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SOME RESULTS ON REVERSED HAZARD RATE

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In view of the growing importance of reversed hazard rate (RHR) in reliability analysis and stochastic modeling, we have considered different implicative relationships with respect to the monotonic behavior of RHR. In that context, a few characterizing properties have also been presented based on expected inactivity time.

1. INTRODUCTION

In reliability analysis, the hazard rate, also known as failure rate, of a life distribution plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distribution and exhibits different monotonic behaviors. The concept of reversed hazard rate (RHR) was initially introduced as the hazard rate in the negative direction and received a cold reception in the literature at the early stage. This was because RHR, being the ratio of probability density function and the corresponding distribution function, was conceived as a dual measure of hazard rate.

Keilson and Sumita [3] were among the first to define RHR and called it the "dual failure function." According to them, hazard rate ordering is the uniform

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stochastic ordering and the RHR ordering is the uniform stochastic ordering in the negative direction. In fact, a variety of simple stochastic settings can be constructed along the line of Keilson and Sumita [3] to demonstrate the importance of RHR orderings.

Shaked and Shanthikumar [9] also presented some nice results relating to RHR ordering. What is more important is the inclusion of some interesting characterizations based on the monotonicity of RHR. This added dimension to the field drew the attention of Block et al. [1], who pointed out that if the RHR is increasing on its interval of support, then that interval must have a finite upper point. Further, they observed that there is no nonnegative random variable having an increasing reversed hazard rate (IRHR) distribution. Increasing failure rate (IFR) distributions like Weibull, gamma, and lognormal distributions were found to be decreasing reversed hazard rate (DRHR) distributions. Block et al. [1] also observed that a decreasing failure rate (DFR) distribution is necessarily DRHR distribution. Sengupta and Nanda [8] presented some related results interconnecting RHR with log-concavity.

In view of the growing importance of the concept of RHR, we made an attempt to examine this subject from several directions. In Section 2, we examined some results on implicative relationship in the context of the monotonic behavior of RHR. A few characterization results have also been presented.

2. RESULTS ON MONOTONIC BEHAVIOR

Let us consider a random variable X with an absolutely continuous cumulative distribution function $F(x)$, probability density function $f(x)$, and survival function $S(x) = 1 - F(x)$. Let $a = \inf\{t: F(t) > 0\}$ and $g = \sup\{t: F(t) < 1\}$. We call (a, b), $-\infty \leq a < b < \infty$, the interval of support of F (or X). Then, the RHR of X at t $(t > a)$ is denoted by $a(t)$ and is defined as

$$a(t) = \frac{d}{dt} \log F(t) = \frac{f(t)}{F(t)}. \tag{2.1}$$

The corresponding hazard rate (HR) of X at t $(t < b)$ is denoted by $r(t)$ and is defined as

$$r(t) = -\frac{d}{dt} \log S(t) = \frac{f(t)}{S(t)}. \tag{2.2}$$

Definition 2.1 (Block et al. [1]): *X is said to have increasing (decreasing) RHR distribution if $a(t)$ is nondecreasing (nonincreasing) on its interval of support.*

Remark 2.1: It may be noted from Corollary 2.1 of Block et al. [1] that X has RHR $a(t)$ on (a, b), $-\infty \leq a \leq b < \infty$, iff $Y = -X$ has HR $r(t) = a(-t)$ on $(-b, -a)$. Further, writing $S_Y(\cdot)$ as the survival function of Y, we have $S_Y(-x) = F(x)$.

Definition 2.2 (Ruiz and Navarro [7]): *The inactivity time (IT) given that failure occurs on or before the time point t $(t > a)$ is given by*

$$X_t = \{t - X | X \leq t\}. \tag{2.3}$$

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The corresponding distribution function of X_t is denoted by $H_t(x)$ where, by definition,

$$H_t(x) = P[t - X \le x | X \le t] = \{F(t) - F(t-x)\}/F(t). \quad \textbf{(2.4)}$$

The probability density function of X_t then works out as

$$h_t(x) = \frac{f(t-x)}{F(t)}. \quad \textbf{(2.5)}$$

To begin our discussion on the monotonic behavior of RHR and related properties, we first present two basic results on the same.

Result 2.1: A random variable X with distribution function $F(x)$ belongs to the DRHR class if and only if the corresponding X_t is stochastically increasing in t.

Result 2.2: If the RHR of X exists, then $F(x)$ is log-concave if and only if the RHR is decreasing in t.

Thus, we have the implicative relationship shown in Figure 1. To expand Figure 1 further, we consider four additional results on RHR properties.

Result 2.3: X_t is smaller than X_s in the likelihood ratio order (denoted by $X_t \le_{\mathrm{LR}} X_s$) for $t < s$ if and only if $\log f(x)$ is concave.

PROOF: By the definition of the likelihood ratio (LR) ordering given in Ross [5], we have

$$\begin{aligned} X_t \le_{\mathrm{LR}} X_s, \quad t < s &\Leftrightarrow h_s(x)/h_t(x) \uparrow x, && t < s \\ &\Leftrightarrow \frac{f(s-x)/F(s)}{f(t-x)/F(t)} \uparrow x, && t < s \text{ (from (2.5))} \\ &\Leftrightarrow \frac{f(s-x)}{f(t-x)} \uparrow x, && t < s \\ &\Leftrightarrow \frac{f(y+h)}{f(y)} \downarrow y, && h > 0, \end{aligned}$$

where $y = t - x$ and $h = s - t > 0$. Thus, $X_t \le_{\mathrm{LR}} X_s$, $t < s$ is equivalent to log-concavity of $f(x)$. ■

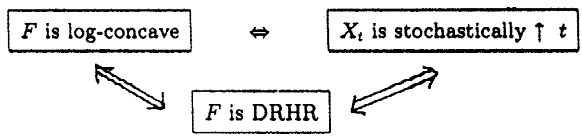


FIGURE 1. Implicative relationship.

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Result 2.3 can be linked up with Figure 1 by noting that the log-concavity of $f(\cdot)$ implies the log-concavity of $F(\cdot)$. However, before we completely enlarge Figure 1, a few other results may be worth pointing out.

Result 2.4: For a random variable X with probability density function $f(x)$, if $f(x)$ is log-concave, then X has both DRHR and IFR distributions.

PROOF: By definition, $f(x)$ is log-concave implies

$$\frac{f(x)}{f(y)} \geq \frac{f(x+c)}{f(y+c)} \quad \forall (x,y) \in (a,b) \text{ where } x > y \text{ and } \forall c > 0 \tag{2.6}$$

$$\Rightarrow \int_y^b \frac{f(x)}{f(y)}\,dx \geq \int_y^b \frac{f(x+c)}{f(y+c)}\,dx \quad \forall y > a,\ \forall c > 0$$

$$\Leftrightarrow \frac{S(y)}{f(y)} \geq \frac{S(y+c)}{f(y+c)} \quad \forall y > a,\ \forall c > 0$$

$$\Leftrightarrow r(y+c) \geq r(y) \quad \forall y > 0,\ \forall c > 0$$

$$\Leftrightarrow X \text{ has IFR distribution.} \tag{2.7}$$

Again, from (2.6), we have

$$\frac{f(y+c)}{f(x+c)} \geq \frac{f(y)}{f(x)} \quad \forall x, y > 0 \text{ where } x > y,\ \forall c > 0$$

$$\Rightarrow \int_0^x \frac{f(y+c)}{f(x+c)}\,dy \geq \int_0^x \frac{f(y)}{f(x)}\,dy \quad \forall x > 0,\ \forall c > 0$$

$$\Leftrightarrow \frac{F(x+c) - F(c)}{f(x+c)} \geq \frac{F(x)}{f(x)} \quad \forall x > 0,\ \forall c > 0$$

$$\Rightarrow a(x) \geq a(x+c) \quad \forall x > 0,\ \forall c > 0$$

$$\Leftrightarrow X \text{ has DRHR distribution.} \tag{2.8}$$

■

Remark 2.2: Since a log-concave probability density function describes a strongly unimodal distribution, we may claim that a strongly unimodal distribution belongs to both IFR and DRHR classes.

Let us denote the expected inactivity time (EIT) by $m(t)$, where

$$m(t) = \frac{1}{F(t)} \int_a^t F(x)\,dx, \tag{2.9}$$

and in terms of $m(t)$, let us present the following results.

Result 2.5: A random variable X has EIT $m(t)$ on (a,b), $-\infty \leq a < b < \infty$, iff $Y = -X$ has mean residual life $\mu_Y(y) = m(-y)$ on $(-b,-a)$.

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Result 2.6: If a random variable X has a DRHR distribution, then $m(t)$ is increasing in t, $t \in (a, b)$.

The proof follows from Result 2.5 and the DMRL implication of IFR property of $Y = -X$.

Now, we are in a position to present the final relational diagram (Figure 2) covering the implicative and equivalent relations presented so far.

It is easy to note that $\{a(t), t \in (a, b)\}$ uniquely determines the underlying distribution function via the relation

$$F(x) = \exp\left[-\int_x^b a(t)\,dt\right]. \tag{2.10}$$

Writing the cumulative RHR of greater than type as

$$A(x) = \int_x^b a(t)\,dt, \tag{2.11}$$

we have

$$F(x) = \exp[-A(x)] \tag{2.12}$$

A similar relation existing between $\{m(t), t \in (a, b)\}$ and $F(x)$ may be seen from the following result.

Result 2.7: For a random variable X, given the EIT sequence $\{m(t), t \in (a, b)\}$, the distribution function is uniquely determined as

$$F(x) = \exp\left[-\int_x^b \frac{1 - m'(t)}{m(t)}\,dt\right], \tag{2.13}$$

where $m'(t) = (d/dt)m(t)$.

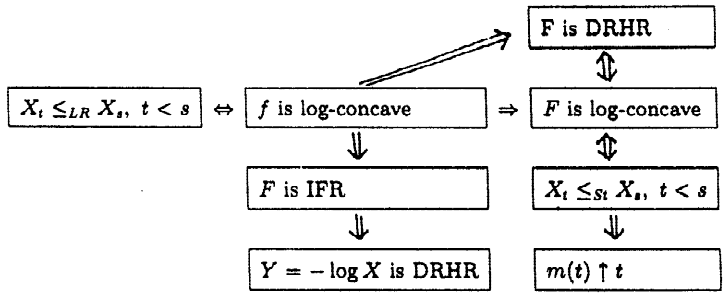


FIGURE 2. Relational diagram showing implicative and equivalent relations.

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The proof follows from Result 2.5 and an application of the following relationship as given in Galambos and Kotz [2]:

$$S_Y(y) = \exp\left[-\int_{-b}^{y} \{(1+\mu'(u))/\mu(u)\}\, du\right]. \tag{2.14}$$

It may be noted from Result 2.7 that

$$f(x) = \exp\left[-\int_{x}^{b} \frac{1-m'(t)}{m(t)}\, dt\right]\left\{\frac{1-m'(x)}{m(x)}\right\}. \tag{2.15}$$

The following characterization also follows from Result 2.7.

Result 2.8: If (a) $m(t)$ is a strictly positive function for positive t with $m'(t) \leq 1$ and (b)

$$\int_{a}^{b} \frac{1-m'(t)}{m(t)}\, dt$$

diverges, then there exists a distribution function $F(t)$ with

$$m(t) = \frac{1}{F(t)} \int_{a}^{t} F(x)\, dx$$

and

$$F(t) = \exp\left[-\int_{t}^{b} \frac{1-m'(x)}{m(x)}\, dx\right].$$

In view of the importance of EIT, let us introduce a few life distribution classes based on the monotonicity of $m(t)$.

Definition 2.3: *A distribution F has increasing expected inactivity time* (*IEIT*) *if* $m(t)$ *is increasing in* $t \in (a,b)$.

Definition 2.4: *A distribution F has decreasing expected inactivity time* (*DEIT*) *if* $m(t)$ *is decreasing in* $t \in (a,b)$.

Definition 2.5: *A distribution F has constant expected inactivity time* (*CEIT*) *if* $m(t)$ *is constant for all* $t \in (a,b)$.

Now, note that $m(t)$ increasing in $t \in (a,b)$ is equivalent to $m'(t) > 0$ for $t \in (a,b)$, which, in turn, is equivalent to $a(t)m(t) < 1$ for $t \in (a,b)$. Thus, for $t \in (a,b)$, $F(t)$ is IEIT is equivalent to $a(t)m(t) < 1$, $F(t)$ is DEIT is equivalent to $a(t)m(t) > 1$, and $F(t)$ is CEIT is equivalent to $a(t)m(t) = 1$.

Motivated by the result of Ruiz and Navarro [6] on the mean residual life (MRL), we state the following result on EIT. Its proof is similar to [6] and is omitted.

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Result 2.9: Let X be a continuous random variable with finite mean μ. $F(x)$ is uniquely determined by $\nu(x) = a(x)m(x)$, $x \in (a, b)$:

$$F(x) = \exp\left[-\int_x^b \frac{\nu(t)}{t - \mu + I(t)}\, dt\right],$$

where

$$I(t) = \int_t^b \nu(x)\, dx.$$

Result 2.10: Let $\nu(t) = c$, a constant, then the following hold:

(i) $c > 1$ if and only if

$$F(x) = \left[\frac{bc - \mu + x(1-c)}{b - \mu}\right]^{C/(1-C)} \quad \text{for } x \in (-\infty, b].$$

(ii) $c < 1$ if and only if

$$F(x) = \left[\frac{bc - \mu + x(1-c)}{b - \mu}\right]^{C/(1-C)} \quad \text{for } x \in \left[\frac{\mu - bc}{1-c}, b\right],$$

which is a finite range distribution.

(iii) $c = 1$ if and only if

$$F(x) = \exp\left[\frac{x - b}{b - \mu}\right] \quad \text{for } x \in (-\infty, b].$$

The proof follows from Result 2.5 and an application of Theorem 4.2 of Mukherjee and Roy [4].

Remark 2.3: It is noted that no nonnegative random variable belongs to DEIT and CEIT classes.

Ruiz and Navarro [6] characterized the distribution function $F(x)$ by the relation $L(x) = K + q(x)r(x)$, where $L(x)$ is the mean of the random variable X truncated on the left and $q(x)$ is a real function satisfying certain conditions. Similar results can be obtained for the right truncated mean $\mu(x) = x - m(x)$ from the following. Its proof is similar to the left truncation part of Ruiz and Navarro [6] and is omitted.

Result 2.11: Let X be a continuous random variable in (a, b), where a can be $-\infty$ and b can be $+\infty$ with probability density function $f(x)$ such that $f(x) \neq 0$. Let there exist $f'(x) = (d/dx)f(x)$ and the right truncated mean $\mu(x)$ for all $x \in (a, b)$. Let K be a constant and let $q(x)$ be a real differentiable function in (a, b) such that $\lim_{x \to a^+} q(x)f(x) = 0$; then, conditions (i) and (ii) are equivalent:

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(i)

$$\frac{f'(x)}{f(x)} = \frac{K - x - q'(x)}{q(x)},$$

where

$$q'(x) = \frac{d}{dx}\, q(x)$$

(ii)

$$\mu(x) = K - a(x)q(x).$$

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