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Linearization of holomorphic germs with quasi-parabolic fixed points

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Abstract. Let f be a germ of a holomorphic diffeomorphism of $\mathbb{C}^n$ with the origin O being a quasi-parabolic fixed point, i.e. the spectrum of df_O consists of 1 and $e^{2i\pi\theta_j}$ with $\theta_j \in \mathbb{R}\backslash\mathbb{Q}$. We show that f is locally holomorphically conjugated to its linear part, if f is of some particular form and its eigenvalues satisfy certain arithmetic conditions. When the spectrum of df_O does not consist of any 1's, this is the classical result of Siegel [C. L. Siegel. Iteration of analytic functions. *Ann. of Math.* **43** (1942), 607–612] and Brjuno [A. D. Brjuno. Analytic form of differential equations. *Trans. Moscow Math. Soc.* **25** (1971), 131–288; **26** (1972), 199–239].

1. *Introduction*

Let f be a germ of a holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O. One of the main questions in the study of local holomorphic dynamics is: does there exist a local holomorphic change of coordinates such that f is conjugated to its linear part? The answer to this question depends on the spectrum $Sp(f)$ of df_O, i.e. the set of eigenvalues of df_O. For more information on this and other aspects of local dynamics, see the survey papers by Abate [**A**] and Bracci [**B**]. In the parabolic case, that is when $Sp(f)$ consists of only roots of unity, the answer is negative unless $f^N \equiv \mathrm{Id}$ for some positive integer N. In the elliptic case, that is when $Sp(f)$ consists of eigenvalues of modulus one but none is a root of unity, Siegel [**S1, S2**] and later Brjuno [**Br**] found sufficient conditions on the eigenvalues for the answer to be positive. In this paper, we deal with the quasi-parabolic case, that is when $Sp(f)$ consists of 1 and $e^{2i\pi\theta_j}$ with $\theta_j \in \mathbb{R}\backslash\mathbb{Q}$.

One way to obtain the desired holomorphic change of coordinates is to start with a formal change of coordinates using formal power series and then to show that the series converges in a neighborhood of the origin. Such a formal power series always exists, and is unique, provided that there are no *resonances*, i.e.

$$\lambda_1^{k_1} \cdot \cdots \cdot \lambda_n^{k_n} - \lambda_i \neq 0, \quad k_1 + \cdots + k_n \geq 2, \quad 1 \leq i \leq n,$$

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where $Sp(f) = \{\lambda_1, \ldots, \lambda_n\}$ and $k_1, \ldots, k_n$ are non-negative integers. We can express the above condition in multi-index notation as

$$\lambda^k - \lambda_i \neq 0, \quad |k| \geq 2, \quad 1 \leq i \leq n, \tag{1}$$

where

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad k = (k_1, \ldots, k_n) \quad \text{and} \quad |k| = k_1 + \cdots + k_n.$$

Due to the presence of resonances, the linearization of a generic map in the quasi-parabolic case is not possible, even formally. Hence, we first need to find a particular form of f to ensure the existence of such a formal solution.

The question of convergence is far more subtle. The differences in (1) enter into the denominators of the coefficients of the formal power series, and hence give rise to the problem of *small divisors*. If they approach zero too rapidly, the power series may diverge (see [**C1, C2, SM**]). Siegel [**S1, S2**] was the first to overcome the difficulty of the small divisors using the majorant method. Brjuno [**Br**] slightly varied the majorant method to allow for a wider class of small divisor estimates. He introduced the following condition:

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \omega(p_{\nu+1})^{-1} < \infty, \tag{2}$$

where

$$\omega(m) = \min_{2 \leq |k| \leq m} \min_{1 \leq i \leq n} |\lambda^k - \lambda_i|, \quad m \geq 2,$$

and $\{p_\nu\}_{\nu=0}^{\infty}$ is a sequence of integers with $1 = p_0 < p_1 < \cdots$.

In this paper, we show that there exists a holomorphic change of coordinates that conjugates some particular quasi-parabolic germ f to its linear part. More precisely we have the following.

THEOREM. *Let f be a germ of a holomorphic diffeomorphism of $\mathbb{C}^n$, with the origin O as a quasi-parabolic fixed point. Let M be an f-invariant smooth analytic variety through O of dimension r, where r is the dimension of the generalized eigenspace of* 1, *such that $f|_M = \mathrm{id}$. Choose local coordinates (x, y) centered in O such that M is given by $\{y = 0\}$. For any $p \in M$, write $df_p = \begin{pmatrix} I & \star \\ O & \Lambda_s(x) \end{pmatrix}$. Assume that $\Lambda_s(x) = \Lambda_s = diag(\lambda_1, \ldots, \lambda_s)$ with $\lambda_j = e^{2\pi i \theta_j}$, $\theta_j \in \mathbb{R} \backslash \mathbb{Q}$, for all $p \in M$. If the λ_j satisfy the Brjuno condition (2), then there exists a local holomorphic change of coordinates ψ such that*

$$f \circ \psi = \psi \circ L, \tag{3}$$

where L is the linear part of f.

2. *Proof of the theorem*

The main new difficulty of the proof is the presence of resonances. Note also that we do not assume the Brjuno condition on all of the eigenvalues.

Choose local coordinates $(x, y) = (x_1, \ldots, x_r, y_1, \ldots, y_s)$ such that $M = \{y = 0\}$. By the assumption of the theorem, we can assume that f is of the form

$$x_i' = x_i + f_i^1(x, y) \qquad (i = 1, \ldots, r),$$
$$y_j' = \lambda_j y_j + f_j^2(x, y) \quad (j = 1, \ldots, s),$$

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where f_i^1 and f_j^2 are holomorphic functions with $\mathrm{ord}_y(f_i^1) \geq 1$ and $\mathrm{ord}_y(f_j^2) \geq 2$. Here $\mathrm{ord}_y(\cdot)$ denotes the lowest value of $|l|$ for terms $x^k y^l$ in the Taylor expansion of a function at the origin. We first show that a formal solution to (3) exists for ψ of the form

$$x_i = u_i + \psi_i^1(u, v) \quad (i = 1, \ldots, r),$$
$$y_j = v_j + \psi_i^2(u, v) \quad (j = 1, \ldots, s),$$

where $(u, v) = (u_1, \ldots, u_r, v_1, \ldots, v_s)$ and ψ_i^1 and ψ_i^2 are holomorphic functions with $\mathrm{ord}_v(\psi_i^1) \geq 1$ and $\mathrm{ord}_v(\psi_i^2) \geq 2$.

Write $f(z) = \Lambda z + \hat{f}(z)$ and $\psi(w) = w + \hat{\psi}(w)$, where $z = (x, y)$ and $w = (u, v)$. Then equation (3) yields

$$\hat{\psi} \circ \Lambda - \Lambda \hat{\psi} = \hat{f} \circ \psi. \tag{4}$$

To obtain a formal solution, write

$$\hat{\psi} = \sum_{|k| \geq 2} \psi_k w^k, \quad \psi_k \in \mathbb{C}^n,$$

where $k = (k_1, \ldots, k_n)$, and

$$\hat{f} = \sum_{|l| \geq 2} f_l z^l, \quad f_l \in \mathbb{C}^n,$$

where $l = (l_1, \ldots, l_n)$. Denote $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ with $\tilde{\lambda}_i = 1$ for $1 \leq i \leq r$ and $\tilde{\lambda}_{r+j} = \lambda_j$ for $1 \leq j \leq s$, so (4) becomes

$$\sum_{|k| \geq 2} E_k \psi_k w^k = \sum_{|l| \geq 2} f_l \left(\sum_{|m| \geq 1} \psi_m w^m \right)^l, \tag{5}$$

where

$$E_k = \tilde{\lambda}^k I_n - \Lambda.$$

The matrices E_k are not all invertible due to the presence of resonances. Write $E_k = \mathrm{diag}(E_k^1, E_k^2)$, then $\det E_k^1 = 0$ for

$$k = (k_1, \ldots, k_r, 0, \ldots, 0) \tag{6}$$

and $\det E_k^2 = 0$ for

$$k = (k_1, \ldots, k_r, 0, \ldots, 0, 1, 0, \ldots, 0). \tag{7}$$

Denote by K_1 the set of k's as in (6) and by K_2 the set of k's as in (7). Write $f_l = (f_l^1, f_l^2)$, $\psi_k = (\psi_k^1, \psi_k^2)$ and $l = (l^1, l^2)$. Set $\psi_k^1 = 0$ for $k \in K_1$ and $\psi_k^2 = 0$ for $k \in K_1 \cup K_2$. Then the right-hand side of (5) is equal to

$$\sum_{|l| \geq 2} f_l \left(u + \sum_{m_1 \notin K_1} \psi_{m_1}^1 w^{m_1} \right)^{l^1} \cdot \left(v + \sum_{m_2 \notin K_1 \cup K_2} \psi_{m_2}^2 w^{m_2} \right)^{l^2}. \tag{8}$$

It is easy to see that the summands in (8) produce terms w^k with $k \in K_1$ only when $l \in K_1$ and terms w^k with $k \in K_2$ only when $l \in K_1 \cup K_2$. Since $f_l^1 = 0$ for $l \in K_1$ and $f_l^2 = 0$ for $l \in K_1 \cup K_2$, we see that equation (5) has a formal solution, with the coefficients ψ_k^1 for $k \notin K_1$ and ψ_k^2 for $k \notin K_1 \cup K_2$ determined by recursion.

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To prove the convergence of this formal solution in a neighborhood of the origin, assuming that the λ_j satisfy the Brjuno condition (2), we have to show that

$$\sup_k \frac{1}{|k|} \log |\psi_k| < \infty. \tag{9}$$

Since f is holomorphic in a neighborhood of the origin, there exists a positive number a such that $|f_l| \le a^{|l|-1}$ for $|l| \ge 2$. The functional equation (3) remains true under the transformation $f(z) \to af(z/a)$, $\psi(w) \to a\psi(w/a)$. Hence we may assume that

$$|f_l| \le 1, \quad |l| \ge 2.$$

It follows from (5) and our assumptions on ψ_k that

$$|\psi_k| \le \epsilon_k^{-1} \sum_{\substack{k_1+\cdots+k_\nu=k \\ \nu\ge 2}} |\psi_{k_1}| \cdot \cdots \cdot |\psi_{k_\nu}|, \quad |k| \ge 2, \quad k \notin K_1, \tag{10}$$

where

$$\epsilon_k = \begin{cases} \min\limits_{1\le i\le n} |\tilde{\lambda}^k - \tilde{\lambda}_i|, & k \notin K_1 \cup K_2, \\ \min\limits_{1\le j\le s} |\lambda_j - 1|, & k \in K_2. \end{cases}$$

We can inductively define, for $j \ge 2$,

$$\sigma_j = \sum_{\substack{j_1+\cdots+j_\nu=j \\ \nu\ge 2}} \sigma_{j_1} \cdot \cdots \cdot \sigma_{j_\nu},$$

and, for $|k| \ge 2$,

$$\delta_k = \begin{cases} \epsilon_k^{-1} \max\limits_{\substack{k_1+\cdots+k_\nu=k \\ \nu\ge 2}} \delta_{k_1} \cdot \cdots \cdot \delta_{k_\nu}, & k \notin K_1, \\ 0, & k \in K_1, \end{cases}$$

with $\sigma_1 = 1$ and $\delta_e = 1$, where e stands for any integer vector k with $|k| = 1$. Then we have, by induction, that

$$|\psi_k| \le \sigma_{|k|}\delta_k, \quad |k| \ge 1.$$

Therefore, it suffices to prove analogous estimates for the σ_j and δ_k to establish (9).

The estimate for the σ_j is easy to obtain. Let $\sigma = \sum_{j\ge 1} \sigma_j t^j$. We have

$$\sigma - t = \sum_{j\ge 2} \sigma_j t^j = \sum_{j\ge 2} \left(\sum_{i\ge 1} \sigma_i t^i \right)^j = \frac{\sigma^2}{1-\sigma}.$$

This equation has a unique holomorphic solution

$$\sigma = \frac{t+1}{4}\left(1 - \sqrt{1 - \frac{8t}{(1+t)^2}}\right), \quad |t| \ll 1.$$

Hence,

$$\sup_j \frac{1}{j} \log \sigma_j < \infty.$$

The estimate for the δ_k is much harder as it involves small divisors.

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Choose $k \notin K_1$ with the decomposition $k = k_1 + \cdots + k_\nu$ such that it obtains the maximum in the expression of δ_k. Obviously we have $k_j \notin K_1$. We can then express δ_{k_j} in terms of $\epsilon_{k_j}^{-1}$ and $\delta_{k_{j'}}$ with $|k_{j'}| < |k_j|$. Proceeding like this, we finally arrive at the following decomposition

$$\delta_k = \epsilon_{l_0}^{-1}\epsilon_{l_1}^{-1} \cdot \cdots \cdot \epsilon_{l_\mu}^{-1}, \quad |k| \geq 2, \quad k \notin K_1,$$

where $l_0 = k$, $|k| > |l_1|, \ldots, |l_\mu| \geq 2$ and $l_j \notin K_1$. Denote by $N_m(k)$ the number of factors $\epsilon_{l_j}^{-1}$ in δ_k with $\epsilon_{l_j} < \frac{1}{2}\tilde{\omega}(m)$, where

$$\tilde{\omega}(m) = \min_{\substack{2 \leq |k| \leq m \\ k \notin K_1}} \epsilon_k, \quad m \geq 2.$$

Note that $\tilde{\omega}(m)$ is non-increasing with respect to m and we are interested in the case where $\tilde{\omega}(m)$ goes to zero as m goes to infinity. The following is the key estimate.

LEMMA. (Brjuno) *For $m \geq 2$ and $k \notin K_1$,*

$$N_m(k) \leq \begin{cases} 0, & |k| \leq m, \\ 2\dfrac{|k|}{m} - 1, & |k| > m. \end{cases}$$

Proof. The proof is done by induction. Since m is constant throughout the whole lemma, we write N for N_m.

For $|k| \leq m$,

$$\epsilon_k \geq \tilde{\omega}(|k|) \geq \tilde{\omega}(m) > \tfrac{1}{2}\tilde{\omega}(m).$$

Hence $N(k) = 0$.

So assume that $|k| > m$. Write

$$\delta_k = \epsilon_k^{-1}\delta_{k_1} \cdot \cdots \cdot \delta_{k_\nu}, \quad k = k_1 + \cdots + k_\nu, \quad \nu \geq 2,$$

with $|k| > |k_1| \geq \cdots \geq |k_\nu|$, and consider the following different cases.

Case 1: $\epsilon_k \geq \frac{1}{2}\tilde{\omega}(m)$. Then

$$N(k) = N(k_1) + \cdots + N(k_\nu).$$

Applying the induction hypotheses to each term, we get $N(k) \leq 2|k|/m - 1$ immediately.

Case 2: $\epsilon_k < \frac{1}{2}\tilde{\omega}(m)$. Then

$$N(k) = 1 + N(k_1) + \cdots + N(k_\nu).$$

There are three different cases.

Case 2.1: $|k_1| \leq m$. Then

$$N(k) = 1 < 2\frac{|k|}{m} - 1.$$

Case 2.2: $|k_1| \geq |k_2| > m$. Then there exists $2 \leq \mu \leq \nu$ such that $|k_\mu| > m \geq |k_{\mu+1}|$. Therefore

$$N(k) = 1 + N(k_1) + \cdots + N(k_\mu) \leq 1 + 2\frac{|k|}{m} - \mu \leq 2\frac{|k|}{m} - 1.$$

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Case 2.3: $|k_1| > m \geq |k_2|$. Then

$$N(k) = 1 + N(k_1).$$

There are two different cases.

Case 2.3.1: $|k_1| \leq |k| - m$. Then

$$N(k) \leq 1 + 2\frac{|k| - m}{m} - 1 < 2\frac{|k|}{m} - 1.$$

Case 2.3.2: $|k_1| > |k| - m$. The crucial observation is that $\epsilon_{k_1}^{-1}$ does not contribute to $N(k_1)$, which is the content of Siegel's lemma.

LEMMA. (Siegel) *If* $k > k_1$, *with* $k, k_1, k - k_1 \notin K_1$, *and*

$$\epsilon_k < \tfrac{1}{2}\tilde{\omega}(m) \quad \textit{and} \quad \epsilon_{k_1} < \tfrac{1}{2}\tilde{\omega}(m),$$

then $|k - k_1| = |k| - |k_1| \geq m$.

Proof. The proof of this lemma is simple. By the assumption, it follows that

$$\begin{aligned}
\tilde{\omega}(m) &> \epsilon_k + \epsilon_{k_1} \\
&= |\tilde{\lambda}^k - \tilde{\lambda}_{i_k}| + |\tilde{\lambda}^{k_1} - \tilde{\lambda}_{i_{k_1}}| \\
&= |\tilde{\lambda}^k - \tilde{\lambda}_{i_k}| + |\tilde{\lambda}^k - \tilde{\lambda}_{i_{k_1}} \cdot \tilde{\lambda}^{k-k_1}| \\
&\geq |\tilde{\lambda}_{i_{k_1}} \cdot \tilde{\lambda}^{k-k_1} - \tilde{\lambda}_{i_k}| \\
&\geq \tilde{\omega}(|k - k_1| + 1).
\end{aligned}$$

By the monotonicity of $\tilde{\omega}$ we have $|k - k_1| \geq m$, which proves Siegel's lemma.

Therefore, Case 1 applies to δ_{k_1} and we have

$$N(k) = 1 + N(k_{1_1}) + \cdots + N(k_{1_{\nu'}}),$$

where $|k| > |k_1| > |k_{1_1}| \geq \cdots \geq |k_{1_{\nu'}}|$ and $k_1 = k_{1_1} + \cdots + k_{1_{\nu'}}$. We can now repeat the analysis of Case 2 for this decomposition, and we are finished unless we run into Case 2.3.2 again. However, this loop can happen at most m times and finally we have to run into a different case. This completes the induction, and the proof of Brjuno's lemma.

Before we proceed, let us take a closer look at the Brjuno condition (2). Denote $\lambda = (\lambda_1, \ldots, \lambda_s)$ and define

$$\omega(m) = \min_{2 \leq |k| \leq m} \min_{1 \leq j \leq s} |\lambda^k - \lambda_j|, \quad m \geq 2.$$

It is clear that

$$\omega(m) \geq \tilde{\omega}(m) \geq \omega(m + 1).$$

By the assumption of the theorem, there exists a sequence of integers $1 = p_0 < p_1 < \cdots$ such that

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \omega(p_{\nu+1})^{-1} < \infty.$$

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Let $q_0 = p_0$ and $q_j = p_j - 1$ for $j \geq 1$. Then

$$\begin{aligned}\sum_{\nu\geq 0} q_\nu^{-1} \log \tilde{\omega}(q_{\nu+1})^{-1} &\leq \sum_{\nu\geq 0} q_\nu^{-1} \log \omega(q_{\nu+1}+1)^{-1} \\ &= p_0^{-1} \log \omega(p_1)^{-1} + \sum_{\nu\geq 1} \frac{p_\nu}{p_\nu - 1} p_\nu^{-1} \log \omega(p_{\nu+1})^{-1} \\ &\leq 2 \sum_{\nu\geq 0} p_\nu^{-1} \log \omega(p_{\nu+1})^{-1} < \infty.\end{aligned} \tag{11}$$

Since $\delta_k = 0$ for $k \in K_1$, we only need to estimate

$$\frac{1}{|k|} \log \delta_k = \sum_{j=0}^{\mu} \frac{1}{|k|} \log \epsilon_{l_j}^{-1}, \quad k \notin K_1.$$

By Brjuno's lemma,

$$\operatorname{card}\left\{0 \leq j \leq \mu : \frac{1}{2}\tilde{\omega}(q_{\nu+1}) \leq \epsilon_{l_j} < \frac{1}{2}\tilde{\omega}(q_\nu)\right\} \leq 2\frac{|k|}{q_\nu}$$

for $\nu \geq 1$. Moreover, it is easy to see from the definition of δ_k that the number of factors ϵ_{l_j} is bounded by $2|k| - 1$. In particular,

$$\operatorname{card}\left\{0 \leq j \leq \mu : \frac{1}{2}\tilde{\omega}(q_1) \leq \epsilon_{l_j}\right\} \leq 2|k| = 2\frac{|k|}{q_0}.$$

Hence,

$$\begin{aligned}\frac{1}{|k|} \log \delta_k &\leq 2 \sum_{\nu\geq 0} q_\nu^{-1} \log 2\tilde{\omega}(q_{\nu+1})^{-1} \\ &= 2\left(\sum_{\nu\geq 0} q_\nu^{-1} \log \tilde{\omega}(q_{\nu+1})^{-1} + \log 2 \sum_{\nu\geq 0} q_\nu^{-1}\right).\end{aligned} \tag{12}$$

Since $\tilde{\omega}(m)$ goes to zero monotonically as m goes to infinity, we can choose some $\bar{m}$ such that $1 > \tilde{\omega}(m)$ for all $m > \bar{m}$. From this we get

$$\sum_{\nu\geq \nu_0} q_\nu^{-1} \leq \log^{-1} \tilde{\omega}(\bar{m})^{-1} \sum_{\nu\geq \nu_0} q_\nu^{-1} \log \tilde{\omega}(q_{\nu+1})^{-1},$$

where ν_0 is defined by the inequalities $q_{\nu_0-1} \leq \bar{m} < q_{\nu_0}$. Thus (11) implies that both series in parentheses in (12) converge. Therefore

$$\sup_k \frac{1}{|k|} \log \delta_k < \infty.$$

This completes the proof of the theorem.

Remark 1. This note is inspired by the papers of Nishimura [**N**] and Pöschel [**P**]. In [**N**], Nishimura had similar assumptions in his theorems on 'semi-attractive' germs. Moreover, in [**P**], Pöschel proved his result using a similar approach.

Remark 2. While the local dynamics in the 'tangent to the identity' case, that is when $df_O = I_n$, has been studied extensively (see [**A, B**]), the quasi-parabolic case has not

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received much attention. Pöschel's results [**P**] certainly apply to this case. He proved that there exists an invariant complex manifold tangent to the eigenspace of the λ_j on which the quasi-parabolic germ is holomorphically linearizable, provided that the λ_j satisfy a certain Brjuno condition. Bracci and Molino [**BM**] studied quasi-parabolic germs in $\mathbb{C}^2$, proving that there exist 'parabolic curves' tangent to the eigenspace of one for germs that are 'dynamically separating' (see [**BM**] for more precise information). A parabolic curve is, roughly speaking, an injective holomorphic image of a simply connected domain in $\mathbb{C}$ with the origin O on its boundary such that it is mapped into itself and is attracted to O. In a forthcoming paper [**R2**], we study quasi-parabolic germs in $\mathbb{C}^n$ using ideas from Hakim [**H1, H2**]. In particular, we will generalize the results of Bracci and Molino [**BM**] to higher dimensions and extend the results of Hakim [**H1, H2**] to the quasi-parabolic case. Note that the germs we study here do not fall into the category of germs that are studied in [**BM**] and [**R2**], and that our theorem shows the existence of 'Siegel cylinders' in a neighborhood of the origin. In particular, there exist no parabolic curves for such germs.

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