corresponding to exponential rate of decay of correlations while the second corresponds to a subexponential decay rate.

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2. *Mixing subsets and the rate of mixing*

We study the rate of mixing of a class of measurable subsets of $\mathbb{T}^2$. We say that the measurable subsets E and F are mixing if

$$\mu(E \cap T^{-t}F) - \mu(E)\mu(F) \to 0, \quad \text{when } |t| \to \infty.$$

If we denote $f = 1_F - \mu(F)$ and $g = 1_E - \mu(E)$, then

$$\mu(E \cap T^{-t}F) - \mu(E)\mu(F) = \langle g, U^t f \rangle. \tag{7}$$

It can be seen that a set E is self-mixing if and only if $g \in \mathcal{H}_{\text{a.c.}}$, that is,

$$\frac{1}{2\pi} \int_0^{2\pi} 1_E(x, y)\, dy = \mu(E). \tag{8}$$

Thus, a self-mixing set is a set of which the x-section has constant length.

In what follows, we denote by $\mathcal{B}_0$ the family of measurable subsets with constant length x-section. Let $\mathcal{B}_1$ be the family of subsets from $\mathcal{B}_0$ of the form $E = \{(x, y) \mid y \in \Delta + f(x)\}$, where Δ is a measurable subset of $\mathbb{T}$, and f is a measurable function from $\mathbb{T}$ to itself. We say that E is a translate of Δ by f. Let $\mathcal{B}_2 \subset \mathcal{B}_0$ such that $E \in \mathcal{B}_2$ if E is a finite disjoint union of E_i, $E_i \in \mathcal{B}_1$. Finally, we denote by $\mathcal{A}_n$ the algebra of measurable subsets generated by squares with sides of length $2\pi/n$, $\mathcal{C}_n = \mathcal{A}_n \cap \mathcal{B}_2$, and $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$. We shall first study the rate of mixing for two elements E, F of $\mathcal{C}_n$.

LEMMA 2. *With the above notation, for any $E, F \in \mathcal{C}$, we have*

$$\langle g, U_t f \rangle = \frac{h(t)}{t},$$

where $h(t)$ is a bounded function which is either null, or with no limit as $|t| \to +\infty$.

Proof. By refining the partitions we may suppose that E and F belong to the same $\mathcal{C}_n$ and, therefore, have the form

$$E = \bigcup_{r=0}^{n-1} \left\{ \left[\frac{r}{n}, \frac{r+1}{n}\right[\times \bigcup_{k \in K_r} \left[\frac{k}{n}, \frac{k+1}{n}\right[\right\},$$

where K_r is a subset of $\{0, 1, \ldots, n-1\}$. F has a similar form with J_r instead of K_r. We shall use formula (4) to calculate the mixing rate (7). We first compute, for $m \neq 0$,

$$g_m(x) = \frac{1}{2\pi} \int_0^{2\pi} g(x, y) e^{-imy}\, dy.$$

For $x \in [r/n, (r+1)/n[$ we have

$$\begin{aligned} g_m(x) &= \frac{1}{2\pi} \int_0^{2\pi} 1_{E_x}(y) e^{-imy}\, dy \\ &= \frac{i}{2\pi m} \sum_{k \in K_r} (e^{-im(k+1)/n} - e^{-imk/n}) \\ &= \frac{i}{2\pi m} (e^{-im/n} - 1) B(r, m), \end{aligned} \tag{9}$$

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where $B(r,m)=\sum_{k\in K_r} e^{-imk/n}$.

We have similar formula for the set F, with $A(r,m)=\sum_{j\in J_r} e^{-imj/n}$, instead of $B(r,m)$. Inserting (9) into (4) we obtain

$$\begin{aligned}\langle g, U^t f\rangle &= \sum_{m\neq 0} e^{imt(t-1)p\alpha/2}\frac{1}{2\pi}\frac{1}{(2\pi m)^2}\times|e^{-im/n}-1|^2\\ &\quad\times\int_0^{2\pi} dx\, e^{itmpx}\left(\sum_{s=0}^{n-1}\overline{B(s,m)}1_{[s/n,(s+1)/n[}(x)\right)\\ &\quad\times\left(\sum_{r=0}^{n-1}A(r,m)1_{[(r/n)-t\alpha,((r+1)/n)-t\alpha[}(x)\right).\end{aligned}\tag{10}$$

For a fixed s, let $r(s,t)$ be the integer such that

$$\frac{r(s,t)}{n}-t\alpha\in\left[\frac{s}{n},\frac{s+1}{n}\right[.$$

We denote $\tilde r(s,t)=(r(s,t)/n)-t\alpha$. For economy of notation, we denote $r(s,t)$ simply by r. Thus,

$$\begin{aligned}&\langle g, U_t f\rangle\\ &= \sum_{m\neq 0} e^{imt(t-1)p\alpha/2}\frac{1}{(2\pi)^3}\frac{|e^{-im/n}-1|^2}{m^2}\sum_{s=0}^{n-1}\overline{B(s,m)}\\ &\quad\times\left[\int_{s/n}^{\tilde r}A(r-1,m)e^{itmpx}\,dx+\int_{\tilde r}^{(s+1)/n}A(r,m)e^{itmpx}\,dx\right]\\ &= \frac{1}{itp}\frac{1}{(2\pi)^3}\sum_{s=0}^{n-1}\sum_{k\in K_s}\sum_{m\neq 0}\frac{1}{m^3}|e^{im/n}-1|^2e^{imk/n}e^{imt(t-1)p\alpha/2}\\ &\quad\times[A(r-1,m)(e^{impt\tilde r}-e^{imtps/n})+A(r,m)(e^{impt(s+1)/n}-e^{impt\tilde r})].\end{aligned}\tag{11}$$

It follows from this equation that $|t\langle g,U_t f\rangle|$ is bounded. Now, consider the integers t such that $t\alpha\in[\ell/n,(\ell+1)/n[$. Therefore, for fixed s and ℓ, and all t such that $\alpha t\in[\ell/n,(\ell+1)/n[$, $r(s,t)$ verifies

$$\frac{s}{n}=\frac{r-(\ell+1)}{n},$$

that is, r depends only on s and $r(s)=s+\ell+1 \bmod n$. Define the function

$$\xi(x)=\frac{1}{(2\pi)^3}\sum_{m\neq 0}\frac{1}{m^3}|e^{im/n}-1|^2e^{imx}.$$

Then we may write (11), for t such that $t\alpha\in[\ell/n,(\ell+1)/n[$, as

$$\begin{aligned}\langle g,U_t f\rangle &= \frac{1}{itp}\sum_{s=0}^{n-1}\sum_{k\in K_s}\left\{\sum_{j\in J_{r(s)-1}}\xi\left(-\frac{p\alpha t^2}{2}+p\left(\frac{r}{n}-\frac{\alpha}{2}\right)t+\frac{k-j}{n}\right)\right.\\ &\quad\left.-\sum_{j\in J_{r(s)-1}}\xi\left(\frac{p\alpha t^2}{2}+p\left(\frac{s}{n}-\frac{\alpha}{2}\right)t+\frac{k-j}{n}\right)\right.\end{aligned}$$

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$$+\sum_{j\in J_{r(s)}}\xi\left(\frac{p\alpha t^2}{2}+p\left(\frac{s+1}{n}-\frac{\alpha}{2}\right)t+\frac{k-j}{n}\right)$$
$$-\sum_{j\in J_{r(s)}}\xi\left(-\frac{p\alpha t^2}{2}+p\left(\frac{r}{n}-\frac{\alpha}{2}\right)t+\frac{k-j}{n}\right)\Bigg\}.$$

We shall suppose that α is an irrational, since $t\langle g, U_t f\rangle$ is periodic for rational α, and the theorem follows in this case.

The third term in the right-hand side can be written as

$$\sum_{s=0}^{n-1}\sum_{k\in K_s}\sum_{j\in J_{r(s)}}\xi\left(\frac{p\alpha t^2}{2}+p\left(\frac{s+1}{n}-\frac{\alpha}{2}\right)t+\frac{k-j}{n}\right)$$
$$=\sum_{s=0}^{n-1}\sum_{k\in K_{s-1}}\sum_{j\in J_{r(s)-1}}\xi\left(p\alpha t^2+p\left(\frac{s}{n}-\frac{\alpha}{2}\right)t+\frac{k-j}{n}\right).$$

Here we used the fact that $r(s-1)=r(s)-1$. Let us write $t=un+v$, then $\alpha t\in[\ell/n,(\ell+1)/n[$ if and only if

$$\alpha u\in\left[\frac{\ell}{n^2}-\frac{\alpha v}{n},\frac{\ell+1}{n^2}-\frac{\alpha v}{n}\right[.$$

Thus, we have for a fixed v,

$$-\frac{p\alpha t^2}{2}+p\left(\frac{r}{n}-\frac{\alpha}{2}\right)t=-\frac{p\alpha}{2}n^2u^2-p\alpha n\left(v+\frac{1}{2}\right)u\pmod{2\pi}.$$

We shall denote the right-hand side of the above equation by $P_1(u)$:

$$P_1(u)=-\frac{p\alpha}{2}n^2u^2-p\alpha n\left(v+\frac{1}{2}\right)u\pmod{2\pi}.$$

Similarly, denote

$$P_2(u)\equiv\frac{p\alpha t^2}{2}+p\left(\frac{s}{n}-\frac{\alpha}{2}\right)t=\frac{p\alpha}{2}n^2u^2+p\alpha n\left(v-\frac{1}{2}\right)u$$
$$H_1(x)\equiv\sum_{s=0}^{n-1}\sum_{k\in K_s}\sum_{j\in J_{r(s)-1}}\xi\left(x+\frac{k-j}{n}\right)$$
$$H_2(x)\equiv\sum_{s=0}^{n-1}\sum_{k\in K_{s-1}}\sum_{j\in J_{r(s)-1}}\xi\left(x+\frac{k-j}{n}\right)$$
$$H_3(x)\equiv\sum_{s=0}^{n-1}\sum_{k\in K_s}\sum_{j\in J_{r(s)}}\xi\left(x+\frac{k-j}{n}\right).$$

Now, as noted above, $r(s-1)=r(s)-1\pmod n$. Therefore, $H_2=H_3$. We write the function $h_1(t)$, defined by $h_1(t)=itp\langle g,U_t f\rangle$, in the form

$$\begin{aligned}h_1(un+v)&=i(un+v)p\langle g,U_{(un+v)}f\rangle\\&=H_1(P_1(u))-H_1(P_2(u))+H_2(P_2(u))-H_3(P_1(u)).\end{aligned}$$

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Denoting $G_1 = H_1 - H_3$, we obtain

$$h_1(un + v) = G_1(P_1(u)) - G_1(P_2(u)). \tag{12}$$

Let $Q_1(u) = \alpha u$ and $Q_2(u) = \alpha u^2$. As shown in Theorem 2, the sequence $(Q_1(u), Q_2(u))$ is uniformly distributed in $([0, 2\pi[)^2$. Then, for any $x \in](\ell/n^2) - (\alpha v/n), ((\ell+1)/n^2) - (\alpha v/n)[$ and any $y \in [0, 2\pi[$, there exist u_k such that $Q_1(u_k) \to x$ and $Q_2(u_k) \to y$ for $k \to \infty$. Thus, for large k,

$$u_k \alpha \in \left]\frac{\ell}{n^2} - \frac{\alpha v}{n}, \frac{\ell+1}{n^2} - \frac{\alpha v}{n}\right[.$$

Suppose now that $tp\langle g, U_t f\rangle$ has a limit L as $t \to \infty$. Then, for any fixed ℓ and v, $h_1(u_k n + v)$ has the limit iL. (12) yields

$$iL = \lim_{k\to\infty} h_1(u_k) = G_1(ay + bx) - G_1(-ay + cx),$$

where $a = -pn^2/2$, $b = -p(v + 1/2)$ and $c = pn(v - 1/2)$.

It is easy to see that for n sufficiently great, the mapping $A : (x, y) \to (ay+bx, -ay+cx)$ will transform the strip

$$\Delta_0 = \left]\frac{\ell}{n^2} - \frac{\alpha v}{n}, \frac{\ell+1}{n^2} - \frac{\alpha v}{n}\right[\times [0, 2\pi[$$

into a parallelogram Δ_1 sufficiently dilated (since the determinant of A is equal to $1/2p^2n^3$) in order that its projection onto the x-axis (respectively y-axis) covers at least an interval of length 2π. Thus, for all $(u, v) \in \Delta_1$ we have

$$G_1(u) = G_1(v) + iL,$$

which implies that G_1 is constant. Now, on account of (12), for any fixed ℓ, v, $h(u) = L = 0$. That is,

$$\langle g, U_t f\rangle = 0.$$

□

THEOREM 3. *For any $E, F \in \mathcal{C}$, the rate of mixing is* $O(1/t)$. *If this rate is* $o(1/t)$, *then for any* $|t| \geq 1$,

$$\mu(E \cap T^{-t}F) = \mu(E)\mu(F). \tag{13}$$

Proof. If $h(t)$ tends to zero, then, according to Lemma 2, $h(t)$ is identically zero. This proves the theorem. □

Two sets satisfying (13) for any $|t| > 1$ are called independent under $T_{\alpha,p}$.

THEOREM 4.

(1) *If E and F are two translates of E_0 and F_0 by the same function $h(x)$, then E and F are independent under $T_{0,p}$.*
(2) *If E and F are two translates of E_0 and F_0 by two constants, then E and F are independent under $T_{\alpha,p}$.*

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(3) *Let $\mathcal{P}$ be a partition of $\mathbb{T}^2$ by horizontal strips, that is, $\mathcal{P} = \{T \times E_i : i = 1, \dots, n\}$, where $\{E_i\}$ is a partition of $\mathbb{T}$. Then $\mathcal{P}$ is pairwise independent under $T_{\alpha,p}$.*

Proof. (1) We shall use (4) with the notation of (7). Then we compute g_m and f_m as

$$\begin{aligned} g_m(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-imy} 1_{E_0+h(x)}(y)\,dy \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-imy} 1_{E_o}(y-h(x))\,dy \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-im(y+h(x))} 1_{E_0}(y)\,dy = e^{-imh(x)} C_m(E_0), \end{aligned} \tag{14}$$

where

$$C_m(E_0) = \frac{1}{2\pi}\int_0^{2\pi} e^{-imy} 1_{E_0}(y)\,dy.$$

Formula (4) gives

$$\langle g, U_t f\rangle = \sum_{m\neq 0} \frac{1}{2\pi}\overline{C_m(E_0)} C_m(F_0) \int_0^{2\pi} e^{-imptx}\,dx = 0.$$

(2) We again use the same calculations as above, with $h_1 = A$ and $h_2 = B$, to obtain

$$\langle g, U_t f\rangle = \sum_{m\neq 0} \frac{1}{2\pi}\overline{C_m(E_0)} C_m(F_0) e^{-im(A-B)} e^{imt(t-1)p\alpha/2} \int_0^{2\pi} e^{-imptx}\,dx = 0.$$

(3) Follows from (2). □

Remark. The same proof works for transformations of the form $T(x, y) = (x, \varphi(x) + y)$, where φ is a measure preserving function. This can also be directly obtained by using Fubini's theorem. In fact, let us denote the section of E at x by

$$E(x) = E_0 + h(x),$$

and similarly the section of F. A straightforward calculation shows that

$$\begin{aligned} \mu(E \cap T^{-n}F) &= \int dx \int 1_{E(x)}(y) 1_{F(x)}(y + n\varphi(x))\,dy \\ &= \int \lambda((E_0 + h(x)) \cap (F_0 + h(x) - n\varphi(x)))\,dx \\ &= \int \lambda(E_0 \cap (F_0 - n\varphi(x))\,dx \\ &= \int 1_{E_0}(y)\,dy \int 1_{F_0-y}(n\varphi(x))\,dx = \lambda(E_0)\lambda(F_0) \\ &= \mu(E)\mu(F). \end{aligned}$$

THEOREM 5. *Let E and F be two translates of E_0 and F_0 by $h_1(x)$ and $h_2(x)$ respectively. If $h_1(x)$ and $h_2(x)$ have analytic continuations in $A(0, r, R)$, then there is $\lambda \in]0, 1[$, $C > 0$ and τ, depending on (λ, E, F), such that for every $|t| > \tau$,*

$$|\mu(E \cap T^{-t}F) - \mu(E)\mu(F)| \leq C\lambda^{|t|}.$$

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Proof. The proof again uses (14) and (4) to obtain

$$\langle g, U^t f\rangle = \sum_{m\neq 0} e^{imt(t-1)p\alpha/2}\frac{1}{2\pi}\times \overline{C_m(E_0)}C_m(F_0)\int_0^{2\pi} e^{imtpx}e^{i(h_1(x)-h_2(x+t\alpha))m}\,dx.$$

Let $\tilde{h}_1(z)$ and $\tilde{h}_2(z)$ be the analytic continuations of $h_1(x)$ and $h_2(x)$ respectively in $A(0, r, R)$, and let $H_1 = \mathrm{Im}(\tilde{h}_1)$, $H_2 = \mathrm{Im}(\tilde{h}_2)$.

Then, the function $\varphi_{m,t}(x) = e^{im(h_1(x)-h_2(x+t\alpha))}$ has analytic continuation in $A(0, r, R)$. We again use (i) of Lemma 1 to obtain a bound of the Fourier coefficients $a_n^{(m,t)}$ of $\varphi_{m,t}$, $|a_n^{(m,t)}| \leq C_{m,t}\rho^{|n|}$ and bound of $C_{m,t}$:

$$\begin{aligned} C_{m,t} &\leq \frac{1}{2\pi}\int_0^{2\pi} |e^{im[\tilde{h}_1(\rho e^{i\theta})-\tilde{h}_2(\rho e^{i\theta}e^{i t\alpha})]}|\,d\theta \\ &\leq \sup[e^{-H_1(\rho e^{i\theta})-H_2(\rho e^{i\theta}e^{it\alpha})}]^m \\ &\leq B^m, \end{aligned}$$

where $B = \sup e^{H_2(\rho e^{i\theta})}\sup e^{-H_1(\rho e^{i\theta})}$. The supremum is taken over θ running in $[0, 2\pi]$, and ρ running in $[\rho_0, 1]$, such that $\max(r, 1/R) < \rho_0 < 1$.

Here, we have only considered the case $m > 0$, since in the case $m < 0$, the corresponding term is the complex conjugated one.

A similar bound is obtained for the circle $e^{i\theta}/\rho$, which we denote D^m. Let $A = \max(B, D)$; this gives $|C_{m,t}| \leq A^m$. Thus,

$$\langle g, U_t f\rangle \leq 2\sum_{m=1}^{\infty}(A\rho^{|tp|})^m.$$

This series converges if and only if

$$\rho < \left(\frac{1}{A}\right)^{1/|tp|}.$$

When $|t| \to \infty$,

$$\left(\frac{1}{A}\right)^{1/|tp|} \to 1.$$

Therefore, for all $|t|$ greater than some $t_0 > 0$,

$$\left(\frac{1}{A}\right)^{1/|tp|} > \max\left(r, \frac{1}{R}\right),$$

so that the series converges for any ρ such that

$$\max\left(r, \frac{1}{R}\right) < \rho < \left(\frac{1}{A}\right)^{1/|t_0 p|}.$$

Thus,

$$|\langle g, U^t f\rangle| \leq 2\frac{A\rho^{|tp|}}{1 - A\rho^{|t_0 p|}}.$$

□

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COROLLARY 1. *Let $E \in \mathcal{B}_2$ be such that E is a finite union of disjoint translate sets by functions with analytic continuations in $A(0, r, R)$. Then, there are $\lambda \in]0, 1[$, $C > 0$ and τ such that for any $|t| > \tau$,*

$$|\mu(E \cap T^{-t}E) - \mu(E)\mu(E)| \leq C\lambda^{|t|}.$$

This corollary follows from an application of Theorem 5.

It is possible to construct subsets with exponential mixing rate that are not independent. Examples can be constructed by taking E as a disjoint union of translates of two disjoint intervals E_0 and F_0 by two functions h_1 and h_2, respectively, such that $h_1 - h_2 = G$ has analytic continuation in some $A(0, a, 1/a)$, $a \in]0, 1[$ with singularities outside this annulus. For example,

$$G(x) = -\varepsilon \sin x,$$

where ε is sufficiently small in order to ensure disjointness of the two translates. Let us show, first for $\alpha = 0$, that $\langle g, U^t f\rangle \neq 0$, for all values of t.

Denote $a_m = C_m(E_0)C_m(F_0)$. It follows from the definition of $C_m(E_0)$, that $|a_m| \leq A/m^2$ for some constant A. The expression of $\langle g, U^t f\rangle$ given in the proof of Theorem 5 becomes, in this example,

$$\langle g, U^t f\rangle = \sum_{m\neq 0} a_m J_{-mtp}(\varepsilon m),$$

where $J_n(x)$ is the Bessel function of order n given by

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}e^{ix\sin\theta}\,d\theta.$$

Now using the identity $J_{-n}(x) = J_n(-x)$, we have

$$\langle g, U^t f\rangle = \sum_{m>0}(a_m + a_{-m})J_{mtp}(\varepsilon m) \equiv F_t(\varepsilon).$$

We shall show that the series of the right-hand side defines a non-identically vanishing analytic function of ε. This will imply that $\langle g, U^t f\rangle$ has only a finite number of zeros. Therefore, outside a countable number of values of ε, all $\langle g, U^t f\rangle$ will be, for all $t \in \mathbb{Z}$, non-vanishing.

We shall use the following inequality [**9**, p. 269]:

$$|J_n(n\rho e^{i\alpha})| \leq \exp(n\rho\sqrt{(\sinh^2 u + \sin^2\alpha)} - nu)$$

for all positive values of u. Hence, for $z = |z|e^{i\alpha}$ we have

$$|J_{mtp}(mz)| \leq \exp(m[|z|\sqrt{(\sinh^2 u + \sin^2\alpha)} - tpu]).$$

Thus, for $t > 0$, $p > 0$, and for any z such that

$$|z| \leq \sup_u \frac{pu}{\cosh u} \equiv \rho_1,$$

we have $|z|\sqrt{(\sinh^2 u + \sin^2\alpha)} - tpu \leq 0$. Therefore, $|J_{mtp}(mz)| \leq 1$. It follows that $|2(a_{+m} + a_{-m})J_{mtp}(mz)|$ is uniformly bounded in the disk with centre 0 and radius ρ_1,

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by $4A/m^2$. This ensures the analyticity of $F_t(z)$ in the same disk. To show it is not identically zero, notice that $J_n(z)$ has a zero of order n at the origin, thus $J_m^{(n)}(0) = 0$ for all $n < m$ and $J_n^{(n)}(0) \neq 0$. Let m_0 be the smallest $m > 0$ such that $a_m \neq 0$, then $F_t^{(m_0 tp)}(0) \neq 0$. Thus, $F_t(z)$ is not identically zero for any $t \in \mathbb{N}$. This completes the proof for $p > 0$ and $t > 0$. The proofs in the other cases follows from the fact that $F_t(-\varepsilon) = F_{-t}(\varepsilon)$.

For the case of irrational α, we can take $h_2 =$ constant, $h_1(x) = h_2 + \varepsilon \sin x$ and $E_0 = F_0$. Then, as above, we get

$$\langle g, U_t f\rangle = \sum_{m\neq 0} \frac{1}{2\pi} a_m 2\cos mt(t-1)p\alpha/2 J_{mtp}(\varepsilon m).$$

We can then repeat the above argument without modification, since $\cos \alpha k$ is non-vanishing for any integer k.

Finally, we show that there are subsets $E \subseteq \mathcal{C}$ such that the self-mixing rate oscillates around $1/t$.

THEOREM 6. *Let* $E_{a,b} = ([0, b[\times[0, 2a[) \cup ([b, 2\pi[\times([0, a[\cup[\pi, \pi + a[)),\ a \in]0, \pi[,$ $b \in]0, 2\pi[$ *and let*

$$\mu_{a,b}(t) = \mu(E_{a,b} \cap T_{0,p}^{-t} E_{a,b}) - \mu(E_{a,b})\mu(E_{a,b}).$$

Then, there are a *and* b *such that* $\mu_{a,b}(t) = h_{a,b}(t)/t$, *and* $h_{a,b}(t)$ *has no limit as* $t \to \infty$.

Proof. Denoting $E_{a,b} = E$, $1_E - \mu(E) = f$ and using (4), we obtain by straightforward calculations that

$$\begin{aligned}\langle f, U^t f\rangle &= \frac{1}{(2\pi)^3}\frac{1}{itp}[-2\varphi(tpb) + \varphi(tpb + a) + \varphi(tpb - a)\\ &\quad -4\varphi(tpb + \pi) + 2\varphi(tpb + a + \pi) + 2\varphi(tpb - a + \pi)]\\ &\equiv \frac{h_{a,b}(t)}{t}, \end{aligned} \tag{15}$$

where

$$\varphi(x) = \sum_{m\neq 0} \frac{1}{m^3} e^{imx}.$$

For a fixed $b/2\pi \notin \mathbb{Q}$, and any $x \in [0, 2\pi[$, there is t_k such that $t_k pb \to x$, as $k \to \infty$. Thus, (15) yields

$$\begin{aligned}\lim_{k\to\infty} h(a, b, t_k) &= -2\varphi(x) + \varphi(x + a) + \varphi(x - a)\\ &\quad -4\varphi(x + \pi) + 2\varphi(x + a + \pi) + 2\varphi(x - a + \pi)\\ &\equiv \Phi(x, a). \end{aligned} \tag{16}$$

As $\varphi(x) = \varphi(-x)$, $\varphi(0) = \varphi(\pi) = 0$ and $\varphi(x + 2\pi) = \varphi(x)$, we have $\Phi(\pi, a) = 0$, for any $a \in]0, \pi[$. Let us suppose that $\lim h(a, b, t)$ exists. Then (16) implies that $\Phi(x, a)$ is independent of x, hence $\Phi(x, a) = 0$, for any $x \in [0, 2\pi[$ and any $a \in I$. But,

$$\Phi(\pi/2, \pi/2) = 2\varphi(\pi/2) \neq 0.$$

Then, there is an interval I such that $\Phi(x, a) \neq 0$, for any $a \in I$ and any x in some interval J. Thus, $0 \to a$ has no limit as $t \to \infty$ for $b \in [0, 2\pi[\setminus Q$ and any $a \in I$. □

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Similar calculations show for $b = \pi/2$ and $a = \pi/2$, that $\xi_t = itp(2\pi)^3\langle f, U^t f\rangle$ is equal to 0 for $t = 4\ell$, or $t = 4\ell + 2$, and equal to $2\sum_{m=1}^{\infty} 1/m^3$ for $t = 4\ell + 1$ and to $-2\sum_{m=1}^{\infty} 1/m^3$ for $t = 4\ell + 3$. That is, $b(t)$ is a periodic function taking four values.

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