Proceedings of the Edinburgh Mathematical Society (2026) **69**, 1314–1322
doi:10.1017/S0013091526101461

ON THE IMAGE OF THE HILBERT MAP

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(Received 27 June 2024 UTC; first published online 9 September 2026 UTC)

Abstract We talk about the image of the Hilbert map. In particular, we show the necessary and sufficient condition for the Hilbert map to be surjective.

Keywords: Hilbert map; Space of Kahler metrics

2020 Mathematics subject classication: 53C55

1. Introduction

Let (X, L) be a polarized Kähler manifold of dimension n. The space $\mathcal{H}$, consisting of all smooth Hermitian metrics on L with positive curvature, is a central playground for many analytic geometers, especially for those who are interested in the extremal metrics. Following the ideas of quantization of Kähler metrics by Donaldson and others, the Hilbert map is often considered. More precisely, let $\mathcal{B}$ denote the space of Hermitian inner products on $H^0(X, L)$, the Hilbert map $\mathrm{Hilb} : \mathcal{H} \to \mathcal{B}$ is defined as follows: $\forall h \in \mathcal{H}$,

$$\langle s_1, s_2\rangle_{\mathrm{Hilb}(h)} = \int_X (s_1, s_2)_h \omega^n/n!,$$

where $\omega = \frac{i}{2}\Theta_h$. More generally, let $\mathcal{B}_k$ denote the space of Hermitian inner products on $H^0(X, L^k)$; we have the Hilbert map $\mathrm{Hilb}_k : \mathcal{H} \to \mathcal{B}_k$. In the applications of the ideas of quantization, Hilb_k and its image in $\mathcal{B}_k$ for k large are mostly considered (see e.g., [3]). In some sense, the images of the Hilbert maps Hilb_k can be seen as the shadows of the complicated infinite-dimensional space $\mathcal{H}$. In this article, we focus on a very ample line bundle L instead of its high tensor powers.

It is believed by many [4] that when L is very ample, the Hilbert map Hilb is surjective. But this is not always true. Actually, the following arguments show that in most cases, the image of the Hilbert map is a proper subset of $\mathcal{B}$. Let $\{s_1, \ldots, s_m\} \subset H^0(X, L)$ be a fixed basis. Then $\mathcal{B}$ can be identified with the space of $m \times m$ positive definite Hermitian matrices. And then $\mathrm{Hilb}(h)$ is a Hermitian matrix with entries:

$$(\mathrm{Hilb}(h))_{ij} = \int_X (s_i, s_j)_h \omega^n/n!$$

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Recall that a set of sections $I \subset H^0(X, L)$ is said to generate the line bundle L if $\forall p \in X$, $\exists s \in I$ such that $s(p) \neq 0$. Assume that $\{s_1, \ldots, s_{m-1}\}$ generate L, then

$$M = \max_{p \in X} \frac{|s_m(p)|_h^2}{\sum_{i=1}^{m-1} |s_i(p)|_h^2} < \infty,$$

which is independent of the metric h. This implies that

$$(\mathrm{Hilb}(h))_{mm} < M \sum_{i=1}^{m-1} (\mathrm{Hilb}(h))_{ii}.$$

Therefore, if $H^0(X, L)$ is more than just generating L, namely a proper subspace generates L, which is the case for most very ample line bundles L, then the Hilbert map is not surjective. Professor Hashimoto has acknowledged this misunderstanding and is preparing an erratum to [4].

Naturally, the next question is what is the image of the Hilbert map? From our previous argument, it is easy to see that we have the following constraints. Let $\{s_1, \ldots, s_l\} \subset H^0(X, L)$ be a set of sections generating L, and let $\{\sigma_1, \ldots, \sigma_q\} \subset H^0(X, L)$ be any finite set of sections. Then

$$M = \max_{p \in X} \frac{\sum_{i=1}^{q} |\sigma_i(p)|^2}{\sum_{i=1}^{l} |s_i(p)|^2} < \infty. \tag{1}$$

So

$$\sum_{i=1}^{q} |\sigma_i|^2_{\mathrm{Hilb}(h)} < M \sum_{i=1}^{l} |s_i|^2_{\mathrm{Hilb}(h)}, \tag{2}$$

for any $h \in \mathcal{H}$. More generally, if $\{s_1, \ldots, s_l\}$ do not necessarily generate L, then the local representations of $\{s_1, \ldots, s_l\}$ generate an ideal sheaf I. Let J be the ideal sheaf generated by $\{\sigma_1, \ldots, \sigma_q\}$. Then if $J \subset I$, we still have inequalities (1) and (2). If we regard $\mathcal{B}$ as an open subset of the space of $m \times m$ Hermitian matrices which is identified with $\mathbb{R}^{m^2}$, then each of inequalities (2) defines a half space of $\mathbb{R}^{m^2}$. We let $\mathcal{A} \subset \mathcal{B}$ denote the subset defined by all these inequalities. Clearly, the image of the Hilbert map is contained in $\mathcal{A}$. We propose the following

Conjecture: 1 $\mathcal{A}$ is the image of the Hilbert map.

Our first result gives some evidence to this conjecture. To state our result, we first fix some notations. Let $V \subset H^0(X, L)$ be a subspace. Let $\mathcal{B}_V$ denote the space of Hermitian inner products on V. Then, composing with the restriction, the Hilbert map gives

$$\mathrm{Hilb}_V : \mathcal{H} \to \mathcal{B}_V.$$

Theorem 1.1. *Let $V \subset H^0(X, L)$ be a subspace that generates L, which is minimal in the sense that no proper subspace of V generates L. Then* Hilb_V *is surjective.*

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Remark: 1 As we will discuss in the section on reduction of constraints, under the assumption on V as in the theorem, the constraints of the form (2) are all redundant. In other words, when $\mathcal{A} = \mathcal{B}$, $\mathcal{A}$ is the image of the Hilbert map.

The general case becomes complicated due to the fact that there are too many constraints, which makes it hard to decide which Hermitian metric satisfies all the constraints. We will give an example in § 3 to give the readers a glance at the subtlety of the problem. This example also shows that it is not clear whether we can reduce the set of the constraints to a compact subset. So it is not clear if $\mathcal{A}$ is open.

We recall that the space $\mathcal{B}$ has a natural convex structure, namely, given $H_0, H_1 \in \mathcal{B}$, $tH_1 + (1-t)H_0$ is in $\mathcal{B}$ for all $t \in [0,1]$. Clearly, $\mathcal{A}$ is also convex.

Theorem 1.2. *The image of* Hilb *is a convex open subset of* $\mathcal{B}$.

Our last result describes the image of the Hilbert map from a different angle.

Let $\bar{\mathcal{B}}$ be the space of semi-positive Hermitian bilinear forms on $H^0(X, L)$, or by abuse of notation, the semi-positive Hermitian inner products on $H^0(X, L)$. Given a Hermitian metric h on L, we have a map $P_h : X \to \bar{\mathcal{B}}$ defined by

$$\langle s_1, s_2 \rangle_{P_h(x)} = (s_1(x), s_2(x))_h$$

We let $\mathcal{F}_h \subset \bar{\mathcal{B}}$ denote the closure of the subset consisting of linear combinations of the form

$$\sum_{i=1}^{t} a_i P_h(x_i), t < +\infty, x_i \in X, a_i > 0.$$

Clearly, $\mathcal{F}_h$ is the cone generated by the closure of the convex hull of $P_h(X)$. It is not hard to see that $\mathcal{F}_h$ actually does not depend on the choice of h, so we will simply use $\mathcal{F}$.

Theorem 1.3. *The closure of the image of the Hilbert map in* $\bar{\mathcal{B}}$ *is* $\mathcal{F}$.

Remark: 2 If the image of the Hilbert map is $\mathcal{A}$, then Theorem 1.3 says that the closure of $\mathcal{A}$ in $\bar{\mathcal{B}}$ is $\mathcal{F}$. Conversely, if the closure of $\mathcal{A}$ in $\bar{\mathcal{B}}$ is $\mathcal{F}$, then by Theorem 2.2, Hilb$(\mathcal{H})$ is a convex dense subset in $\mathcal{A}$, hence the image of the Hilbert map being $\mathcal{A}$.

The structure of this article is as follows. We will first prove Theorem 1.1, then we prove Theorem 1.3. Then, after some discussion of the reduction of the linear constraints, we will prove Theorem 1.2.

2. Proof of the main results

We fix a Kähler form ω on X. Let $\varphi \in C^\infty(X)$, then the Monge–Ampère equation is

$$\mathrm{MA}(\varphi) = \frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n}$$

We recall the famous theorem by Aubin and Yau (see [1] or [7] for example)

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Theorem 2.1 (Aubin–Yau). *Given $f \in C^\infty(X)$, the Monge–Ampère equation*

$$\log \mathrm{MA}(\varphi) - \varphi = f$$

has a unique solution φ, which is in $C^\infty(X)$, satisfying $\omega + i\partial\bar\partial\varphi > 0$.

Theorem 2.2. *The image of the Hilbert map is a convex subset of $\mathcal{B}$.*

Proof. We fix a metric $h \in \mathcal{H}$ with $\omega = \frac{i}{2}\Theta_h$. Then $\mathcal{H}$ can be identified with the set $\mathcal{H}_h = \{\varphi \in C^\infty(X) | \omega + i\partial\bar\partial\varphi > 0\}$. Let $H_0 = \mathrm{Hilb}(h_0)$ and $H_1 = \mathrm{Hilb}(h_1)$, with $h_0 = he^{-\varphi_0}$ and $h_1 = he^{-\varphi_1}$. For $t \in (0,1)$, we need to find φ_t such that $\omega + i\partial\bar\partial\varphi_t > 0$ and

$$\int_X (s_i, s_j)_h e^{-\varphi_t} \frac{(\omega + i\partial\bar\partial\varphi_t)^n}{n!} = (tH_1 + (1-t)H_0)_{ij},$$

namely

$$\int_X (s_i, s_j)_h e^{-\varphi_t} \frac{(\omega + i\partial\bar\partial\varphi_t)^n}{n!} = \int_X \frac{(s_i, s_j)_h}{n!} [te^{-\varphi_1}(\omega + i\partial\bar\partial\varphi_1)^n + (1-t)e^{-\varphi_0}(\omega + i\partial\bar\partial\varphi_0)^n]$$

If we write $e^{f_t}\omega^n = te^{-\varphi_1}(\omega + i\partial\bar\partial\varphi_1)^n + (1-t)e^{-\varphi_0}(\omega + i\partial\bar\partial\varphi_0)^n$, then we only need to solve the Monge–Ampère equation

$$\log \mathrm{MA}(\varphi) - \varphi = f_t,$$

which, by the theorem of Aubin and Yau, has a unique solution $\varphi_t \in C^\infty(X)$ satisfying $\omega + i\partial\bar\partial\varphi_t > 0$. And the theorem is proved. □

Fixing a basis $\{s_1, \ldots, s_m\}$, $\bar{\mathcal{B}}$ then corresponds to the space of positive semi-definite $m \times m$ matrices.

Now we prove Theorem 1.1:

Theorem 2.3 (Theorem 1.1). *Let $V \subset H^0(X, L)$ be a subspace that generates L, which is minimal in the sense that no proper subspace of V generates L. Then Hilb_V is surjective.*

Proof. We prove that the closure of $\mathrm{Hilb}(\mathcal{H})$ in $\bar{\mathcal{B}}$ is $\bar{\mathcal{B}}$. Then by Theorem 2.2, $\mathrm{Hilb}(\mathcal{H})$ is a convex dense subset of $\mathcal{B}$, which must be $\mathcal{B}$ itself.

Clearly the closure of $\mathrm{Hilb}(\mathcal{H})$ in $\bar{\mathcal{B}}$ is still convex. Given any basis $\{s'_1, \ldots, s'_m\}$, let $H_i \in \bar{\mathcal{B}}$ be defined as

$$|s'_j|^2_{H_i} = \delta^i_j$$

By unitary transformations, in order to show that the closure $\mathcal{D}$ of $\mathrm{Hilb}(\mathcal{H})$ in $\bar{\mathcal{B}}$ is $\bar{\mathcal{B}}$, it suffices to show that $H_i \in \mathcal{D}, 1 \le i \le m$. We use the idea of Demailly from [2]. Fix i, by the assumption on V, $\exists p_i \in X$ such that $s'_i(p_i) \neq 0$, while $s'_j(p_i) = 0$ for $j \neq i$. Fix a local coordinate $(z_1, \ldots, z_n)$ in a neighbourhood U around p_i. $\forall \varepsilon > 0$, we can find a positive smooth function f_ε such that

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- $\int_X f_\varepsilon \omega^n = n!$
- $\int_{|z|<\varepsilon} f_\varepsilon \omega^n > n!(1-\varepsilon)$

So when $\varepsilon \to 0$, $f_\varepsilon \frac{\omega^n}{n!}$ converges to the delta measure δ_{p_i}. Again, we can solve the Monge–Ampère equation to get φ_ε satisfying $\omega + i\partial\bar\partial\varphi_\varepsilon > 0$ and $e^{-\varphi_\varepsilon}(\omega + i\partial\bar\partial\varphi_\varepsilon)^n = f_\varepsilon \omega^n$. Let $a = |s_i'(p_i)|_h^2$ and $h_\varepsilon = he^{-\varphi_\varepsilon}$. Then as $\varepsilon \to 0$, we have

$$\mathrm{Hilb}(h_\varepsilon) \to aH_i.$$

By rescaling, we see that $H_i \in \mathcal{D}$. And we have finished the proof. □

We then prove Theorem 1.3.

Theorem 2.4 (Theorem 1.3). *The closure of the image of the Hilbert map in $\bar{\mathcal{B}}$ is $\mathcal{F}$.*

Proof of theorem 1.3. We fix a Hermitian metric h on L with corresponding Kähler form ω. Then for each $x \in X$, we can approximate the point measure δ_x with volume forms $f_\varepsilon \frac{\omega^n}{n!}$. Then again by the theorem by Aubin and Yau, we can find φ_ε solving the equation $e^{-\varphi_\varepsilon}(\omega + i\partial\bar\partial\varphi_\varepsilon)^n = f_\varepsilon \omega^n$. So $P_h(x)$ is contained in the closure of the image of Hilb in $\bar{\mathcal{B}}$. Since the closure of the image of Hilb is still convex, one sees that $\mathcal{F}$ is contained in the closure of the image of Hilb.

Conversely, $\forall h_\varphi = he^{-\varphi} \in \mathcal{H}$, we claim that we can find a sequence of measures $\{m_a\}$ which are all supported on sets of discrete points satisfying

$$\lim_{a\to\infty} \int_X (s_i, s_j)_h m_a = \int_X (s_i, s_j)_h e^{-\varphi} w'^n,$$

for $1 \le i, j \le m$. One way to do this is to use Shiffman–Zelditch's result in [5, 6], where they proved that the normalized zero currents of sections in $H^0(X, L^k)$ almost surely converge to $\omega' = \omega + i\partial\bar\partial\varphi$. When $n = 1$, the zero currents of course are measures supported on sets of discrete points. When $n > 1$, by Corollary 1.3 in [5] and Bertini's theorem on linear series, we can first approximate ω' with a sequence of currents represented by smooth hypersurfaces Σ_b, that is

$$\lim_{b\to\infty} \int_{\Sigma_b} \alpha = \int_X \alpha \wedge \omega',$$

for all smooth $(n-1, n-1)$-forms α. So $\forall \varepsilon > 0$, we can find a smooth hypersurface Σ_b so that

$$\sup_{1\le i,j\le m} \Big| \int_X (s_i, s_j)_h e^{-\varphi} w'^n - \int_{\Sigma_b} (s_i, s_j)_h e^{-\varphi} w'^{n-1} \Big| < \varepsilon.$$

Then we can use induction to find a measure m_a supported on a set of discrete points in Σ_b satisfying

$$\sup_{1\le i,j\le m} \Big| \int_{\Sigma_b} (s_i, s_j)_h m_a - \int_{\Sigma_b} (s_i, s_j)_h e^{-\varphi} w'^{n-1} \Big| < \varepsilon.$$

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So we can indeed approximate $\mathrm{Hilb}(h_\varphi)$ by points in $\mathcal{F}$. Therefore the image of Hilb in $\bar{\mathcal{B}}$ is contained in $\mathcal{F}$. □

3. Discussion of the constraints

We now consider the linear constraints of the form

$$\sum_{i=1}^{q} |\sigma_i|^2_{\mathrm{Hilb}(h)} < M \sum_{i=1}^{l} |s_i|^2_{\mathrm{Hilb}(h)}.$$

The first observation is that we can assume that $(\sigma_1, \ldots, \sigma_q)$ and $(s_1, \ldots, s_l)$ are both linearly independent. This is easy to see. For example, if $s_j \in \langle s_1, \ldots, s_\alpha \rangle$, then after a unitary transformation, we can get $(s'_1, \ldots, s'_\alpha)$ linearly independent such that $\sum_{i=1}^{\alpha} |s_i|^2 = \sum_{i=1}^{\alpha} |s'_i|^2$ and $s'_1 = a s_j$. So

$$|s_j|^2 + \sum_{i=1}^{\alpha} |s_i|^2 = |\sqrt{1+|a|^2}\, s'_1|^2 + \sum_{i=2}^{\alpha} |s'_i|^2.$$

Then by induction, we can make $\sum_{i=1}^{l} |s_i|^2 = \sum_{i=1}^{\alpha} |s'_i|^2$ with $(s'_1, \ldots, s'_\alpha)$ linearly independent.

Assume $\langle s_1, \ldots, s_l \rangle$ minimally generates L or does not generate L. If $\langle \sigma_1, \ldots, \sigma_q \rangle \subseteq \langle s_1, \ldots, s_l \rangle$, then by Theorem 1.1, the constraint (2) is redundant. Therefore, we only need to consider the constraints given by the set of sections $(\sigma_1, \ldots, \sigma_q)$ such that $\langle \sigma_1, \ldots, \sigma_q \rangle$ is not contained in $\langle s_1, \ldots, s_l \rangle$. Without this assumption on $\langle s_1, \ldots, s_l \rangle$, it is not clear if we can ignore the constraints produced by those $\{\sigma_1, \ldots, \sigma_q\}$ satisfying $\langle \sigma_1, \ldots, \sigma_q \rangle \subseteq \langle s_1, \ldots, s_l \rangle$.

In general, suppose that the max of $\frac{\sum_{i=1}^{q} |\sigma_i(p)|^2}{\sum_{i=1}^{l} |s_i(p)|^2}$ is attained at p_0. We, by abuse of notation, let $s'_1(z) = \frac{\sum \bar{s_i}(p_0) s_i(z)}{\sqrt{\sum |s_i(p_0)|^2}}$. Of course, the vector $\frac{1}{|s_i(p)|^2}(\bar{s_1}(p_0), \ldots, \bar{s_l}(p_0))$ is independent of the choice of local frame. Then $\forall s(z) = \sum a_i s_i(z)$ such that $\sum a_i s_i(p_0) = 0$, we have $s(p_0) = 0$. Therefore, we can make an unitary transformation $(s_1, \ldots, s_l) \to (s'_1, \ldots, s'_l)$ so that $\sum |s_i(z)|^2 = \sum |s'_i(z)|^2$ and $s'_i(p_0) = 0$ for $i \geq 2$. So the linear constraints $\sum_{i=1}^{q} |\sigma_i|^2_{\mathrm{Hilb}(h)} < M(|s'_1|^2_{\mathrm{Hilb}(h)} + \sum_{i=2}^{l} \lambda_i |s'_i|^2_{\mathrm{Hilb}(h)})$ for $\lambda_i \geq 1$ are reduced to one constraint

$$\sum_{i=1}^{q} |\sigma_i|^2_{\mathrm{Hilb}(h)} < M \sum_{i=1}^{l} |s'_i|^2_{\mathrm{Hilb}(h)}$$

Now we prove Theorem 1.2.

Theorem 3.1 (Theorem 1.2). *The image of* Hilb *is a convex open subset of* $\mathcal{B}$.

Proof of Theorem 1.2. The tangent space of $\mathcal{H}$ at a point h is $C^\infty(X)$, and the tangent space of $\mathcal{B}$ at the identity is the space H_m of $m \times m$ Hermitian matrices as a real m^2-dimensional vector space. We choose a basis $\{s_1, \ldots, s_m\}$ so that

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$\mathrm{Hilb}(h) = I$. We then look at the tangent map of the Hilbert map. Since $(\mathrm{Hilb}(he^{-f}))_{ij} = \int_X (s_i, s_j)_h e^{-f} \frac{(\omega + i\partial\bar{\partial} f)^n}{n!}$ for each $f \in C^\infty(X)$ such that $\omega + i\partial\bar{\partial} f > 0$, and

$$\frac{d}{d\delta}|_{\delta=0} \frac{(\omega + \delta i\partial\bar{\partial}\varphi)^n}{n!} = \Delta\varphi \frac{\omega^n}{n!},$$

we have that

$$(d\mathrm{Hilb}_h(\varphi))_{ij} = \int_X (s_i, s_j)_h (\Delta\varphi - \varphi) \frac{\omega^n}{n!}.$$

Then if we can show that the tangent map is surjective in this case, then we can find $\varphi_i \in C^\infty(X)$, $1 \leq i \leq m^2$ such that $\{d\mathrm{Hilb}_h(\varphi_i)\}$ is a basis of the space of $m \times m$ Hermitian matrices. Since for each i, $\exists \delta_i > 0$ such that $\omega + t\sqrt{-1}\partial\bar{\partial}\varphi_i > 0$ for $t \in (-\delta_i, \delta_i)$, the set

$$Y = \{\sum_{i=1}^{m^2} a_i \delta_i \varphi_i \big| \sum_{i=1}^{N} |a_i| < 1\}$$

is an open set in $\mathbb{R}^{m^2}$ satisfying that $\omega + \delta i\partial\bar{\partial}\varphi > 0$ for each $\varphi \in M$. Then the restriction of Hilb to Y has an invertible tangent map at the origin. Then the inverse function theorem implies that the image of Hilb contains an open neighbourhood of I.

Now we show that the tangent map is surjective. We claim that the functions $(s_i, s_j)_h$, $1 \leq i, j \leq m$, are $\mathbb{R}$-linear independent. When $n = 1$, we can choose local coordinates z around a point p such that $z = 0$. Let k be the minimal vanishing order at p of all the s_i's. Then after some nonsingular transformation, we can assume that the vanishing order of s_1 is k while the vanishing orders of the other s_i's are $> k$ (simply minus a multiple of s_1). Then we can repeat this process to get that the vanishing orders of $s_i, 1 \leq i \leq m$ at p are strictly increasing as i increases.

Let f_i denote the local holomorphic function representing s_i, and let n_i denote the vanishing order of f_i at p. We can then go backwards from m to 1 to make that f_i has no degree n_j-term for $j > i$. Then since the functions $\{z^a \bar{z}^b\}_{a \geq 0, b \geq 0}$ are linearly independent, it is easy to see that the functions $(s_i, s_j)_h$, $1 \leq i, j \leq m$ are linearly independent. For $n > 1$, we can choose a smooth curve passing through a point p so that the sections $s_i, 1 \leq i \leq m$ are still linearly independent on this curve. Then the same argument shows that the functions $(s_i, s_j)_h$, $1 \leq i, j \leq m$ are linearly independent.

We claim that the linear map $T_h : C^\infty(X) \to H_m$ defined by $(T_h(f))_{ij} = \int_X (s_i, s_j)_h f \frac{\omega^n}{n!}$ is surjective. Assume this is not the case, then the image satisfies a nontrivial linear equation $\sum_{i,j} a_{ij} (T_h(f))_{ij} = 0$, which is the same as saying that

$$\int_X gf \frac{\omega^n}{n!} = 0,$$

where $g = \sum_{i,j} a_{ij} (s_i, s_j)_h$, for all $f \in C^\infty(X)$. So $g = 0$, contraditing the linear independence of $(s_i, s_j)_h$, $1 \leq i, j \leq m$.

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Since the eigenvalues of Δ are nonpositive and discrete, by using eigenfunctions as an orthornormal basis, one sees that $\Delta\varphi - \varphi = f$ has a solution for any $f \in C^\infty(X)$. And we have proved that the tangent map is surjective. □

Example: 1 Let $X = \mathbb{CP}^1$ and $L = O(2)$. Then in homogeneous coordinates $Z = [Z_0, Z_1]$, we have three sections of L: $s_1 = Z_0^2$, $s_2 = \sqrt{2}Z_0Z_1$ and $s_3 = Z_2^2$. Let Q denote the Hermitian metric on $H^0(X, L)$ that have $\{s_1, s_2, s_3\}$ as an orthonormal basis. Then clearly, $Q \in \mathrm{Hilb}(\mathcal{H})$.

- Let $Q_\varepsilon \in \mathcal{B}$ be the metric whose matrix with $\{s_1, s_2, s_3\}$ as a basis is $\mathrm{diag}(1, \varepsilon, 1)$. We have $Q_\varepsilon \in \mathcal{A}$ for small ε. To see this, one only needs to mimic the proof of Theorem 1.1 to make the mass of the volume form $e^{-\varphi}(\omega+\sqrt{-1}\partial\bar\partial\varphi)$ to be equally concentrated around the two points $[0, 1]$ and $[1, 0]$. And then add a suitable scalar to φ to get Q_ε.
- Let $Q'_\varepsilon \in \mathcal{B}$ be the metric whose matrix with $\{s_1, s_2, s_3\}$ as a basis is $\mathrm{diag}(\varepsilon, 1, 1)$. It seems that we should also have $Q_\varepsilon \in \mathcal{A}$ for small ε. But actually for ε small enough, we have

$$Q_\varepsilon \notin \mathrm{Hilb}(\mathcal{H}).$$

  To see this, one just needs to notice that if $Q_\varepsilon = \mathrm{Hilb}(\varphi)$ then the mass of the volume form $e^{-\varphi}(\omega + \sqrt{-1}\partial\bar\partial\varphi)$ must be concentrated in a small neighbourhood of $[0, 1]$. But this would imply that the norm of s_2 should also be small. So it is now clear that $Q'_\varepsilon \notin \mathrm{Hilb}(\mathcal{H})$ for ε small enough. But actually $Q'_\varepsilon \notin \mathcal{A}$ for ε small enough. Clearly, $\{\varepsilon^{-1/2}s_1, s_3\}$ generate L. We then see that the maximum of $\frac{|s_2|^2}{\varepsilon^{-1}|s_1|^2+|s_3|^2}$ is $\sqrt{\varepsilon}$. So we must have

$$|s_2|^2_{\mathrm{Hilb}(h)} < \sqrt{\varepsilon}(\varepsilon^{-1}|s_1|^2_{\mathrm{Hilb}(h)} + |s_3|^2_{\mathrm{Hilb}(h)}),$$

  which is violated by Q'_ε.
- One might doubt that the difference of the two examples comes from the fact that $\{s_2, s_3\}$ do not generate L. We can consider the basis $\{s_1, s_2 + as_1, s_3\}$ instead. Then by Theorem 1.2, for $|a|$ small enough, there exists $P \in \mathrm{Hilb}(\mathcal{H})$ under which $H^0(X, L)$ has $\{s_1, s_2 + as_1, s_3\}$ as an orthonormal basis. Then $\{s_2 + as_1, s_3\}$ does generate L. Let $P'_\varepsilon \in \mathcal{B}$ be the metric whose matrix with $\{s_1, s_2 + as_1, s_3\}$ as a basis is $\mathrm{diag}(\varepsilon, 1, 1)$. Then we have $|s_2|^2_{P'_\varepsilon} = 1 + |a|^2\varepsilon$, which again violates the constraint violated by Q'_ε.

Acknowledgments The author would like to thank Professors Steve Zelditch, Yoshinori Hashimoto and Siarhei Finski for their interest in this article. The author would also like to thank the referee, whose many suggestions make this article much more readable. The author is partially supported by NNSF of China no. 11701353.

https://doi.org/10.1017/S0013091526101461 Published online by Cambridge University Press

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https://doi.org/10.1017/S0013091526101461 Published online by Cambridge University Press