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A LINEAR SCHEME TO APPROXIMATE NONLINEAR CROSS-DIFFUSION SYSTEMS *

Hideki Murakawa [1]

Abstract. This paper proposes a linear discrete-time scheme for general nonlinear cross-diffusion systems. The scheme can be regarded as an extension of a linear scheme based on the nonlinear Chernoff formula for the degenerate parabolic equations, which proposed by Berger *et al.* [*RAIRO Anal. Numer.* **13** (1979) 297–312]. We analyze stability and convergence of the linear scheme. To this end, we apply the theory of reaction-diffusion system approximation. After discretizing the scheme in space, we obtain a versatile, easy to implement and efficient numerical scheme for the cross-diffusion systems. Numerical experiments are carried out to demonstrate the effectiveness of the proposed scheme.

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1. Introduction

In this paper, we consider discrete-time schemes and numerical schemes to approximate the following nonlinear diffusion problems: Find $\boldsymbol{z} = (z_1, \dots, z_M) : \overline{\Omega} \times [0,T) \to \mathbb{R}^M$ $(M \in \mathbb{N})$ such that

$$\begin{cases} \dfrac{\partial \boldsymbol{z}}{\partial t} = \Delta \boldsymbol{\beta}(\boldsymbol{z}) + \boldsymbol{f}(\boldsymbol{z}) & \text{in} \quad Q := \Omega \times (0,T), \\ \dfrac{\partial \boldsymbol{\beta}(\boldsymbol{z})}{\partial \nu} = \boldsymbol{0} & \text{on} \quad \partial\Omega \times (0,T), \\ \boldsymbol{z}(\cdot,0) = \boldsymbol{z}_0 & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d$ $(d \in \mathbb{N})$ is a bounded domain with smooth boundary $\partial\Omega$, T is a positive constant, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_M)$, $\boldsymbol{f} = (f_1, \dots, f_M)$: $\mathbb{R}^M \to \mathbb{R}^M$ and $\boldsymbol{z}_0 = (z_{01}, \dots, z_{0M})$: $\Omega \to \mathbb{R}^M$ are given functions. We note that the diffusivity β_i of the ith component depends not only on the ith variable but also on the jth $(j \neq i)$ variables in general. This mixture of diffusion terms is called cross-diffusion. Numerous problems of this type have been proposed in the literature, especially in the area of population ecology.

Keywords and phrases. Cross-diffusion systems, nonlinear diffusion, discrete-time schemes, numerical schemes, Reaction-diffusion system approximations.

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In the case $\beta_i(\boldsymbol{z}) = a_i z_i$ for non-negative constants a_i $(i = 1, 2, \ldots, M)$, the problem (1.1) is reduced to a semilinear reaction-diffusion system which are studied in many fields of applications. If the reaction term $\boldsymbol{f}$ is given by

$$f_i(\boldsymbol{z}) = \left(g_{i0} - \sum_{j=1}^{M} g_{ij} z_j \right) z_i, \qquad i = 1, 2, \ldots, M \tag{1.2}$$

for non-negative constants g_{ij}, the system is the well-known Lotka-Volterra competition-diffusion system. Here, z_i represents the population density of ith species. The constant a_i is the diffusion rate, g_{i0} denotes the intrinsic growth rate, g_{ii} accounts for the intraspecific competition coefficient, and g_{ij} $(i \neq j, j \neq 0)$ are the interspecific competition rates. Qualitative properties of non-negative solutions of this problem have been extensively studied from the mathematical point of view. However, it covers only the case where the diffusive motions of different species are mutually non-interfering. Apart from the point-interactions between species described in (1.2), Kerner [13] considered a motional type of interaction, that recognizes the bias of the motion of predator toward prey and of prey away from predator. He proposed the following cross-diffusion system:

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = a_1 \Delta z_1 + b_{12} \Delta z_2 + f_1(\boldsymbol{z}), \\ \dfrac{\partial z_2}{\partial t} = a_2 \Delta z_2 + b_{21} \Delta z_1 + f_2(\boldsymbol{z}), \end{cases} \tag{1.3}$$

where a_i $(i = 1, 2)$ are non-negative constants, b_{12} and b_{21} can be either positive or negative. The condition $b_{ij} > 0$ denotes the movement of ith species in the direction of lower concentration of jth species, while $b_{ij} < 0$ implies that the flux is directed toward increasing population density of the jth species. Gurtin [9] investigated the effect of the cross-diffusion and showed that the effect may give rise to segregation of the two species.

One of the typical problems with cross-diffusion is the following model which was proposed by Shigesada *et al.* [24] to understand spatial and temporal behaviours of two animal species under the influence of the population pressure due to intra- and interspecific interferences:

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = \Delta \left[(a_1 + b_1 z_1 + c_1 z_2) z_1 \right] + f_1(\boldsymbol{z}), \\ \dfrac{\partial z_2}{\partial t} = \Delta \left[(a_2 + b_2 z_2 + c_2 z_1) z_2 \right] + f_2(\boldsymbol{z}), \end{cases} \tag{1.4}$$

where a_i, b_i, c_i $(i = 1, 2)$ are non-negative constants. The virtual diffusivity of the ith species $a_i + b_i z_i + c_i z_j$ $(j \neq i)$ is dependent on intra- and interspecific population pressure. The cross-diffusion terms describe tendencies such that the ith species keeps away from high-density areas of the jth species. The spatially segregating coexistence of two competing species occurs by the cross-diffusion effect, which is called cross-diffusion induced instability [15].

There are a number of other exciting cross-diffusion models in population ecology. For instance, Kadota and Kuto [12] investigated the following prey-predator cross-diffusion system.

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = \Delta \left[(1 + b_1 z_2) z_1 \right] + \left(g_{10} - g_{11} z_1 - g_{12} z_2 \right) z_1, \\ \dfrac{\partial z_2}{\partial t} = \Delta \left[\left(a + \dfrac{1}{1 + b_2 z_1} \right) z_2 \right] + \left(g_{20} + g_{21} z_1 - g_{22} z_2 \right) z_2, \end{cases} \tag{1.5}$$

where a and g_{ij} $(i = 1, 2, j = 0, 1, 2)$ are positive constants, and b_1 and b_2 are non-negative constants. In the system, the diffusivity of the prey is the same type of (1.4), but that of the predator is a fractional type which implies the population pressure of the predator weakens in high-density areas of the prey, *i.e.*, the predator migrates towards areas of high concentration of the prey. The same type of nonlinear diffusivity was treated by

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Pang and Wang [23]. They studied a two-predator-one-prey cross-diffusion system in which the prey exercises a defence switching and the predators collaboratively take advantage of the prey's strategy.

Thus, there are a lot of interesting and important cross-diffusion systems. We would like to carry out numerical experiments for various type of nonlinearities. However, there are few results on numerical methods for the cross-diffusion systems. Moreover, numerical methods have been constructed and analyzed for each individual problem separately. To the best of the author's knowledge, there is no effective numerical method for general cross-diffusion systems. The forward difference scheme is versatile and it is very easy to implement. However, the scheme is unstable. The time step size has to be taken to be very small if explicit schemes are used. We need versatile and stable numerical methods for general cross-diffusion systems.

We discuss discrete-time approximations to the problem (1.1). They are simpler than fully-discrete approximations but play a crucial role in developing numerical methods. In this paper, we denote by $\tau = T/N_T$ ($N_T \in \mathbb{N}$) the time step size, and by $\bar{D}_\tau$ the backward difference operator, *i.e.*, $\bar{D}_\tau Z^n = (Z^n - Z^{n-1})/\tau$ for any given family $\{Z^n\}_n$.

We introduce known studies about numerical methods for the Shigesada-Kawasaki-Teramoto model (1.4). Galiano *et al.* [7] carried out one-dimensional numerical simulations. First they rewrote the diffusion terms and the Neumann boundary conditions as follows:

$$\begin{aligned}\frac{\partial z_i}{\partial t} &= (a_i + 2b_i z_i + c_i z_j)\Delta z_i + z_i \Delta z_j + 2(b_i \nabla z_i + \nabla z_j)\cdot \nabla z_i,\\ \frac{\partial}{\partial \nu}[(a_i + b_i z_i + c_i z_j) z_i] &= (a_i + 2b_i z_i + c_i z_j)\frac{\partial}{\partial \nu} z_i + c_i z_i \frac{\partial}{\partial \nu} z_j = 0\end{aligned}$$

for $(i,j) \in \{(1,2),(2,1)\}$. Then, they chose the following discretization in time.

$$\begin{aligned}\bar{D}_\tau Z_i^n = (a_i + 2b_i Z_i^{n-1} + c_i Z_j^{n-1})\Delta Z_i^n + Z_i^n \Delta Z_j^{n-1} + 2(b_i \nabla Z_i^{n-1} + \nabla Z_j^{n-1})\cdot \nabla Z_i^n,\\ (a_i + 2b_i Z_i^{n-1} + c_i Z_j^{n-1})\frac{\partial}{\partial \nu} Z_i^n + c_i Z_i^n \frac{\partial}{\partial \nu} Z_j^{n-1} = 0\end{aligned} \tag{1.6}$$

for $n = 1, 2, \ldots, N_T$ with the initial data Z_i^0. This is linear scheme. So, after discretizing in space, the implementation is easy and it's some efficient. However, when the other problem such as (1.5) is considered, another scheme has to be constructed again. Moreover, there is no mathematical results on this scheme. There are theoretical results on numerical methods for the Shigesada-Kawasaki-Teramoto model (1.4). Galiano *et al.* [8] considered a nonlinear implicit scheme. They proved that convergence in one space dimension. Barrett and Blowey [1] considered a fully discrete finite element approximation with a regularization technique. Their method is also implicit scheme. They showed that convergence in space dimensions $d \leq 3$, and presented numerical experiments in one space dimension. Implicit schemes show better stability and accuracy properties in general. Therefore, their schemes might be efficient. However, for three or four or more components system, the implementation is complicated. Moreover, their analysis can not apply to other cross-diffusion systems. There may be cases where many numerical simulations are carried out by changing not only the coefficients but also the nonlinearity itself. These schemes are not suitable in such cases.

We would like to construct a versatile, easily implemented and efficient numerical method for the general cross-diffusion system (1.1). Investigating numerical methods for single nonlinear diffusion equation $\frac{\partial z}{\partial t} = \Delta \beta(z)$, $z(x,t) \in \mathbb{R}$ may give us a hint. The single equation is still important because this framework is so general as to include the Stefan problems and the porous medium equations. Thereby, a lot of numerical methods have been suggested and analyzed.

Implicit methods of type $\bar{D}_\tau Z^n = \Delta \beta(Z^n)$ or $\bar{D}_\tau \beta^{-1}(U^n) = \Delta U^n$ have been extensively studied and they are efficient (see, *e.g.*, [2,20,22,26] and references therein). However, as I mentioned before, the implementation becomes complicated when systems are treated. Implicit methods are good for precise examination of each individual problem.

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Berger *et al.* [3] suggested the following linear scheme:

$$\begin{cases} U^n - \dfrac{\tau}{\mu}\Delta U^n = \beta(Z^{n-1}) & \text{in} \quad \Omega, \\ \dfrac{\partial U^n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ Z^n := Z^{n-1} + \mu(U^n - \beta(Z^{n-1})) & \text{in} \quad \Omega. \end{cases} \tag{1.7}$$

Here, μ is a positive constant. This scheme is based on nonlinear Chernoff formula arises in nonlinear semigroup theory. This scheme consists of solving a linear elliptic equation and updating Z. Many authors investigated this scheme because of its effectiveness (*e.g.*, [10,11,14,18,19,21,26]). This scheme can be implemented very easily. Implementation and computational cost are almost same as those for the implicit method for the linear heat equation. When you want to deal with other nonlinearity, all you have to do is to rewrite the function beta. This is the method which we have required. Therefore, we would like to extend this scheme to systems.

The linear scheme (1.7) for the single equation can be extended to that for the nonlinear cross-diffusion system (1.1) immediately. We just replace U, Z and β with boldface, that is, vectors; namely,

$$\begin{cases} \boldsymbol{U}^n - \dfrac{\tau}{\mu}\Delta \boldsymbol{U}^n = \boldsymbol{\beta}(\boldsymbol{Z}^{n-1}) + \dfrac{\tau}{\mu}\boldsymbol{f}(\boldsymbol{Z}^{n-1}) & \text{in} \quad \Omega, \\ \dfrac{\partial \boldsymbol{U}^n}{\partial \nu} = \boldsymbol{0} & \text{on} \quad \partial\Omega, \\ \boldsymbol{Z}^n := \boldsymbol{Z}^{n-1} + \mu(\boldsymbol{U}^n - \boldsymbol{\beta}(\boldsymbol{Z}^{n-1})) & \text{in} \quad \Omega. \end{cases} \tag{1.8}$$

This scheme is quite simple. The scheme amounts to solving M linear elliptic equations, followed by explicit algebraic corrections at each time step. The boundary condition is also considerably simpler than that in (1.6). However, the analyses of (1.7) can not be applied to the system (1.8) because they need maximum principle or the primitive function of β. The maximum principle does not hold for systems in general. And it is difficult to construct the primitive functions of $\boldsymbol{\beta}$ because $\beta_i(\boldsymbol{z})$ depends on z_j $(j \neq i)$.

Fortunately, we have a clue to solving this problem. We focus on the following semilinear reaction-diffusion system:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} = \dfrac{1}{\mu}\Delta \boldsymbol{u} - \dfrac{1}{\varepsilon}(\boldsymbol{u} - \boldsymbol{\beta}(\mu\boldsymbol{u} + \boldsymbol{v})) + \dfrac{1}{\mu}\boldsymbol{f}(\mu\boldsymbol{u} + \boldsymbol{v}) & \text{in} \quad Q, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} = \dfrac{\mu}{\varepsilon}(\boldsymbol{u} - \boldsymbol{\beta}(\mu\boldsymbol{u} + \boldsymbol{v})) & \text{in} \quad Q, \\ \dfrac{\partial \boldsymbol{u}}{\partial \nu} = \boldsymbol{0} & \text{on} \quad \partial\Omega \times (0,T), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0^\varepsilon, \quad \boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0^\varepsilon & \text{in} \quad \Omega, \end{cases} \tag{1.9}$$

where μ and ε are positive parameters. This reaction-diffusion system was proposed by the author to avoid the nonlinearity of the diffusion [16]. Let $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a weak solution of (1.9). The convergence of $\boldsymbol{z}^\varepsilon := \mu\boldsymbol{u}^\varepsilon + \boldsymbol{v}^\varepsilon$ to the weak solution $\boldsymbol{z}$ of (1.1) with initial datum $\boldsymbol{z}_0 = \lim_{\varepsilon\to 0}(\mu\boldsymbol{u}_0^\varepsilon + \boldsymbol{v}_0^\varepsilon)$ as ε tends to zero has been shown provided that β_i depends only on the ith variable. Hence, cross-diffusion systems were not treated. We extend the results to the cross-diffusion system in this paper. The convergence of $\boldsymbol{z}^\varepsilon$ to the weak solution of the cross-diffusion systems are shown by imitating a relevant study by the author [17]. In [17], the system (1.1) is approximated by a system of $2M$ semilinear PDEs while (1.9) consists of M PDEs coupled with M ODEs. The computational costs for ODEs are generally cheaper than those for PDEs. So, the approximation (1.9) is more suitable than the approximation in [17] for constructions of efficient numerical schemes.

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The linear scheme (1.8) can be regarded as a particular time discretization of the reaction-diffusion system (1.9). We consider the following semi-implicit time discretization.

$$\begin{cases} \bar{D}_\tau \boldsymbol{U}^n = \dfrac{1}{\mu}\Delta \boldsymbol{U}^n - \dfrac{1}{\varepsilon}(\boldsymbol{U}^{n-1} - \boldsymbol{\beta}(\mu \boldsymbol{U}^{n-1} + \boldsymbol{V}^{n-1})) + \dfrac{1}{\mu}\boldsymbol{f}(\mu \boldsymbol{U}^{n-1} + \boldsymbol{V}^{n-1}) & \text{in} \quad \Omega, \\ \dfrac{\partial \boldsymbol{U}^n}{\partial \nu} = \boldsymbol{0} & \text{on} \quad \partial\Omega, \\ \bar{D}_\tau \boldsymbol{V}^n = \dfrac{\mu}{\varepsilon}(\boldsymbol{U}^{n-1} - \boldsymbol{\beta}(\mu \boldsymbol{U}^{n-1} + \boldsymbol{V}^{n-1})) & \text{in} \quad \Omega. \end{cases} \tag{1.10}$$

Put $\boldsymbol{Z}^n = \mu \boldsymbol{U}^n + \boldsymbol{V}^n$. Then, this scheme corresponds to the linear scheme (1.8) if $\varepsilon = \tau$. Indeed, the first equation of (1.10) coincides with the first equation of (1.8) since $\varepsilon = \tau$. The third equation of (1.10) implies

$$\boldsymbol{V}^n = \boldsymbol{Z}^{n-1} - \mu \boldsymbol{\beta}(\boldsymbol{Z}^{n-1}).$$

Add $\mu \boldsymbol{U}^n$ both sides to obtain the third relation of (1.8). Thus, (1.8) is regarded as a time discretization of (1.9). Therefore, the idea of proof of the approximation theory can be applied to proof of convergence of the linear scheme (1.8). The aim of this paper is to show stability and convergence of (1.10). As a consequence, we find that $\varepsilon = \tau$ is the best choice while the semi-implicit discretization (1.10) is considered.

The organization of this paper is as follows. In the next section, we state the assumptions and problems precisely. Then, our main result, that is, the convergence of the solution of the discrete-time scheme (1.10) to that of (1.1), is presented. The reaction-diffusion system (1.9) is treated in Section 3. We establish a convergence result of (1.9). Our proof is simple and is based on conventional method: energy estimate and compactness argument. Stability and convergence results of (1.10) are proved in Section 4. In Section 5, numerical simulations are carried out to demonstrate the effectiveness of the linear scheme (1.8).

2. Formulation and main results

In this section we establish the assumptions on the data, state precisely the problems and present our results.

2.1. Assumptions

The general cross-diffusion system are quite difficult to deal with. Even for the problem (1.4), only partial results are available on the existence of solutions (see [5,6] and references therein). We impose the following assumptions:

(H1) $\boldsymbol{\beta}$ is a Lipschitz continuous function satisfying $\boldsymbol{\beta}(\boldsymbol{0}) = \boldsymbol{0}$.

(H2) There exists a positive constant a such that

$$\sum_{i=1}^{M}\sum_{j=1}^{M}(\beta_i)_j(\boldsymbol{\eta})\xi_i\xi_j \geq a|\boldsymbol{\xi}|^2$$

for almost all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^M$.

Here, $(\beta_i)_j$ denotes the derivative of the ith component of $\boldsymbol{\beta}$ with respect to the jth variable.

(H3) $\boldsymbol{f}$ is a Lipschitz continuous function.

Let L be a positive constant satisfying

$$|(\beta_i)_j(\boldsymbol{\eta}) - a\delta_{ij}| \leq L$$

for almost all $\boldsymbol{\eta} \in \mathbb{R}^M$ and all $i, j \in \{1, 2, \ldots, M\}$, where δ_{ij} is the Kronecker delta.

(H4) μ satisfies

$$0 < \mu < \frac{a}{a^2 + M^2L^2}.$$

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(H5) $\boldsymbol{z}_0 \in L^2(\Omega)^M$.
(H6) $\tau \le \varepsilon$ and $\boldsymbol{u}_0^{\varepsilon,\tau}, \boldsymbol{v}_0^{\varepsilon,\tau} \in H^1(\Omega)^M$ satisfy

$$\|\mu \boldsymbol{u}_0^{\varepsilon,\tau} + \boldsymbol{v}_0^{\varepsilon,\tau}\|_{L^2(\Omega)^M} + \sqrt{\tau}\,\|\mu \boldsymbol{u}_0^{\varepsilon,\tau} + \boldsymbol{v}_0^{\varepsilon,\tau}\|_{H^1(\Omega)^M} + \sqrt{\varepsilon-\tau}\,\|\boldsymbol{v}_0^{\varepsilon,\tau}\|_{H^1(\Omega)^M} \le C$$

for some positive constant C independent of ε and τ. Moreover,

$$\mu \boldsymbol{u}_0^{\varepsilon,\tau} + \boldsymbol{v}_0^{\varepsilon,\tau} \rightharpoonup \boldsymbol{z}_0 \quad \text{weakly in } L^2(\Omega)^M \text{ as } \varepsilon, \tau \to 0.$$

2.2. Weak formulations

In this paper, $\langle \cdot, \cdot \rangle$ denotes both the inner product in $L^2(\Omega)$ and the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$. The problem (1.1) will be understood in the sense of the following weak form:

Definition 2.1. A function $\boldsymbol{z} \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M$ is said to be a weak solution of (1.1) if it fulfils

$$\int_0^T \left\langle \frac{\partial z_i}{\partial t}, \varphi_i \right\rangle + \int_0^T \langle \nabla \beta_i(\boldsymbol{z}), \nabla \varphi_i \rangle = \int_0^T \langle f_i(\boldsymbol{z}), \varphi_i \rangle$$

for all functions $\varphi_i \in L^2(0,T;H^1(\Omega))$, $i = 1, 2, \ldots, M$, and

$$\boldsymbol{z}(\cdot, 0) = \boldsymbol{z}_0 \quad \text{a.e. in } \Omega.$$

We also introduce a weak form of the time discretization scheme to be studied.

Problem 2.2. We prescribe the initial conditions

$$\boldsymbol{U}^0 = \boldsymbol{u}_0^{\varepsilon,\tau} \in H^1(\Omega)^M, \quad \boldsymbol{V}^0 = \boldsymbol{v}_0^{\varepsilon,\tau} \in H^1(\Omega)^M.$$

For $i = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N_T$, find $U_i^n \in H^1(\Omega)$ and $V_i^n \in H^1(\Omega)$ such that

$$\langle \bar{D}_\tau U_i^n, \varphi \rangle + \frac{1}{\mu} \langle \nabla U_i^n, \nabla \varphi \rangle = \frac{1}{\mu} \langle f_i(\boldsymbol{Z}^{n-1}), \varphi \rangle - \frac{1}{\varepsilon} \langle U_i^{n-1} - \beta_i(\mu \boldsymbol{U}^{n-1} + \boldsymbol{V}^{n-1}), \varphi \rangle \tag{2.1}$$

for all $\varphi \in H^1(\Omega)$, and

$$\bar{D}_\tau V_i^n = \frac{\mu}{\varepsilon}(U_i^{n-1} - \beta_i(\mu \boldsymbol{U}^{n-1} + \boldsymbol{V}^{n-1})) \qquad \text{a.e. in } \Omega. \tag{2.2}$$

Unique solvability of (2.1) besides $U_i^n \in H^3(\Omega)$ are well known (see, *e.g.*, [4]).

2.3. Main results

We now state our main results.

Theorem 2.3. *Assume that* (H1)–(H6) *are satisfied. Let* $\{\boldsymbol{U}^n, \boldsymbol{V}^n\}_{n=0}^{N_T}$ *be the solution of Problem* 2.2. *We denote by* $\boldsymbol{U}^{(\varepsilon,\tau)}$ *and* $\boldsymbol{V}^{(\varepsilon,\tau)}$ *the functions obtained by piecewise constant interpolation in time of* $\{\boldsymbol{U}^n\}$ *and* $\{\boldsymbol{V}^n\}$, *respectively. Then, there exist subsequences* $\{\boldsymbol{U}^{(\varepsilon_k,\tau_k)}\}, \{\boldsymbol{V}^{(\varepsilon_k,\tau_k)}\}$ *of* $\{\boldsymbol{U}^{(\varepsilon,\tau)}\}, \{\boldsymbol{V}^{(\varepsilon,\tau)}\}$ *and a weak solution* $\boldsymbol{z}$ *of* (1.1) *such that*

$$\begin{aligned} &\mu \boldsymbol{U}^{(\varepsilon_k,\tau_k)} + \boldsymbol{V}^{(\varepsilon_k,\tau_k)} \to \boldsymbol{z} && \text{strongly in } L^2(Q)^M, \text{ a.e. in } Q \text{ and weakly in } L^2(0,T;H^1(\Omega))^M, \\ &\boldsymbol{U}^{(\varepsilon_k,\tau_k)} \rightharpoonup \boldsymbol{\beta}(\boldsymbol{z}) && \text{weakly in } L^2(0,T;H^1(\Omega))^M \end{aligned}$$

as ε_k *and* τ_k *tend to zero.*

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The conditions (H1) and (H2) imply that the systems are uniformly parabolic. Therefore, Theorem 2.3 shows that weak solutions of the uniformly parabolic cross-diffusion systems can be approximated by those of the linear scheme (1.8). The cross-diffusion system (1.4) with non-negative bounded solutions is uniformly parabolic if

$$a_i > 0,\ 8b_1c_2 > c_1^2,\ 8b_2c_1 > c_2^2.$$

The system (1.4) with

$$a_i > 0,\ b_i, c_2 \geq 0,\ c_1 = 0$$

is also taken into account. In this case, the system is not always uniformly parabolic, but it is weakly coupled. We can treat more general problems including both cases. We deal with the system (1.1) which possess block triangular diffusion matrices. The diagonal blocks of these matrices have the same structure as the above. We use the following notations: $\mathbb{R}^M = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_l}$,

$$\boldsymbol{z} = (z_1, \ldots, z_n) = (z_{11}, \ldots, z_{1m_1}, z_{21}, \ldots, z_{2m_2}, \ldots, z_{l1}, \ldots, z_{lm_l}),$$
$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) = (\beta_{11}, \ldots, \beta_{1m_1}, \beta_{21}, \ldots, \beta_{2m_2}, \ldots, \beta_{l1}, \ldots, \beta_{lm_l})$$

and so on. The following condition is imposed on $\boldsymbol{\beta}$ instead of (H2):

(H2)′ For each $i = 1, \ldots, l-1$ and $j = 1, \ldots, m_i$, the function β_{ij} is independent of the rsth ($r > i, s = 1, \ldots, m_r$) variables. Moreover, $\boldsymbol{\beta}$ satisfies

$$\sum_{j=1}^{m_i}\sum_{s=1}^{m_i}(\beta_{ij})_{is}(\boldsymbol{\eta})\xi_{ij}\xi_{is} \geq a\sum_{j=1}^{m_i}|\xi_{ij}|^2$$

for a positive constant a, all $i = 1, \ldots, l$ and almost all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^M$.

Combining the proof of Theorem 2.3 with the idea of the proof in [17], we obtain the following result.

Corollary 2.4. *Assume that* (H1), (H2)′ *and* (H3)–(H6) *are satisfied. Let* $\{\boldsymbol{U}^n,\ \boldsymbol{V}^n\}_{n=0}^{N_T}$ *be the solution of Problem 2.2. We denote by* $\boldsymbol{U}^{(\varepsilon,\tau)}$ *and* $\boldsymbol{V}^{(\varepsilon,\tau)}$ *the functions obtained by piecewise constant interpolation in time of* $\{\boldsymbol{U}^n\}$ *and* $\{\boldsymbol{V}^n\}$, *respectively. Then, there exist subsequences* $\{\boldsymbol{U}^{(\varepsilon_k,\tau_k)}\}, \{\boldsymbol{V}^{(\varepsilon_k,\tau_k)}\}$ *of* $\{\boldsymbol{U}^{(\varepsilon,\tau)}\}, \{\boldsymbol{V}^{(\varepsilon,\tau)}\}$ *and a weak solution* $\boldsymbol{z}$ *of* (1.1) *such that*

$$\begin{aligned} &\mu\boldsymbol{U}^{(\varepsilon_k,\tau_k)} + \boldsymbol{V}^{(\varepsilon_k,\tau_k)} \to \boldsymbol{z} && \text{strongly in } L^2(Q)^M,\ \text{a.e. in } Q \text{ and weakly in } L^2(0,T;H^1(\Omega))^M,\\ &\boldsymbol{U}^{(\varepsilon_k,\tau_k)} \rightharpoonup \boldsymbol{\beta}(\boldsymbol{z}) && \text{weakly in } L^2(0,T;H^1(\Omega))^M \end{aligned}$$

as ε_k *and* τ_k *tend to zero.*

3. A reaction-diffusion system approximation

In this section, we analyze the reaction-diffusion system (1.9).

3.1. Formulation and results

The following condition is imposed on the initial data instead of (H6):

(H6)′ $\boldsymbol{u}_0^\varepsilon \in L^2(\Omega)^M$ and $\boldsymbol{v}_0^\varepsilon \in H^1(\Omega)^M$ satisfy

$$\|\mu\boldsymbol{u}_0^\varepsilon + \boldsymbol{v}_0^\varepsilon\|_{L^2(\Omega)^M} + \sqrt{\varepsilon}\,\|\boldsymbol{v}_0^\varepsilon\|_{H^1(\Omega)^M} \leq C$$

for some positive constant C independent of ε, and

$$\mu\boldsymbol{u}_0^\varepsilon + \boldsymbol{v}_0^\varepsilon \rightharpoonup \boldsymbol{z}_0 \quad \text{weakly in } L^2(\Omega)^M.$$

We introduce a weak formulation of (1.9).

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Definition 3.1. We call $(\boldsymbol{u},\boldsymbol{v}) \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M \times H^1(Q)^M$ a weak solution of (1.9) if it satisfies

$$\int_0^T \left\langle \frac{\partial u_i}{\partial t}, \varphi_i \right\rangle + \frac{1}{\mu}\int_0^T \langle \nabla u_i, \nabla \varphi_i \rangle = \frac{1}{\mu}\int_0^T \langle f_i(\mu \boldsymbol{u} + \boldsymbol{v}), \varphi_i \rangle - \frac{1}{\varepsilon}\int_0^T \langle u_i - \beta_i(\mu \boldsymbol{u} + \boldsymbol{v}), \varphi_i \rangle \tag{3.1}$$

for all functions $\varphi_i \in L^2(0,T;H^1(\Omega))$ and

$$\frac{\partial v_i}{\partial t} = \frac{\mu}{\varepsilon}\left(u_i - \beta_i(\mu \boldsymbol{u} + \boldsymbol{v})\right) \quad \text{a.e. in } Q. \tag{3.2}$$

for $i = 1, 2, \ldots, M$, and

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0^\varepsilon, \quad \boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0^\varepsilon \quad \text{a.e. in } \Omega. \tag{3.3}$$

The unique existence of the weak solution of (1.9) can be established by means of the conventional method (see also Rem. 4.2 for existence). For any $\varepsilon > 0$ the problem (1.9) has one and only one weak solution $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ such that

$$\begin{aligned} &\boldsymbol{u}^\varepsilon \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M, \\ &\boldsymbol{v}^\varepsilon \in (L^\infty(0,T;H^1(\Omega)) \cap H^1(Q))^M. \end{aligned}$$

We have prepared sufficiently to state our results.

Theorem 3.2. *Assume that* (H1)–(H5) *and* (H6)′ *hold. Let* $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ *be the weak solution of* (1.9)*. Then, there exist a weak solution* $\boldsymbol{z} \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M$ *of* (1.1) *and subsequences* $\{\boldsymbol{u}^{\varepsilon_k}\}$, $\{\boldsymbol{v}^{\varepsilon_k}\}$ *of* $\{\boldsymbol{u}^\varepsilon\}$, $\{\boldsymbol{v}^\varepsilon\}$ *such that*

$$\begin{aligned} &\mu \boldsymbol{u}^{\varepsilon_k} + \boldsymbol{v}^{\varepsilon_k} \to \boldsymbol{z} && \text{strongly in } L^2(Q)^M, \text{ a.e. in } Q, \\ &&& \text{and weakly in } L^2(0,T;H^1(\Omega))^M \text{ and } H^1(0,T;H^1(\Omega)^*)^M, \\ &\boldsymbol{u}^{\varepsilon_k} \rightharpoonup \boldsymbol{\beta}(\boldsymbol{z}) && \text{weakly in } L^2(0,T;H^1(\Omega))^M \end{aligned}$$

as ε_k *tends to zero.*

Our results show that the weak solutions of the cross-diffusion systems can be approximated by those of the reaction-diffusion system (1.9).

We can deal with the general problem so that (H2)′ is satisfied instead of (H2).

Corollary 3.3. *Assume that* (H1), (H2)′, (H3)–(H5) *and* (H6)′ *hold. Let* $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ *be the weak solution of* (1.9)*. Then, there exist a weak solution* $\boldsymbol{z} \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M$ *of* (1.1) *and subsequences* $\{\boldsymbol{u}^{\varepsilon_k}\}$, $\{\boldsymbol{v}^{\varepsilon_k}\}$ *of* $\{\boldsymbol{u}^\varepsilon\}$, $\{\boldsymbol{v}^\varepsilon\}$ *such that*

$$\begin{aligned} &\mu \boldsymbol{u}^{\varepsilon_k} + \boldsymbol{v}^{\varepsilon_k} \to \boldsymbol{z} && \text{strongly in } L^2(Q)^M, \text{ a.e. in } Q, \\ &&& \text{and weakly in } L^2(0,T;H^1(\Omega))^M \text{ and } H^1(0,T;H^1(\Omega)^*)^M, \\ &\boldsymbol{u}^{\varepsilon_k} \rightharpoonup \boldsymbol{\beta}(\boldsymbol{z}) && \text{weakly in } L^2(0,T;H^1(\Omega))^M \end{aligned}$$

as ε_k *tends to zero.*

3.2. *A priori* estimates

A priori estimates are presented for the weak solution $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ of the system (1.9) and a function $\boldsymbol{z}^\varepsilon = \mu \boldsymbol{u}^\varepsilon + \boldsymbol{v}^\varepsilon$. In this section, C denotes a generic positive constant independent of ε.

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Lemma 3.4. *Assume that* (H1)–(H4) *and* (H6)′ *are satisfied. Then there exists a positive constant* C *independent of* ε *such that*

$$\|\boldsymbol{z}^\varepsilon\|_{(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)^*))^M} + \|\boldsymbol{u}^\varepsilon\|_{L^2(0,T;H^1(\Omega))^M} + \|\boldsymbol{v}^\varepsilon\|_{L^2(0,T;H^1(\Omega))^M} + \sqrt{\varepsilon}\,\|\boldsymbol{v}^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))^M} \le C.$$

Proof. From Definition 3.1, the functions $\boldsymbol{z}^\varepsilon$ and $\boldsymbol{u}^\varepsilon$ satisfy

$$\int_0^T \left\langle \frac{\partial z_i^\varepsilon}{\partial t}, \varphi_i \right\rangle + \int_0^T \langle \nabla u_i^\varepsilon, \nabla \varphi_i \rangle = \int_0^T \langle f_i(\boldsymbol{z}^\varepsilon), \varphi_i \rangle \tag{3.4}$$

for all functions $\varphi_i \in L^2(0,T;H^1(\Omega))$ and $i = 1,\dots,M$. For an arbitrary point $t_0 \in (0,T)$, take

$$\varphi_i(\cdot,t) = \begin{cases} z_i^\varepsilon(\cdot,t) & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise} \end{cases}$$

to obtain

$$\frac{1}{2}\|z_i^\varepsilon(t_0)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|z_{0i}^\varepsilon\|_{L^2(\Omega)}^2 + \int_0^{t_0} \langle \nabla u_i^\varepsilon, \nabla z_i^\varepsilon \rangle = \int_0^{t_0} \langle f_i(\boldsymbol{z}^\varepsilon), z_i^\varepsilon \rangle \le C\left(\|\boldsymbol{z}^\varepsilon\|_{L^2(0,t_0;L^2(\Omega))^M}^2 + 1\right). \tag{3.5}$$

Here, we use the Lipschitz continuity of f_i and the following elementary inequality

$$2bc \le b^2\alpha + c^2/\alpha \quad \text{for } b,c \in \mathbb{R} \text{ and } \alpha > 0. \tag{3.6}$$

This relation will often be used in the sequel. It follows from (3.2) that

$$\begin{aligned}\langle \nabla u_i^\varepsilon, \nabla z_i^\varepsilon \rangle &= \mu\|u_i^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_i^\varepsilon\|_{L^2(\Omega)}^2 + \langle \nabla \beta_i(\boldsymbol{z}^\varepsilon), \nabla z_i^\varepsilon \rangle - \mu\langle \nabla \beta_i(\boldsymbol{z}^\varepsilon), \nabla u_i^\varepsilon \rangle \\ &= \mu\|u_i^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_i^\varepsilon\|_{L^2(\Omega)}^2 + a\|\nabla z_i^\varepsilon\|_{L^2(\Omega)}^2 + \langle \nabla \phi_i(\boldsymbol{z}^\varepsilon), \nabla z_i^\varepsilon \rangle \\ &\quad - \mu a\langle \nabla z_i^\varepsilon, \nabla u_i^\varepsilon \rangle - \mu\langle \nabla \phi_i(\boldsymbol{z}^\varepsilon), \nabla u_i^\varepsilon \rangle. \end{aligned} \tag{3.7}$$

Here, $\phi_i(\boldsymbol{\eta}) = \beta_i(\boldsymbol{\eta}) - a\eta_i$ for all $\boldsymbol{\eta} \in \mathbb{R}^M$. The last term of the right-hand side of (3.7) can be estimated as follows:

$$|\mu\langle \nabla \phi_i(\boldsymbol{z}^\varepsilon), \nabla u_i^\varepsilon \rangle| \le \frac{\mu}{2}\|\nabla u_i^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\mu M L^2}{2}\sum_{j=1}^M \|\nabla z_j^\varepsilon\|_{L^2(\Omega)}^2. \tag{3.8}$$

Collect the previous estimates and sum over i from 1 to M to get

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{z}^\varepsilon(t_0)\|_{L^2(\Omega)^M}^2 + \frac{1}{1+\mu a}\left(a - \frac{\mu M^2 L^2}{2}\right)\|\nabla \boldsymbol{z}^\varepsilon\|_{L^2(0,t_0;L^2(\Omega))}^2 + \frac{\mu}{2(1+\mu a)}\|\nabla \boldsymbol{u}^\varepsilon\|_{L^2(0,t_0;L^2(\Omega))}^2 \\ &\quad + \frac{\varepsilon}{2\mu(1+\mu a)}\|\nabla \boldsymbol{v}^\varepsilon(t_0)\|_{L^2(\Omega)}^2 + \frac{1}{1+\mu a}\int_0^{t_0}\sum_{i=1}^M \langle \nabla \phi_i(\boldsymbol{z}^\varepsilon), \nabla z_i^\varepsilon \rangle \\ &\le \frac{1}{2}\|\boldsymbol{z}_0^\varepsilon\|_{L^2(\Omega)^M}^2 + \frac{\varepsilon}{2\mu(1+\mu a)}\|\nabla \boldsymbol{v}_0^\varepsilon\|_{L^2(\Omega)}^2 + C\left(\|\boldsymbol{z}^\varepsilon\|_{L^2(0,t_0;L^2(\Omega))^M}^2 + 1\right). \end{aligned}$$

The last term of the left-hand side is positive due to the hypothesis (H2). Therefore, using (H4), (H6)′ and the Gronwall inequality yields

$$\|\boldsymbol{z}^\varepsilon\|_{(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)))^M} + \|\nabla \boldsymbol{u}^\varepsilon\|_{L^2(Q)^M} + \|\nabla \boldsymbol{v}^\varepsilon\|_{L^2(Q)^M} + \sqrt{\varepsilon}\,\|\nabla \boldsymbol{v}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))^M} \le C. \tag{3.9}$$

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Multiplying (3.2) by $\varepsilon v_i^\varepsilon$ and integrating both sides over space, we obtain

$$\begin{aligned}\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_i^\varepsilon\|_{L^2(\Omega)}^2 &= \mu\,\langle u_i^\varepsilon, v_i^\varepsilon\rangle - \mu\,\langle \beta_i(\boldsymbol{z}^\varepsilon), v_i^\varepsilon\rangle \\ &= \frac{1}{2}\|z_i^\varepsilon\|_{L^2(\Omega)}^2 - \frac{\mu^2}{2}\|u_i^\varepsilon\|_{L^2(\Omega)}^2 - \frac{1}{2}\|v_i^\varepsilon\|_{L^2(\Omega)}^2 - \mu\,\langle \beta_i(\boldsymbol{z}^\varepsilon), v_i^\varepsilon\rangle \\ &\le \frac{1}{2}\|z_i^\varepsilon\|_{L^2(\Omega)}^2 - \frac{\mu^2}{2}\|u_i^\varepsilon\|_{L^2(\Omega)}^2 - \frac{1}{4}\|v_i^\varepsilon\|_{L^2(\Omega)}^2 + \mu^2\|\beta_i(\boldsymbol{z}^\varepsilon)\|_{L^2(\Omega)}^2.\end{aligned}$$

We deduce from (H6)′ and (3.9) that

$$\|\boldsymbol{u}^\varepsilon\|_{L^2(Q)^M} + \|\boldsymbol{v}^\varepsilon\|_{L^2(Q)^M} + \sqrt{\varepsilon}\,\|\boldsymbol{v}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))^M} \le C.$$

It follows from (3.4) and (3.9) that for $i = 1, \dots, M$ and $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\left|\int_0^T \left\langle \frac{\partial z_i^\varepsilon}{\partial t}, \varphi \right\rangle\right| \le \|\nabla u_i^\varepsilon\|_{L^2(Q)}\,\|\nabla\varphi\|_{L^2(Q)} + \|f_i(\boldsymbol{z}^\varepsilon)\|_{L^2(Q)}\,\|\varphi\|_{L^2(Q)} \le C\,\|\varphi\|_{L^2(0,T;H^1(\Omega))}.$$

This means that

$$\left\|\frac{\partial \boldsymbol{z}^\varepsilon}{\partial t}\right\|_{L^2(0,T;H^1(\Omega)^*)^M} \le C,$$

which completes the proof. □

3.3. **Proof of Theorem 3.2**

In this section, we prove Theorem 3.2.

Proof of Theorem 3.2. By virtue of Lemma 3.4 and the compactness of the embedding $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*) \subset L^2(Q)$ ([25], Thm. 2.1), there exist subsequences $\{\boldsymbol{z}^{\varepsilon_k}\}$ and $\{\boldsymbol{u}^{\varepsilon_k}\}$ of $\{\boldsymbol{z}^\varepsilon\}$ and $\{\boldsymbol{u}^\varepsilon\}$ and functions $\boldsymbol{z}^*$, $\boldsymbol{u}^*$ such that

$$\begin{aligned}&\boldsymbol{z}^* \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M, \\ &\boldsymbol{u}^* \in L^2(0,T;H^1(\Omega))^M,\end{aligned}$$

$$\begin{aligned}&\boldsymbol{z}^{\varepsilon_k} \to \boldsymbol{z}^* \quad \text{strongly in } L^2(Q)^M, \text{ a.e. in } Q, \\ &\qquad\quad \text{and weakly in } L^2(0,T;H^1(\Omega))^M \text{ and } H^1(0,T;H^1(\Omega)^*)^M, \\ &\boldsymbol{u}^{\varepsilon_k} \rightharpoonup \boldsymbol{u}^* \quad \text{weakly in } L^2(0,T;H^1(\Omega))^M\end{aligned}$$

as ε_k tends to zero. The Lipschitz continuities of $\boldsymbol{\beta}$ and $\boldsymbol{f}$ imply that $\boldsymbol{\beta}(\boldsymbol{z}^{\varepsilon_k})$ and $\boldsymbol{f}(\boldsymbol{z}^{\varepsilon_k})$ also converge to $\boldsymbol{\beta}(\boldsymbol{z}^*)$ and $\boldsymbol{f}(\boldsymbol{z}^*)$ strongly in $L^2(Q)^M$ and a.e. in Q, respectively. Taking $\varphi_i = \varepsilon_k\zeta$, $\zeta \in C_0^\infty(Q)$ in (3.1) and letting to the limit in ε_k yield

$$\int_0^T \langle u_i^* - \beta_i(\boldsymbol{z}^*), \zeta\rangle = 0 \quad \text{for all } \zeta \in C_0^\infty(Q).$$

Therefore, $\boldsymbol{u}^* = \boldsymbol{\beta}(\boldsymbol{z}^*)$ holds a.e. in Q. From the initial condition (3.3), we have

$$\int_0^T \left\langle \frac{\partial z_i^\varepsilon}{\partial t}, \varphi_i \right\rangle + \int_0^T \left\langle z_i^\varepsilon - (\mu u_{0i}^\varepsilon + v_{0i}^\varepsilon), \frac{\partial \varphi_i}{\partial t} \right\rangle = 0 \tag{3.10}$$

for all functions $\varphi_i \in H^1(Q)$ with $\varphi_i(\cdot,T) = 0$. Passing to the limit along the subsequences in (3.4) and (3.10), we observe that $\boldsymbol{z}^*$ is a weak solution of (1.1). Thus, the proof is complete. □

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4. Proof of Theorem 2.3

We prove the following estimates for the solution $\{\boldsymbol{U}^n, \boldsymbol{V}^n\}$ of Problem 2.2 and a function $\boldsymbol{Z}^n = \mu\boldsymbol{U}^n + \boldsymbol{V}^n$.

Lemma 4.1. *Suppose* (H1)–(H4) *and* (H6) *are satisfied. Then there exists a positive constant C independent of ε and τ such that*

$$\begin{aligned}
&\max_{0\le n\le N_T}\|\boldsymbol{Z}^n\|_{L^2(\Omega)^M}^2 + \tau\sum_{n=1}^{N_T}\|\boldsymbol{Z}^n\|_{H^1(\Omega)^M}^2 + \tau\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{Z}^n\|_{(H^1(\Omega)^*)^M}^2 + \tau^2\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{Z}^n\|_{L^2(\Omega)^M}^2 \\
&\quad+\tau^3\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{Z}^n\|_{H^1(\Omega)^M}^2 + \tau\sum_{n=1}^{N_T}\|\boldsymbol{U}^n\|_{H^1(\Omega)^M}^2 + \tau^2\sum_{n=1}^{N_T}\|\Delta\boldsymbol{U}^n\|_{L^2(\Omega)^M}^2 + \tau\sum_{n=1}^{N_T}\|\boldsymbol{V}^n\|_{H^1(\Omega)^M}^2 \\
&\quad+\varepsilon\max_{0\le n\le N_T}\|\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2 + (\varepsilon-\tau)\max_{0\le n\le N_T}\|\boldsymbol{V}^n\|_{H^1(\Omega)^M}^2 + \varepsilon^2\tau\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2 \\
&\quad+(\varepsilon-\tau)\tau^2\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{V}^n\|_{H^1(\Omega)^M}^2 \le C.
\end{aligned}$$

In this section, we denote by C a generic positive constant independent of ε and τ.

This lemma implies that the linear scheme (1.10) is stable while $\tau \le \varepsilon$. A better numerical solution may be obtained when ε is taken to be small because the solution of (1.9) is expected to be close to that of (1.1) while ε is small. Therefore, $\varepsilon = \tau$ is the best choice.

Proof. Note that (2.1) and (2.2) can be combined to give

$$\left\langle Z_i^n - Z_i^{n-1}, \varphi\right\rangle + \tau\left\langle\nabla U_i^n, \nabla\varphi\right\rangle = \tau\left\langle f_i(\boldsymbol{Z}^{n-1}), \varphi\right\rangle \tag{4.1}$$

for all $\varphi \in H^1(\Omega)$. Take $\varphi = Z_i^n$ to get

$$\left\langle Z_i^n - Z_i^{n-1}, Z_i^n\right\rangle + \tau\left\langle\nabla U_i^n, \nabla Z_i^n\right\rangle = \tau\left\langle f_i(\boldsymbol{Z}^{n-1}), Z_i^n\right\rangle. \tag{4.2}$$

Using the elementary relations $2b(b-c) = b^2 - c^2 + (b-c)^2$ for $b, c \in \mathbb{R}$ and (3.6), the first term of the left-hand side and the right-hand side can be estimated as follows:

$$\begin{aligned}
2\left\langle Z_i^n - Z_i^{n-1}, Z_i^n\right\rangle &= \|Z_i^n\|_{L^2(\Omega)}^2 - \|Z_i^{n-1}\|_{L^2(\Omega)}^2 + \|Z_i^n - Z_i^{n-1}\|_{L^2(\Omega)}^2, \\
\tau\left\langle f_i(\boldsymbol{Z}^{n-1}), Z_i^n\right\rangle &\le \frac{\tau}{2}\|f_i(\boldsymbol{Z}^{n-1})\|_{L^2(\Omega)}^2 + \frac{\tau}{2}\|Z_i^n\|_{L^2(\Omega)}^2.
\end{aligned}$$

One of the key points of the proof of Lemma 3.4 was that 1 is divided into $1/(1+\mu a)$ and $\mu a/(1+\mu a)$ in order to estimate $\langle\nabla u_i^\varepsilon, \nabla z_i^\varepsilon\rangle$. Hence, we estimate the second term of the left-hand side of (4.2) twice in different ways. It follows from (2.2) that

$$\begin{aligned}
U_i^n &= U_i^n - U_i^{n-1} + \beta_i(\boldsymbol{Z}^{n-1}) + \frac{\varepsilon}{\mu}\bar{D}_\tau V_i^n \\
&= \frac{Z_i^n - Z_i^{n-1}}{\mu} + \beta_i(\boldsymbol{Z}^{n-1}) + \frac{1}{\mu}(\varepsilon-\tau)\bar{D}_\tau V_i^n.
\end{aligned}$$

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Therefore, we obtain

$$\begin{aligned}\tau\langle\nabla U_i^n,\nabla Z_i^n\rangle &= \mu\tau\,\|\nabla U_i^n\|_{L^2(\Omega)}^2+\frac{1}{\mu}\tau\left\langle\nabla(Z_i^n-Z_i^{n-1}),\nabla Z_i^n\right\rangle-\tau\left\langle\nabla(Z_i^n-Z_i^{n-1}),\nabla U_i^n\right\rangle\\ &\quad+\tau\left\langle\nabla\beta_i(\boldsymbol{Z}^{n-1}),\nabla V_i^n\right\rangle+\frac{1}{\mu}(\varepsilon-\tau)\left\langle\nabla(V_i^n-V_i^{n-1}),\nabla V_i^n\right\rangle\\ &=\mu\tau\,\|\nabla U_i^n\|_{L^2(\Omega)}^2+\frac{1}{2\mu}\tau\left(\|\nabla Z_i^n\|_{L^2(\Omega)}^2-\|\nabla Z_i^{n-1}\|_{L^2(\Omega)}^2+\|\nabla(Z_i^n-Z_i^{n-1})\|_{L^2(\Omega)}^2\right)\\ &\quad-\tau\left\langle\nabla(Z_i^n-Z_i^{n-1}),\nabla U_i^n\right\rangle+a\tau\left\|\nabla Z_i^{n-1}\right\|_{L^2(\Omega)}^2+\tau\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla Z_i^{n-1}\right\rangle\\ &\quad+a\tau\left\langle\nabla Z_i^{n-1},\nabla(Z_i^n-Z_i^{n-1})\right\rangle+\tau\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla(Z_i^n-Z_i^{n-1})\right\rangle-\mu a\tau\left\langle\nabla Z_i^{n-1},\nabla U_i^n\right\rangle\\ &\quad-\mu\tau\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla U_i^n\right\rangle+\frac{1}{2\mu}(\varepsilon-\tau)\left(\|\nabla V_i^n\|_{L^2(\Omega)}^2-\|\nabla V_i^{n-1}\|_{L^2(\Omega)}^2+\|\nabla(V_i^n-V_i^{n-1})\|_{L^2(\Omega)}^2\right). \qquad (4.3)\end{aligned}$$

On the other hand, we have

$$\tau\left\langle\nabla U_i^n,\nabla Z_i^n\right\rangle=\tau\left\langle\nabla U_i^n,\nabla(Z_i^n-Z_i^{n-1})\right\rangle+\tau\left\langle\nabla U_i^n,\nabla Z_i^{n-1}\right\rangle. \qquad (4.4)$$

Add up (4.3) and (4.4) weighted by μa and sum over n from 1 to $m\le N_T$. Then, summation by parts yields

$$\begin{aligned}(1+\mu a)\tau\sum_{n=1}^m\langle\nabla U_i^n,\nabla Z_i^n\rangle &= \mu\tau\sum_{n=1}^m\|\nabla U_i^n\|_{L^2(\Omega)}^2+\frac{1}{2}\left(\frac{1}{\mu}+a\right)\tau\left(\|\nabla Z_i^m\|_{L^2(\Omega)}^2-\|\nabla Z_i^0\|_{L^2(\Omega)}^2\right)\\ &\quad+\frac{1}{2}\left(\frac{1}{\mu}-a\right)\sum_{n=1}^m\|\nabla(Z_i^n-Z_i^{n-1})\|_{L^2(\Omega)}^2-(1-\mu a)\tau\sum_{n=1}^m\left\langle\nabla(Z_i^n-Z_i^{n-1}),\nabla U_i^n\right\rangle+a\tau\sum_{n=0}^{m-1}\|\nabla Z_i^n\|_{L^2(\Omega)}^2\\ &\quad+\tau\sum_{n=0}^{m-1}\langle\nabla\phi_i(\boldsymbol{Z}^n),\nabla Z_i^n\rangle+\tau\sum_{n=1}^m\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla(Z_i^n-Z_i^{n-1})\right\rangle-\mu\tau\sum_{n=1}^m\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla U_i^n\right\rangle\\ &\quad+\frac{1}{2\mu}(\varepsilon-\tau)\left(\|\nabla V_i^m\|_{L^2(\Omega)}^2-\|\nabla V_i^0\|_{L^2(\Omega)}^2+\sum_{n=1}^m\|\nabla(V_i^n-V_i^{n-1})\|_{L^2(\Omega)}^2\right). \qquad (4.5)\end{aligned}$$

Note that $\mu a<1$ and $U_i^n\in H^3(\Omega)$. In view of (4.1) and (3.6), we can estimate as follows:

$$\begin{aligned}-(1-\mu a)\tau\sum_{n=1}^m\left\langle\nabla(Z_i^n-Z_i^{n-1}),\nabla U_i^n\right\rangle &= (1-\mu a)\tau^2\sum_{n=1}^m\left(\|\Delta U_i^n\|_{L^2(\Omega)}^2+\left\langle f_i(\boldsymbol{Z}^{n-1}),\Delta U_i^n\right\rangle\right)\\ &\ge\frac{1-\mu a}{2}\tau^2\sum_{n=1}^m\|\Delta U_i^n\|_{L^2(\Omega)}^2-\frac{1-\mu a}{2}\tau^2\sum_{n=0}^{m-1}\|f_i(\boldsymbol{Z}^n)\|_{L^2(\Omega)}^2,\end{aligned}$$

$$\begin{aligned}\left|\tau\sum_{n=1}^m\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla(Z_i^n-Z_i^{n-1})\right\rangle\right| &\le\frac{2\mu}{3(1-\mu a)}\tau\sum_{n=0}^{m-1}\|\nabla\phi_i(\boldsymbol{Z}^n)\|_{L^2(\Omega)}^2+\frac{3(1-\mu a)}{8\mu}\tau\sum_{n=1}^m\left\|\nabla(Z_i^n-Z_i^{n-1})\right\|_{L^2(\Omega)}^2\\ &\le\frac{2\mu ML^2}{3(1-\mu a)}\tau\sum_{n=0}^{m-1}\sum_{j=1}^M\left\|\nabla Z_j^n\right\|_{L^2(\Omega)}^2+\frac{3(1-\mu a)}{8\mu}\tau\sum_{n=1}^m\left\|\nabla(Z_i^n-Z_i^{n-1})\right\|_{L^2(\Omega)}^2,\end{aligned}$$

$$\left|\mu\tau\sum_{n=1}^m\left\langle\nabla\phi_i(\boldsymbol{Z}^{n-1}),\nabla U_i^n\right\rangle\right|\le\frac{\mu ML^2}{3}\tau\sum_{n=0}^{m-1}\sum_{j=1}^M\left\|\nabla Z_j^n\right\|_{L^2(\Omega)}^2+\frac{3\mu}{4}\tau\sum_{n=1}^m\|\nabla U_i^n\|_{L^2(\Omega)}^2.$$

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Collecting all the previous bounds and summing over i from 1 to M, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{Z}^m\|_{L^2(\Omega)^M}^2+\frac{1}{2}\sum_{n=1}^m\|\boldsymbol{Z}^n-\boldsymbol{Z}^{n-1}\|_{L^2(\Omega)^M}^2+\frac{\mu}{4(1+\mu a)}\tau\sum_{n=1}^m\|\nabla\boldsymbol{U}^n\|_{L^2(\Omega)^M}^2+\frac{1}{2\mu}\tau\|\nabla\boldsymbol{Z}^m\|_{L^2(\Omega)^M}^2\\
&\quad+\frac{1-\mu a}{8\mu(1+\mu a)}\tau\sum_{n=1}^m\|\nabla(\boldsymbol{Z}^n-\boldsymbol{Z}^{n-1})\|_{L^2(\Omega)^M}^2+\frac{1-\mu a}{2(1+\mu a)}\tau^2\sum_{n=1}^m\|\Delta\boldsymbol{U}^n\|_{L^2(\Omega)^M}^2\\
&\quad+\frac{1}{1+\mu a}\left(a-\frac{\mu(3-\mu a)M^2L^2}{3(1-\mu a)}\right)\tau\sum_{n=0}^{m-1}\|\nabla\boldsymbol{Z}^n\|_{L^2(\Omega)^M}^2+\frac{1}{1+\mu a}\tau\sum_{n=0}^{m-1}\sum_{i=1}^M\langle\nabla\phi_i(\boldsymbol{Z}^n),\nabla Z_i^n\rangle\\
&\quad+\frac{1}{2\mu(1+\mu a)}(\varepsilon-\tau)\left(\|\nabla\boldsymbol{V}^m\|_{L^2(\Omega)^M}^2+\sum_{n=1}^m\|\nabla(\boldsymbol{V}^n-\boldsymbol{V}^{n-1})\|_{L^2(\Omega)^M}^2\right)\\
&\le C\left(1+\tau\sum_{n=0}^m\|\boldsymbol{Z}^n\|_{L^2(\Omega)^M}^2+\|\boldsymbol{Z}^0\|_{L^2(\Omega)}^2+\frac{1}{2\mu}\tau\|\nabla\boldsymbol{Z}^0\|_{L^2(\Omega)^M}^2+(\varepsilon-\tau)\|\nabla\boldsymbol{V}^0\|_{L^2(\Omega)^M}^2\right).
\end{aligned}$$

By virtue of (H2) and (H4), the seventh and eighth terms are positive. We deduce from (H6) and the discrete Gronwall inequality that

$$\begin{aligned}
&\max_{0\le n\le N_T}\|\boldsymbol{Z}^n\|_{L^2(\Omega)^M}^2+\tau\sum_{n=1}^{N_T}\|\boldsymbol{Z}^n\|_{H^1(\Omega)^M}^2+\tau^2\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{Z}^n\|_{L^2(\Omega)^M}^2+\tau^3\sum_{n=1}^{N_T}\|\bar{D}_\tau\boldsymbol{Z}^n\|_{H^1(\Omega)^M}^2\\
&\quad+\tau\sum_{n=1}^{N_T}\|\nabla\boldsymbol{U}^n\|_{L^2(\Omega)^M}^2+\tau^2\sum_{n=1}^{N_T}\|\Delta\boldsymbol{U}^n\|_{L^2(\Omega)^M}^2+\tau\sum_{n=1}^{N_T}\|\nabla\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2\\
&\quad+(\varepsilon-\tau)\max_{0\le n\le N_T}\|\nabla\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2+(\varepsilon-\tau)\tau^2\sum_{n=1}^{N_T}\|\nabla\bar{D}_\tau\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2\le C.
\end{aligned}\tag{4.6}$$

Multiplying (2.2) by $2\varepsilon\tau V_i^n$, we have

$$\begin{aligned}
&\varepsilon\left(\|V_i^n\|_{L^2(\Omega)}^2-\|V_i^{n-1}\|_{L^2(\Omega)}^2+\|V_i^n-V_i^{n-1}\|_{L^2(\Omega)}^2\right)\\
&\quad=2\mu\tau\left\langle U_i^{n-1},V_i^{n-1}\right\rangle+2\mu\tau\left\langle U_i^{n-1},V_i^n-V_i^{n-1}\right\rangle-2\mu\tau\left\langle\beta_i(\boldsymbol{Z}^{n-1}),V_i^n\right\rangle\\
&\quad=\tau\|Z_i^{n-1}\|_{L^2(\Omega)}^2-\mu^2\tau\|U_i^{n-1}\|_{L^2(\Omega)}^2-\tau\|V_i^{n-1}\|_{L^2(\Omega)}^2+2\mu\tau\left\langle U_i^{n-1},V_i^n-V_i^{n-1}\right\rangle-2\mu\tau\left\langle\beta_i(\boldsymbol{Z}^{n-1}),V_i^n\right\rangle.
\end{aligned}$$

Sum over n from 1 to $m\le N_T$ and apply (3.6) to get

$$\begin{aligned}
&\frac{\varepsilon}{2}\|V_i^m\|_{L^2(\Omega)}^2+(\varepsilon-\tau)\sum_{n=1}^m\left\|V_i^n-V_i^{n-1}\right\|_{L^2(\Omega)}^2+\tau\sum_{n=0}^{m-1}\|V_i^n\|_{L^2(\Omega)}^2\\
&\qquad\le\varepsilon\|V_i^0\|_{L^2(\Omega)}^2+\tau\sum_{n=0}^{m-1}\|Z_i^n\|_{L^2(\Omega)}^2+2\mu^2\tau\sum_{n=0}^{m-1}\|\beta_i(\boldsymbol{Z}^n)\|_{L^2(\Omega)}^2.
\end{aligned}$$

The assumption (H6) and the estimate (4.6) imply

$$\varepsilon\max_{0\le n\le N_T}\|\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2+\tau\sum_{n=1}^{N_T}\|\boldsymbol{V}^n\|_{L^2(\Omega)^M}^2+\tau\sum_{n=1}^{N_T}\|\boldsymbol{U}^n\|_{L^2(\Omega)^M}^2\le C.$$

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Multiplying (2.2) by $\tau\bar{D}_\tau V_i^n$ and summing the result over all $n \in \{1, 2, \dots, N_T\}$ yield

$$\tau \sum_{n=1}^{N_T} \left\| \bar{D}_\tau V_i^n \right\|_{L^2(\Omega)}^2 \le \frac{2\mu^2}{\varepsilon^2} \tau \sum_{n=0}^{N_T-1} \left(\|U_i^n\|_{L^2(\Omega)}^2 + \|\beta_i(\boldsymbol{Z}^n)\|_{L^2(\Omega)}^2 \right).$$

Therefore, we obtain

$$\varepsilon^2 \tau \sum_{n=1}^{N_T} \left\| \bar{D}_\tau \boldsymbol{V}^n \right\|_{L^2(\Omega)^M}^2 \le C.$$

We deduce from (4.1) that

$$\tau \sum_{n=1}^{N_T} \left| \langle \bar{D}_\tau Z_i^n, \varphi \rangle \right|^2 \le 2\tau \sum_{n=1}^{N_T} \|\nabla U_i^n\|_{L^2(\Omega)}^2 \|\nabla \varphi\|_{L^2(\Omega)}^2 + 2\tau \sum_{n=1}^{N_T} \left\| f_i(\boldsymbol{Z}^{n-1}) \right\|_{L^2(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 \le C \|\varphi\|_{H^1(\Omega)}^2$$

for all $\varphi \in H^1(\Omega)$, which implies

$$\tau \sum_{n=1}^{N_T} \left\| \bar{D}_\tau \boldsymbol{Z}^n \right\|_{(H^1(\Omega)^*)^M}^2 \le C.$$

Thus, the proof is complete. □

We denote by $\widehat{\boldsymbol{Z}}^{(\varepsilon,\tau)}$ the function obtained by linear interpolation in time of $\{\boldsymbol{Z}^n\}$, and by $\boldsymbol{Z}^{(\varepsilon,\tau)}$ the piecewise constant function in time; namely,

$$\widehat{Z}_i^{(\varepsilon,\tau)}(\cdot, t) = Z_i^{n-1} + \frac{t - (n-1)\tau}{\tau} (Z_i^n - Z_i^{n-1})$$

for $t \in ((n-1)\tau, n\tau]$, $n = 1, 2, \dots, N_T$ and $i = 1, 2, \dots, M$,

$$Z_i^{(\varepsilon,\tau)}(\cdot, t) = Z_i^n$$

for $t \in ((n-1)\tau, n\tau]$, $n = 0, 1, 2, \dots, N_T$ and $i = 1, 2, \dots, M$. Functions $\widehat{\boldsymbol{U}}^{(\varepsilon,\tau)}$, $\widehat{\boldsymbol{V}}^{(\varepsilon,\tau)}$, $\boldsymbol{U}^{(\varepsilon,\tau)}$ and $\boldsymbol{V}^{(\varepsilon,\tau)}$ are similarly defined. Then, (2.1), (2.2) and (4.1) can be rewritten in the form

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial \widehat{U}_i^{(\varepsilon,\tau)}}{\partial t}(t), \varphi(t) \right\rangle \mathrm{d}t + \frac{1}{\mu} \int_0^T \left\langle \nabla U_i^{(\varepsilon,\tau)}(t), \nabla \varphi(t) \right\rangle \mathrm{d}t \\ &\qquad = \frac{1}{\mu} \int_0^T \left\langle f_i(\boldsymbol{Z}^{(\varepsilon,\tau)}(t-\tau)), \varphi(t) \right\rangle \mathrm{d}t - \frac{1}{\varepsilon} \int_0^T \left\langle U_i^{(\varepsilon,\tau)}(t-\tau) - \beta_i(\boldsymbol{Z}_i^{(\varepsilon,\tau)}(t-\tau)), \varphi(t) \right\rangle \mathrm{d}t, \end{aligned} \tag{4.7}$$

$$\int_0^T \left\langle \frac{\partial \widehat{V}_i^{(\varepsilon,\tau)}}{\partial t}(t), \psi(t) \right\rangle \mathrm{d}t = \frac{\mu}{\varepsilon} \int_0^T \left\langle U_i^{(\varepsilon,\tau)}(t-\tau) - \beta_i(\boldsymbol{Z}_i^{(\varepsilon,\tau)}(t-\tau)), \psi(t) \right\rangle \mathrm{d}t, \tag{4.8}$$

$$\int_0^T \left\langle \frac{\partial \widehat{Z}_i^{(\varepsilon,\tau)}}{\partial t}(t), \varphi(t) \right\rangle \mathrm{d}t + \int_0^T \left\langle \nabla U_i^{(\varepsilon,\tau)}(t), \nabla \varphi(t) \right\rangle \mathrm{d}t = \int_0^T \left\langle f_i(\boldsymbol{Z}^{(\varepsilon,\tau)}(t-\tau)), \varphi(t) \right\rangle \mathrm{d}t \tag{4.9}$$

for all $\varphi \in L^2(0, T; H^1(\Omega))$ and $\psi \in L^2(Q)$, $i = 1, 2, \dots, M$.

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Proof of Theorem 2.3. The proof is analogous to the proof of Theorem 3.2. Lemma 4.1 implies

$$\left\|\widehat{\boldsymbol{Z}}^{(\varepsilon,\tau)}\right\|_{(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)^*))^M} + \left\|\boldsymbol{Z}^{(\varepsilon,\tau)}\right\|_{(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)))^M} + \left\|\boldsymbol{U}^{(\varepsilon,\tau)}\right\|_{L^2(0,T;H^1(\Omega))^M} + \left\|\boldsymbol{V}^{(\varepsilon,\tau)}\right\|_{L^2(0,T;H^1(\Omega))^M} \le C.$$

Here, C is independent of ε and τ. Therefore, there exist subsequences $\{\widehat{\boldsymbol{Z}}^{(\varepsilon_k,\tau_k)}\}$, $\{\boldsymbol{Z}^{(\varepsilon_k,\tau_k)}\}$ and $\{\boldsymbol{U}^{(\varepsilon_k,\tau_k)}\}$ of $\{\widehat{\boldsymbol{Z}}^{(\varepsilon,\tau)}\}$, $\{\boldsymbol{Z}^{(\varepsilon,\tau)}\}$ and $\{\boldsymbol{U}^{(\varepsilon,\tau)}\}$, respectively, and functions $\boldsymbol{z}^*$, $\boldsymbol{u}^*$ such that

$$\begin{aligned}&\boldsymbol{z}^* \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*))^M,\\ &\boldsymbol{u}^* \in L^2(0,T;H^1(\Omega))^M,\end{aligned}$$

$$\begin{aligned}&\widehat{\boldsymbol{Z}}^{(\varepsilon,\tau)} \to \boldsymbol{z}^* && \text{strongly in } L^2(Q)^M, \text{ a.e. in } Q,\\ &&& \text{and weakly in } L^2(0,T;H^1(\Omega))^M \text{ and } H^1(0,T;H^1(\Omega)^*)^M,\\ &\boldsymbol{Z}^{(\varepsilon,\tau)}, \boldsymbol{Z}^{(\varepsilon,\tau)}(\cdot,\cdot-\tau) \to \boldsymbol{z}^* && \text{strongly in } L^2(Q)^M, \text{ a.e. in } Q, \text{and weakly in } L^2(0,T;H^1(\Omega))^M,\\ &\boldsymbol{U}^{(\varepsilon,\tau)}, \boldsymbol{U}^{(\varepsilon,\tau)}(\cdot,\cdot-\tau) \rightharpoonup \boldsymbol{u}^* && \text{weakly in } L^2(0,T;H^1(\Omega))^M\end{aligned}$$

as $\varepsilon_k, \tau_k \to 0$. Noticing (4.9) and using the same strategy as in the proof of Theorem 3.2, we find that $\boldsymbol{u}^* = \boldsymbol{\beta}(\boldsymbol{z}^*)$ and $\boldsymbol{z}^*$ is a weak solution of (1.1). □

Remark 4.2. It follows from Lemma 4.1 that for each ε, there exists a positive constant C which does not depend on sufficiently small τ such that

$$\left\|\widehat{\boldsymbol{U}}^{(\varepsilon,\tau)}\right\|_{(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)^*))^M} + \left\|\boldsymbol{U}^{(\varepsilon,\tau)}\right\|_{(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)))^M} + \left\|\widehat{\boldsymbol{V}}^{(\varepsilon,\tau)}\right\|_{(L^\infty(0,T;H^1(\Omega))\cap H^1(Q))^M} + \left\|\boldsymbol{V}^{(\varepsilon,\tau)}\right\|_{(L^\infty(0,T;H^1(\Omega)))^M} \le C.$$

Hence, passing to the limit in τ along subsequences in (4.7) and (4.8), we can prove the existence of the weak solution of (1.9).

5. Numerical experiments

In this section, we give some numerical results in one and two space dimensions to demonstrate the effectiveness of our scheme.

5.1. 1D experiments

We deal with a problem treated numerically by Galiano *et al.* [7] and by Barrett and Blowey [1]. To this end, we introduce the following cross-diffusion-convection system.

$$\begin{cases}\dfrac{\partial z_i}{\partial t} = \Delta\beta_i(\boldsymbol{z}) + d_i \mathrm{div}(z_i \nabla p) & \text{in} \quad Q,\\ \dfrac{\partial \beta_i(\boldsymbol{z})}{\partial \nu} + d_i z_i \dfrac{\partial p}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T),\\ z_i(\cdot,0) = z_{0i} & \text{in} \quad \Omega\end{cases} \tag{5.1}$$

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for $i = 1, 2, \dots, M$, where d_i are non-negative constants and p is a given function. The following linear scheme is proposed to approximate the solution of (5.1). Set $Z_i^0 = \mu u_{0i}^{\varepsilon,\tau} + v_{0i}^{\varepsilon,\tau}$ and calculate

$$\begin{cases} U_i^n - \dfrac{\tau}{\mu}\Delta U_i^n - d_i\tau \mathrm{div}\,(U_i^n \nabla p) = \beta_i(\boldsymbol{Z}^{n-1}) + \dfrac{d_i\tau}{\mu}\mathrm{div}\,\big((Z_i^{n-1} - \mu\beta_i(\boldsymbol{Z}^{n-1}))\nabla p\big) & \text{in} \quad \Omega, \\ Z_i^n = Z_i^{n-1} + \mu(U_i^n - \beta_i(\boldsymbol{Z}^{n-1})) & \text{in} \quad \Omega, \\ \dfrac{\partial U_i^n}{\partial \nu} + d_i Z_i^n \dfrac{\partial p}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.2}$$

for $i = 1, 2, \dots, M$, $n = 1, 2, \dots, N_T$. The convection terms $\mathrm{div}(z_i \nabla p)$ are calculated by $\mathrm{div}(Z_i^n \nabla p) = \mathrm{div}((Z_i^{n-1} + \mu(U_i^n - \beta_i(\boldsymbol{Z}^{n-1})))\nabla p)$. We can also choose $\mathrm{div}(Z_i^{n-1}\nabla p)$ as the time discretization of the convection terms. As a consequence of our analyses, we obtain convergence results for (5.2) as well as the corresponding reaction-diffusion-convection system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \dfrac{1}{\mu}\Delta u_i - \dfrac{1}{\varepsilon}(u_i - \beta_i(\mu\boldsymbol{u} + \boldsymbol{v})) + \dfrac{d_i}{\mu}\mathrm{div}((\mu u_i + v_i)\nabla p) & \text{in} \quad Q, \\ \dfrac{\partial v_i}{\partial t} = \dfrac{\mu}{\varepsilon}(u_i - \beta_i(\mu\boldsymbol{u} + \boldsymbol{v})) & \text{in} \quad Q, \\ \dfrac{\partial u_i}{\partial \nu} + d_i(\mu u_i + v_i)\dfrac{\partial p}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_i(\cdot, 0) = u_{0i}^{\varepsilon}, \quad v_i(\cdot, 0) = v_{0i}^{\varepsilon} & \text{in} \quad \Omega \end{cases} \tag{5.3}$$

for $i = 1, 2, \dots, M$.

Corollary 5.1. *Assume that* (H1), (H2)′, (H4)–(H6) *and* (H6)′ *are satisfied. Suppose* $p \in W^{2,\infty}(\Omega)$. *Let* $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ *be the weak solution of* (5.3) *and* $\{\boldsymbol{Z}^n,\ \boldsymbol{U}^n\}_{n=0}^{N_T}$ *be that of* (5.2). *We denote by* $\boldsymbol{Z}^{(\tau)}$ *and* $\boldsymbol{U}^{(\tau)}$ *the functions obtained by piecewise constant interpolation in time of* $\{\boldsymbol{Z}^n\}$ *and* $\{\boldsymbol{U}^n\}$, *respectively. Then, there exist subsequences* $\{\boldsymbol{u}^{\varepsilon_k}\}$, $\{\boldsymbol{v}^{\varepsilon_k}\}$, $\{\boldsymbol{Z}^{(\tau_k)}\}, \{\boldsymbol{U}^{(\tau_k)}\}$ *of* $\{\boldsymbol{u}^\varepsilon\}$, $\{\boldsymbol{v}^\varepsilon\}$, $\{\boldsymbol{Z}^{(\tau)}\}, \{\boldsymbol{U}^{(\tau)}\}$, *respectively, and a weak solution* $\boldsymbol{z}$ *of* (5.1) *such that*

$$\begin{array}{ll} \mu\boldsymbol{u}^{\varepsilon_k} + \boldsymbol{v}^{\varepsilon_k} \to \boldsymbol{z} & \textit{strongly in } L^2(Q)^M, \ \textit{a.e. in } Q, \\ & \textit{and weakly in } L^2(0,T; H^1(\Omega))^M \textit{ and } H^1(0,T; H^1(\Omega)^*)^M, \\ \boldsymbol{u}^{\varepsilon_k} \rightharpoonup \boldsymbol{\beta}(\boldsymbol{z}) & \textit{weakly in } L^2(0,T; H^1(\Omega))^M, \\ \boldsymbol{Z}^{(\tau_k)} \to \boldsymbol{z} & \textit{strongly in } L^2(Q)^M, \ \textit{a.e. in } Q \textit{ and weakly in } L^2(0,T; H^1(\Omega))^M, \\ \boldsymbol{U}^{(\tau_k)} \rightharpoonup \boldsymbol{\beta}(\boldsymbol{z}) & \textit{weakly in } L^2(0,T; H^1(\Omega))^M \end{array}$$

as $\varepsilon_k, \tau_k \to 0$.

One-dimensional numerical experiments are carried out in a finite domain $\Omega = (0, L) = (0, 3)$ and a time interval $(0, T] = (0, 10]$. We employ the finite-difference method to discretize (5.2) in space. The spatial mesh size is denoted by $h = L/N_X$, where $N_X + 1$ is the number of mesh points. Let $Z_i^{j,n}$ be the numerical approximation of $z_i(jh, n\tau)$. For given $\{Z_i^{j,n-1}\}_{i=1,\dots,M,\ j=0,\dots,N_X}$ $(n = 1, \dots, N_T)$, solve the following linear

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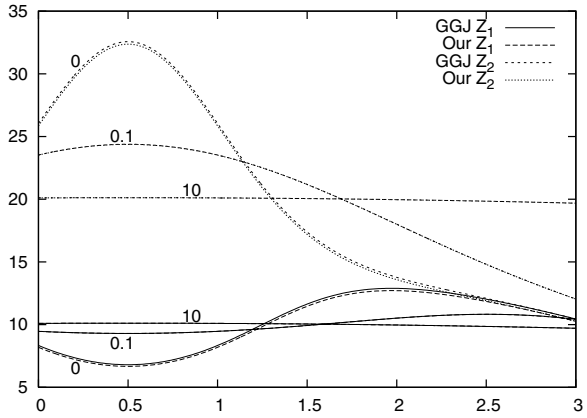


FIGURE 1. Numerical solutions of (5.1) corresponding to case A. Different curves are labelled with the corresponding b_i values.

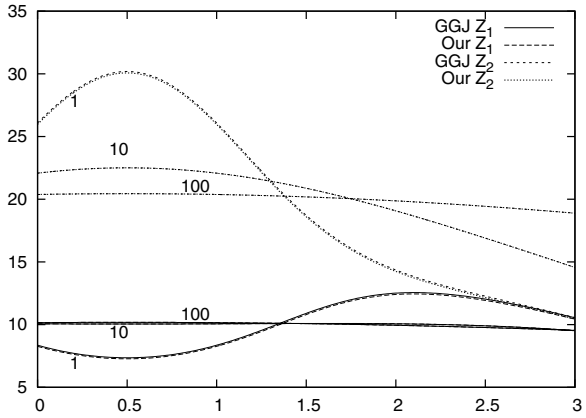


FIGURE 2. Numerical solutions of (5.1) corresponding to case B. Different curves are labelled with the corresponding a_i values.

system to find $\{U_i^{j,n}\}_{i=1,\dots,M,\ j=0,\dots,N_X}$:

$$
\begin{aligned}
&U_i^{j,n} - \frac{\tau}{\mu h^2}(U_i^{j+1,n} - 2U_i^{j,n} + U_i^{j-1,n}) - \frac{d_i\tau}{2h}(U_i^{j+1,n} - U_i^{j-1,n})p_x(jh) - d_i\tau U_i^{j,n}p_{xx}(jh)\\
&\quad = \beta_i(\boldsymbol{Z}^{j,n-1}) + \frac{d_i\tau}{2\mu h}\left(Z_i^{j+1,n-1} - Z_i^{j-1,n-1} - \mu\beta_i(\boldsymbol{Z}^{j+1,n-1}) + \mu\beta_i(\boldsymbol{Z}^{j-1,n-1})\right)p_x(jh)\\
&\qquad + \frac{d_i\tau}{\mu}\left(Z_i^{j,n-1} - \mu\beta_i(\boldsymbol{Z}^{j,n-1})\right)p_{xx}(jh),\\
&\frac{U_i^{-1,n} - U_i^{1,n}}{2h} - d_i(Z_i^{0,n-1} - \mu(U_i^{0,n} - \beta_i(\boldsymbol{Z}^{0,n-1})))p_x(0) = 0,\\
&\frac{U_i^{N_X+1,n} - U_i^{N_X-1,n}}{2h} + d_i(Z_i^{N_X,n-1} - \mu(U_i^{N_X,n} - \beta_i(\boldsymbol{Z}^{N_X,n-1})))p_x(L) = 0.
\end{aligned}
$$

Thereafter, compute $\{Z_i^{j,n}\}_{i=1,\dots,M,\ j=0,\dots,N_X}$ by

$$Z_i^{j,n} = Z_i^{j,n-1} + \mu(U_i^{j,n} - \beta_i(\boldsymbol{Z}^{j,n-1})).$$

We repeat the experiments by Galiano *et al.* [7]. In each experiment, we set $M = 2$, $\beta_i(\boldsymbol{z}) = (a_i + b_i z_i + c_i z_j)z_i$ for $(i,j) \in \{(1,2),(2,1)\}$, $p(x) = 1.5(x-0.5)^2$, $Z_1^{\cdot,0} \equiv 10$, $Z_2^{\cdot,0} \equiv 20$. We take $h = 10^{-2}$, $\tau = 10^{-3}$ and $\mu = 10^{-3}$. Numerical solutions by our method (denoted by Our) are compared with those by the method by Galiano *et al.* [7] (denoted by GGJ). We introduced the mind of GGJ in Section 1, but see [7] for the details.

We carry out numerical simulations in the following cases:

- A: Large and small cross-diffusion terms compared to self-diffusion terms. The coefficients $a_i = c_i = d_i = 1$ are fixed. Figure 1 shows the numerical solutions corresponding to $b_i = 0, 0.1, 10$.
- B: Large diffusion coefficients a_i compared to b_i. The coefficients $b_i = 0.01$, $c_i = d_i = 1$ are fixed. Figure 2 shows the numerical solutions corresponding to $a_i = 1, 10, 100$.

The numerical solutions by our method and by GGJ agree well (see also [1]). Figures 3 and 4 show close up view of Figure 1 near $(x,z) = (2, 12.8)$ and $(x,z) = (0.5, 32.5)$, respectively. A fine grid solution with $h = 10^{-3}$ and $\tau = 10^{-4}$ by GGJ is also drawn in order to compare our method with GGJ in the accuracy. Our numerical solution shows excellent agreement with the fine grid solution. We carried out the experiments in

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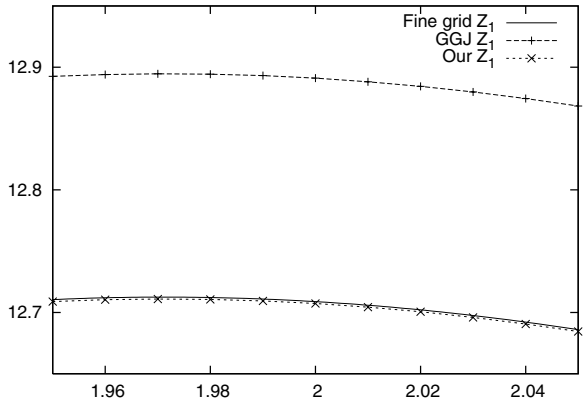


FIGURE 3. Close-up view of Figure 1 with a fine grid solution.

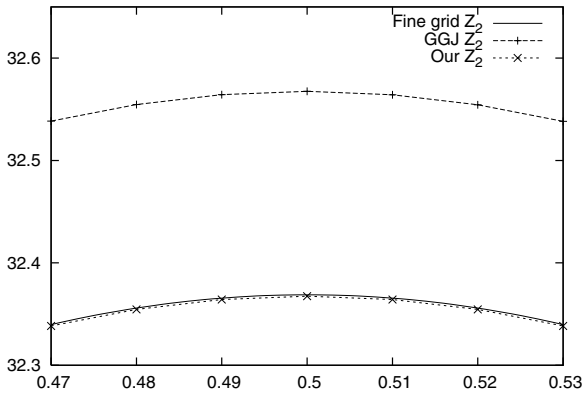


FIGURE 4. Close-up view of Figure 1 with a fine grid solution.

other cases treated in [1,7]. In all of those results, we observed that our numerical solutions agree well with fine grid solutions by GGJ. The numerical evidence demonstrates the effectiveness of our method. Furthermore, our method requires less CPU time than GGJ because the matrix dose not change in our method, *i.e.*, we can take advantage of the LU decomposition to solve the linear systems.

5.2. **2D simulations**

The advantages of our method are its simplicity and versatility. We can easily carry out two and three dimensional simulations for various type of nonlinearities. We present two dimensional simulations. In each simulation, we take the mesh sizes $h_x = h_y = 1/500$, the time step size $\tau = 10^{-4}$ and the parameter $\mu = 0.2$. The following problem is considered.

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = \Delta[(0.04 + 0.04\alpha z_2)z_1] + (2.8 - 1.1z_1 - z_2)\, z_1 & \text{in} \quad Q = (0,1)^2 \times (0,200], \\ \dfrac{\partial z_2}{\partial t} = \Delta[(0.04 + 2\alpha z_1)z_2] + (3.0 - 1.1z_2 - z_1)\, z_2 & \text{in} \quad Q \end{cases} \tag{5.4}$$

with the homogeneous Neumann boundary conditions. If the cross-diffusion terms are not, that is, $\alpha = 0$, the constant steady state $(z_1, z_2) \equiv (8/21, 50/21)$ is stable. The stationary solution perturbed with 1% random noise is used for the initial datum. Figure 5 presents the numerical result with $\alpha = 1$. The gray and the transparent light gray surfaces denote the numerical solutions Z_1 and Z_2, respectively. The stationary solution becomes unstable and a spatial pattern appears by the cross-diffusion effect.

We simulate the following cooperative system:

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = \Delta\left[\left(\dfrac{1}{0.2 + 0.02z_2^2}\right) z_1\right] + (4 - z_1)z_1 & \text{in} \quad Q = (0,1)^2 \times (0,50], \\ \dfrac{\partial z_2}{\partial t} = 0.005\Delta z_2 + (0.5 + z_1 - 0.5z_2)z_2 & \text{in} \quad Q \end{cases} \tag{5.5}$$

with the homogeneous Neumann boundary conditions. The constant steady state $(z_1, z_2) \equiv (4, 9)$ is stable in this problem if the cross-diffusion term is not. However, the stationary solution becomes unstable and aggregation occurs by the cross-diffusion effect (Fig. 6). Thus, we can simulate such complicated phenomena very easily by using the proposed scheme.

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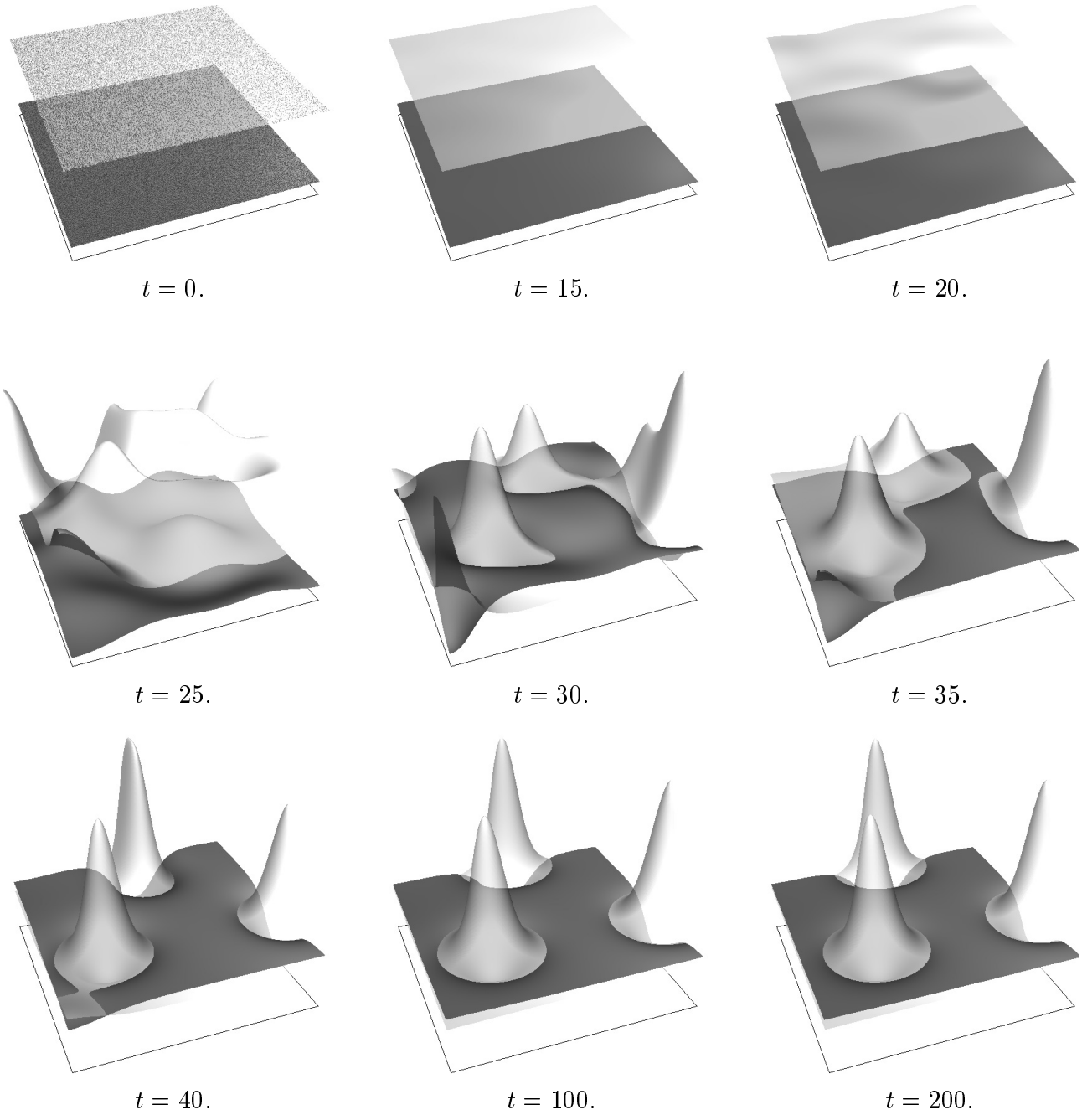


FIGURE 5. Numerical solutions Z_1 (gray) and Z_2 (light gray) of (5.4).

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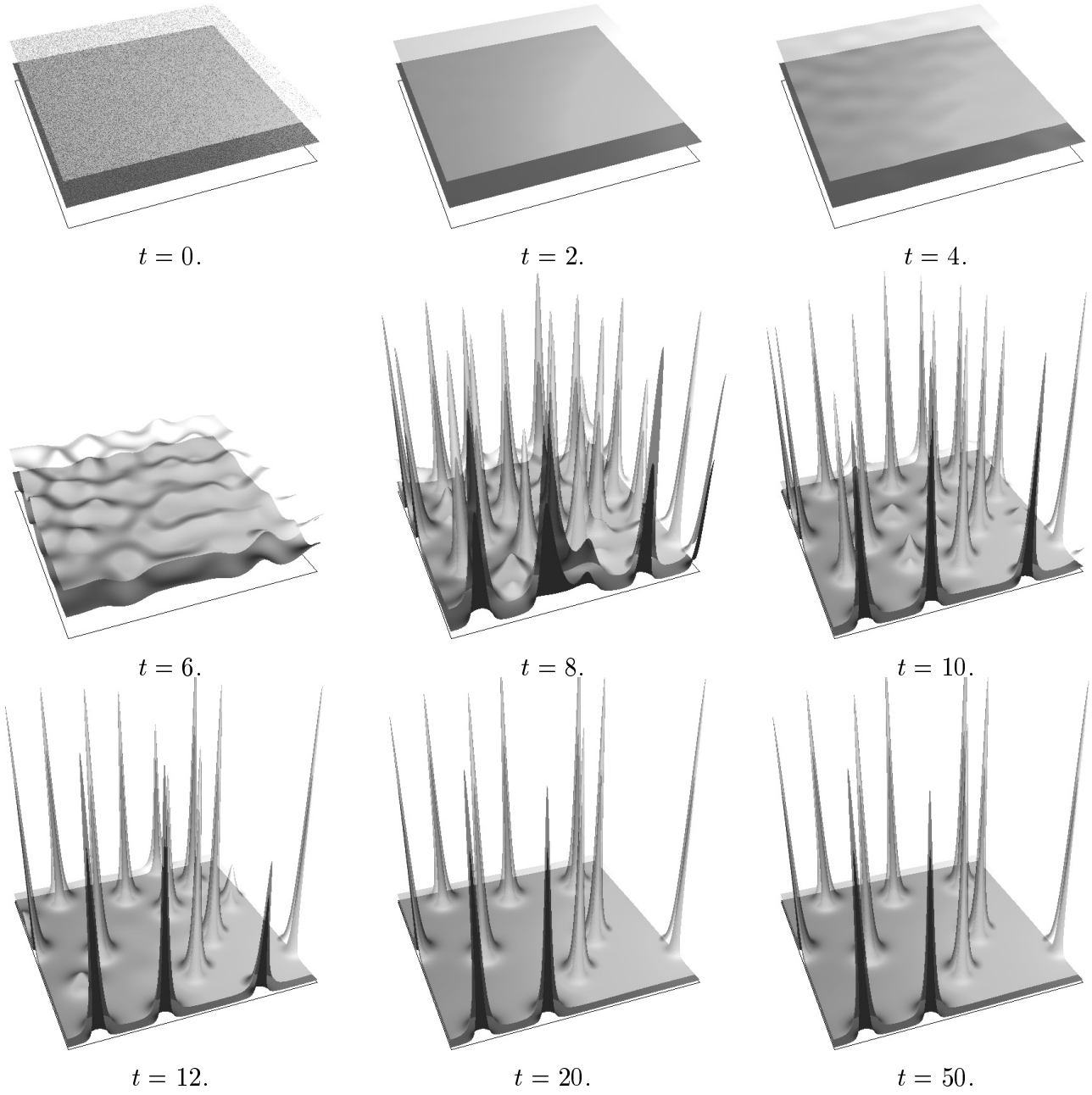


FIGURE 6. Numerical solutions Z_1 (gray) and Z_2 (light gray) of (5.5).

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