10 The Physics Underlying Quantum Phenomena

At this point, after having studied the previous chapters, you should be sufficiently familiar with quantum formalism to be ready to pause for a moment and delve into the underlying physics. It is likely that along the way you have been puzzled by some strange or counter-intuitive quantum features and have kept your puzzlement to yourself. This chapter is devoted to explaining some of the basic physics behind the theory, and in this respect it differs from most textbooks. Although you can skip this material if you are only interested in a more formal or conventional course in QM, we strongly recommend that you take the opportunity to join us in this exciting and timely endeavor. The aim is that you come away with a clearer picture of quantum behavior and its causes, which will also hopefully make it easier for you to understand the further reading.

We have learned from previous chapters that randomness enters the description in the form of "quantum fluctuations," considered to be a signature of quantumness. A number of different stochastic theories have been developed over the years to shed light on the nature of quantum stochasticity. In particular, *stochastic quantum mechanics* (SQM) and *stochastic electrodynamics* (SED) provide valuable and complementary insights, as we shall see.

The phenomenological approach of SQM, which is schematically presented in the next section, uses simple and straightforward mathematics with a clear physical meaning. However, like any phenomenological theory, it has the drawback that the source of the stochasticity remains unidentified.

We recall that already in Section 2.9.1 the interaction with the fluctuating electromagnetic vacuum or zero-point field (ZPF) was pointed out as a natural cause for both, the quantum stochasticity and the presence of wave-like phenomena; this is the starting point for SED. In this chapter we will show that it is indeed possible to clarify central aspects of QM by considering the coupling of matter with the radiation field, including the zero-point component. In particular, the intriguing operator formalism finds a first-principles explanation that gives substance to the basic postulates introduced by Heisenberg a century ago.

Finally, Section 5 introduces the hydrodynamic quantum analogue (HQA) of a walking droplet, a striking macroscopic example of a particle-wave system capable of expressing quantum-like features under appropriate conditions. The discussion of characteristic wave-like manifestations arising from the particle–field interaction in the case of quantum systems is left to future chapters.

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10.1 Quantum Mechanics as a Stochastic Theory. The Nelson Theory

The best known description of QM as a stochastic process, called *stochastic mechanics*, was put forward by Nelson[1] (1966), who considered the underlying process as classical and Brownian in nature.[2] Quantum mechanics and Brownian motion do indeed lend themselves to a stochastic description in x-space, with notable features in common, the most important being that they both can be treated as Markov (memoryless) processes. Put simply, in a Markov process the probability of each event depends only on the state reached by the system in just the last previous event, which means that the system has no memory of its past and predictions about future outcomes depend only on its current state.[3]

However, Brownian motion and QM are two *different* stochastic processes; in particular, while the latter is time-reversible and conservative, the former is irreversible and dissipative. The need to distinguish between the two led the present authors to develop an alternative version of the theory called *stochastic quantum mechanics* (SQM), which is briefly presented here. The stochastic nature of the dynamics requires a statistical treatment, which is carried out in x-space; the description is general enough to encompass a range of physical phenomena in which the stochastic process is Markovian.

The basic kinematic elements are obtained by averaging over the ensemble of particles that cross a given point $\boldsymbol{x}$ at time t. By taking a time interval Δt around t that is small but *finite*, and expanding in a Taylor series up to the second order, two different velocities are obtained, namely the *flow* or *systematic velocity* $\boldsymbol{v}$, which is a local mean measure of the nearly regular flow,

$$\boldsymbol{v}(\boldsymbol{x},t)=\frac{\overline{\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t-\Delta t)}}{2\Delta t}=\hat{\mathcal{D}}_c\boldsymbol{x}, \tag{10.1}$$

where

$$\hat{\mathcal{D}}_c=\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}, \tag{10.2}$$

[1] Edward Nelson (1932–2014) was a US mathematician, well known for his original work on mathematical physics and mathematical logic.

[2] The term *Brownian motion* originally referred to the random movement of pollen particles suspended in water, observed by the British botanist Robert Brown in 1827. In 1905, Einstein developed a theory that explained Brown's observations by recognizing that the (unobservable) impact of the water molecules on the particle would produce irregular motions that could be directly observed under the microscope. Einstein related the $x-$diffusion coefficient of such motions to the mean square displacement of the particle, as in Eq. (10.5), and to measurable physical quantities, thus laying the theoretical foundations for precision measurements that could reveal the reality of atoms, which was the leitmotif of this Einsteinian investigation.

[3] As we will see in Section 3, the detailed dynamics is not strictly memoryless, but the current (coarse-grained) quantum description is.

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and a *diffusive velocity* $\boldsymbol{u}$ resulting from the dispersion,

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{\overline{\boldsymbol{x}(t+\Delta t)+\boldsymbol{x}(t-\Delta t)-2\boldsymbol{x}(t)}}{2\Delta t}=\hat{\mathcal{D}}_s\boldsymbol{x}, \tag{10.3}$$

where

$$\hat{\mathcal{D}}_s=\boldsymbol{u}\cdot\boldsymbol{\nabla}+D\boldsymbol{\nabla}^2. \tag{10.4}$$

The symbol $\overline{(\cdot)}$ denotes the ensemble averaging over the finite time interval Δt; it is thus a *coarse-grained* average. Note that $\boldsymbol{u}(\boldsymbol{x},t)$ is only different from zero if the mean differences $\overline{\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)}$ and $\overline{\boldsymbol{x}(t)-\boldsymbol{x}(t-\Delta t)}$ are different, which for small Δt only occurs due to fluctuations in the positions $\boldsymbol{x}(t-\Delta t)$ and $\boldsymbol{x}(t+\Delta t)$ among the particles of the ensemble.

The diffusion coefficient,

$$D=\frac{\overline{(\Delta\boldsymbol{x})^2}}{2\Delta t}=\frac{\sigma_x^2}{2\Delta t}, \tag{10.5}$$

is taken as a constant scalar. The Taylor series expansion has been truncated to make the definitions of the mean velocities $\boldsymbol{v}$ and $\boldsymbol{u}$ consistent with the Markovian assumption: the position $x(t)$ does not depend on the positions x at times prior to $t-\Delta t$.

From the generality of the preceding procedure it follows that, given a dynamical system containing a stochastic source, *both velocities* $\boldsymbol{v}$ *and* $\boldsymbol{u}$ *are required* to describe it. For this reason, it is common in the theory of Markov processes to refer to the diffusive velocity (associated with the *osmotic* force, which will be discussed later) in addition to the flow velocity.

10.1.1 Quantum mechanics as a Markov process

The two time-derivative operators (10.2) and (10.4), applied to the velocities (10.1) and (10.3), give four different accelerations, which in turn lead to a pair of equations of motion: the first is invariant with respect to time reversal, whereas the second changes sign with time reversal,

$$m\left(\hat{\mathcal{D}}_c\boldsymbol{v}-\lambda\hat{\mathcal{D}}_s\boldsymbol{u}\right)=\boldsymbol{f}, \tag{10.6}$$

$$m\left(\hat{\mathcal{D}}_c\boldsymbol{u}+\hat{\mathcal{D}}_s\boldsymbol{v}\right)=0, \tag{10.7}$$

where λ is a real parameter and $\boldsymbol{f}$ is the external force acting on the particle. If the force contains a component $\boldsymbol{f}_-$ that changes sign on time reversal, this must be added to the right-hand side of Eq. (10.7).

Note that Eq. (10.6) is a stochastic generalization of Newton's second law; Eq. (10.7), on the other hand, is the gradient of the continuity equation expressing the local conservation of particles,

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0, \tag{10.8}$$

with the diffusive velocity $\boldsymbol{u}(\boldsymbol{x},t)$ given by

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$$\boldsymbol{u}(\boldsymbol{x},t) = D\frac{\boldsymbol{\nabla}\rho}{\rho}, \tag{10.9}$$

as is confirmed in what follows. Equations (10.6) and (10.7) can be combined into a single one by multiplying the second equation by $\sqrt{-\lambda}$; this gives

$$\hat{\mathcal{D}}_\lambda \boldsymbol{p}_\lambda = \boldsymbol{f}, \tag{10.10}$$

where

$$\boldsymbol{p}_\lambda = m(\boldsymbol{v} + \sqrt{-\lambda}\boldsymbol{u}), \tag{10.11}$$

and

$$\hat{\mathcal{D}}_\lambda = \hat{\mathcal{D}}_c + \sqrt{-\lambda}\hat{\mathcal{D}}_s. \tag{10.12}$$

These equations show that the specific dynamical properties of the system depend strongly on the sign of λ. Since the magnitude of λ can be absorbed in the value of D, we take $\lambda = \pm 1$. The value $\lambda = -1$ describes a classical process of the Brownian type. However, as Brownian motion is dissipative, this method is not the most suitable for a simple description, since the inclusion of the required (velocity-dependent) dissipative term leads to a nonlinear, parabolic differential equation after integrating Eq. (10.10).

In contrast, with $\lambda = 1$ and setting

$$D = \frac{\hbar}{2m}, \tag{10.13}$$

we obtain the *Schrödinger* equation

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi(\boldsymbol{x},t) + V(\boldsymbol{x})\psi(\boldsymbol{x},t) = i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} \tag{10.14}$$

and its complex conjugate, where $\psi(\boldsymbol{x},t)$ is a complex function such that

$$\rho(\boldsymbol{x},t) = |\psi(\boldsymbol{x},t)|^2, \tag{10.15}$$

$$\boldsymbol{v} = i\frac{\hbar}{2m}\left(\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi}\right), \tag{10.16}$$

and

$$\boldsymbol{u} = \frac{\hbar}{2m}\left(\frac{\nabla\psi^*}{\psi^*} + \frac{\nabla\psi}{\psi}\right) = \frac{\hbar}{2m}\frac{\nabla\rho}{\rho}. \tag{10.17}$$

From these equations we obtain

$$-i\hbar\nabla\psi = \widehat{\boldsymbol{p}}\psi = m(\boldsymbol{v} - i\boldsymbol{u})\psi, \tag{10.18}$$

which shows that the diffusive velocity $\boldsymbol{u}$ appears naturally (and more often than you might suppose) in the quantum description, although it is usually overlooked; see Exercise E10.1 and Section 4.4.

The value of the diffusion coefficient D has been set by hand to satisfy the needs of the Schrödinger equation. To *derive* Eq. (10.13) we need to lay the theory on a physical foundation, i.e., to specify the cause of stochasticity. This is where SED comes in, as we will see below.

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Exercise E10.1. Role of the diffusive velocity $\boldsymbol{u}$ in QM

Compare the SQM and QM formulas obtained for the variance of the momentum and the mean kinetic energy, and discuss the results.

Solution. Writing the wave function in polar form,

$$\psi(\boldsymbol{x},t) = \sqrt{\rho(\boldsymbol{x},t)}\, e^{iS(\boldsymbol{x},t)}, \tag{10.19}$$

where S is a real function of $\boldsymbol{x}, t$, we obtain from Eq. (10.16)

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla S. \tag{10.20}$$

This equation together with Eq. (10.17) give the correct quantum result when introduced in Eq. (10.18),

$$\langle \widehat{\boldsymbol{p}} \rangle = \hbar \langle \nabla S \rangle = m \langle \boldsymbol{v} \rangle\,; \quad \langle \boldsymbol{u} \rangle = 0. \tag{10.21}$$

On the other hand, the quantum expectation value of $\widehat{\mathbf{p}}^2$ gives

$$\begin{aligned}
\langle \widehat{\mathbf{p}}^2 \rangle &= -\hbar^2 \int \psi^* \nabla^2 \psi \, dx \\
&= -\hbar^2 \int \left[\sqrt{\rho}\nabla^2\sqrt{\rho} - \rho(\nabla S)^2 + 2i\sqrt{\rho}\nabla S \cdot \nabla\sqrt{\rho} + i\rho\nabla^2 S \right] dx \\
&= -\hbar^2 \int \left[\sqrt{\rho}\nabla^2\sqrt{\rho} - \rho(\nabla S)^2 \right] dx = \hbar^2 \left\langle (\nabla S)^2 \right\rangle + \tfrac{1}{4}\hbar^2 \left\langle \left(\frac{\nabla\rho}{\rho} \right)^2 \right\rangle,
\end{aligned} \tag{10.22}$$

since the imaginary contribution cancels out by an integration by parts, and

$$\int \sqrt{\rho}\nabla^2\sqrt{\rho} dx = \left\langle \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \right\rangle = -\int \rho \left(\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}} \right)^2 dx = -\tfrac{1}{4} \left\langle \left(\frac{\nabla\rho}{\rho} \right)^2 \right\rangle. \tag{10.23}$$

Therefore,

$$\langle \hat{\boldsymbol{p}}^2 \rangle = \langle (m\boldsymbol{v})^2 \rangle + \tfrac{1}{4}\hbar^2 \left\langle \left(\frac{\nabla\rho}{\rho} \right)^2 \right\rangle. \tag{10.24}$$

With $\boldsymbol{u}$ given by Eq. (10.17), we obtain two contributions to the kinetic energy,

$$\langle T \rangle = \tfrac{1}{2} m \langle \boldsymbol{v}^2 + \boldsymbol{u}^2 \rangle. \tag{10.25}$$

Similarly, the resulting variance of the momentum,

$$(\Delta \boldsymbol{p})^2 = \langle \left(\Delta \hat{\boldsymbol{p}} \right)^2 \rangle = m^2 \langle (\Delta \boldsymbol{v})^2 \rangle + m^2 \langle (\Delta \boldsymbol{u})^2 \rangle, \tag{10.26}$$

consists of an extrinsic part due to the dispersion of the flow velocity (e.g., produced by a distribution of the initial velocities) and an intrinsic part due to the unavoidable dispersion of the diffusive velocity. These results show that the velocity $\boldsymbol{u}$ plays a decisive role in QM. Among other things, the dispersion of $\boldsymbol{u}$ is found to be responsible for both the nonzero ground-state energy and the nonzero minimum value of the Heisenberg inequality.

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10.1.2 Evidence of diffusion in quantum mechanics nonlocality of the description

Another distinctive and persistent manifestation of the diffusive velocity $\boldsymbol{u}$ in QM is the so-called *quantum potential*, which is derived from the time-dependent Schrödinger equation (see Section 7.2),

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \tag{10.27}$$

by multiplying it by ψ^* and separating it into real and imaginary parts. Using Eq. (10.19), the real part gives the continuity equation, $\frac{\partial\rho}{\partial t} + \nabla\cdot(\boldsymbol{v}\rho) = 0$, and the imaginary part gives the dynamical equation

$$\hbar\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{10.28}$$

which can be rewritten as

$$\hbar\frac{\partial S}{\partial t} + \frac{1}{2}m\boldsymbol{v}^2 + V + V_Q = 0, \tag{10.29}$$

where

$$V_Q = -\frac{\hbar^2}{2m}\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{10.30}$$

A formal analogy to classical mechanics suggests to interpret Eq. (10.29) as a Hamilton–Jacobi equation with an additional potential V_Q given by Eq. (10.30). This term, known as the quantum potential, is the basis of Bohm's causal theory of QM (Bohm 1951). In this interpretation, $\boldsymbol{v}$ is taken to be the velocity field of a *single* particle located at $\boldsymbol{x}$ at time t and describing a real trajectory,

$$\frac{d\boldsymbol{x}}{dt} = \frac{\hbar}{m}\,\nabla S(\boldsymbol{x},t)|_{\boldsymbol{x}=\boldsymbol{x}(t)}\,, \tag{10.31}$$

where $S(\boldsymbol{x},t)$ is the action function of the problem, and the wave function is taken to represent a real field that *guides* the particle, known thus as the *pilot wave*, following a suggestion by de Broglie. Trajectories are sensitive to the initial conditions and therefore produce patterns such as the one illustrated in Fig. 10.1 for a uniformly distributed packet scattered by two Gaussian slits.

The pilot-wave equation (10.31) was introduced in 1925–1928 by de Broglie in his efforts to unify the laws of mechanics and optics, following his fundamental proposal on the wave behavior of particles in motion. For Bohm, on the other hand, the starting point is Eq. (10.28), which contains the quantum potential. What is known as Bohmian mechanics is a mixture of the two theories and is recognized for its success in restoring realism, objectivity and determinism to QM.

However, this success comes at the cost of the nonlocality embodied in the theory. Formally, the only difference between a classical problem and the corresponding quantum problem is the presence of V_Q in the latter; but unlike the local classical potential V, V_Q depends on the spatial distribution $\rho(\boldsymbol{x},t)$ (see Eq. (10.30)), which in turn is influenced by V_Q, in a kind of feedback loop. The dependence of V_Q on $\rho(\boldsymbol{x},t)$ gives

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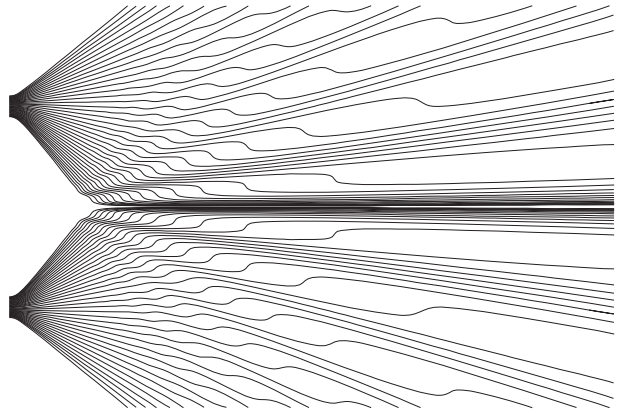


Figure 10.1 The trajectories of an initially uniform distribution of particles scattered by two Gaussian slits, according to Bohm's theory. Adapted from Philippidis et al. (1979).

(10.29) a nonlocal and statistical content. This nonlocality becomes manifest as soon V_Q comes into play.

By introducing Eq. (10.17) into (10.30), we get

$$V_Q = -\frac{1}{2}\left(m\boldsymbol{u}^2 + \hbar\nabla\cdot\boldsymbol{u}\right), \tag{10.32}$$

which suggests reading the "quantum potential" as a nonlocal kinematic term associated with the diffusive nature of the dynamics, i.e., as an energy contribution due to "quantum fluctuations." Whichever view is adopted, the term is clearly of paramount importance in determining much of the quantum behavior.

Exercise E10.2. Estimating the time resolution of the quantum description
One of the important lessons to be learned from the stochastic formulation of QM is the confirmation that the quantum description has a time resolution limit. Using the formulas derived in this section, obtain an estimate for Δt.

Solution. Indeed, the development of the stochastic theory required the introduction of a time interval Δt, which by definition is very short but not zero; no better estimate of it is proposed. This Δt represents a minimum value for the precision of the description of the temporal behavior. Thus, any phenomenon or detail shorter in time than Δt remains outside the possibilities of (nonrelativistic) quantum theory.

Let us consider the fluctuations of the position variable x; according to Eq. (10.5) the variance of x is given by

$$\overline{(\Delta x)^2} = 2D\Delta t, \tag{10.33}$$

where the diffusion coefficient of the position variable D is phenomenologically ascribed the value $D = \hbar/2m$ in the quantum case, according to Eq. (10.13). Combining this with the above equation gives

$$\overline{(\Delta x)^2} = (\hbar/m)\,\Delta t. \tag{10.34}$$

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So all we need to get an estimate for Δt is to assign a minimum characteristic value to the mean velocity $\bar{v} = \sqrt{\overline{(\Delta x)^2}}/\Delta t$. To do this, consider an orbiting atomic electron; its kinetic energy is of the order $\alpha^2 mc^2$, where as usual, α is the fine structure constant (see Section 12.5); hence

$$\overline{(\Delta x)^2}/\,(\Delta t)^2 = \frac{\hbar}{m\Delta t} \simeq \alpha^2 c^2. \tag{10.35}$$

It follows that

$$\Delta t \simeq \frac{\hbar}{m\alpha^2 c^2} = \frac{1}{\alpha^2 \omega_C} = \frac{1}{2\pi\alpha^2}\tau_C, \tag{10.36}$$

where τ_C is the Compton time of the electron. With the above estimates, Δt is about 3×10^3 times the Compton time for the electron, i.e., $\Delta t \sim 10^{-17}\, s$.

Note further that if we assume the dispersion of p to be of the order of $m\bar{v}$, then by combining the above equations we get

$$\overline{(\Delta x)^2}\,\overline{(\Delta p)^2} \simeq \hbar^2, \tag{10.37}$$

implying that the limited time resolution results in a limited resolution in x and p.

10.2 Stochastic Electrodynamics. Basic Properties of the ZPF

Stochastic electrodynamics has been developed on the basis of the existence of the ZPF as a real, omnipresent, fluctuating Maxwellian field in permanent interaction with matter. A number of authors have contributed to SED, with different approaches and views on its relation to classical mechanics, QM, SQM and even QED (for a few examples see the bibliographical notes at the end of the chapter). Over the years, important results have been obtained, some of which will be presented in this and following chapters, using procedures that are generally more transparent and direct than the corresponding quantum-mechanical and QED ones.

The vacuum or zero-point radiation field, i.e., the nonthermal field proposed by Planck – corresponding to "zero photons" in the language of quantum theory – has an average energy per normal mode of value $\hbar\omega/2$, which implies a spectral energy density of value

$$\rho_0(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3}, \tag{10.38}$$

as explained in Section 2.9 of Chapter 2. The spectral energy density $\rho \sim \omega^3$ is the *only* one consistent with relativity, as was shown by Einstein and Hopf (1910). The argument is as follows.

When a charged or polarizable body moving at a constant velocity $\boldsymbol{v}$ interacts with an electromagnetic field of spectral energy density $\rho(\omega)$, the field exerts on it a force $\boldsymbol{F}$ due to the Doppler effect, given by

$$\boldsymbol{F} = A\left(\rho - \tfrac{1}{3}\omega\frac{\partial\rho}{\partial\omega}\right)\boldsymbol{v}, \tag{10.39}$$

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where the constant A depends on the geometry of the body. But the vacuum field must not exert any force on a body in uniform motion; this is true for any velocity $\boldsymbol{v}$ only if $\rho \sim \omega^3$.

From Eq. (10.38) it follows that the total energy of the ZPF is infinite, since the field contains in principle modes of frequencies ω covering the entire spectrum (0 to ∞). Much has been written about this problem, as it is part of contemporary cosmology and quantum theory in general, which deals with a variety of vacuum fields in addition to the electromagnetic one. Several plausible solutions have been proposed, including the possibility that some (or all) of the dark energy in the universe is associated with the vacua, whose energy is positive in some cases and negative in others. Without getting into unresolved cosmological arguments, it is a fact that atomic particles do not respond to extremely high frequencies; atoms are known not to radiate or absorb at frequencies above a cut-off frequency, that can be set around the Compton frequency $\omega_c = mc^2/\hbar$ of the electron ($\sim 10^{21}\,\mathrm{s}^{-1}$). This sets a natural limit to the energy content of the ZPF, which is large but not infinite. In Chapter 17 we will find further justification for the introduction of this frequency cut-off.

To describe the basic statistical properties of the ZPF, we introduce the Fourier transform of a (Cartesian) electric field component $E_i(t)$, $i = 1, 2, 3$,

$$E_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_i(t)\, e^{-i\omega t} dt, \tag{10.40}$$

$$E_i(t) = \int_{-\infty}^{\infty} E_i(\omega)\, e^{i\omega t} d\omega, \tag{10.41}$$

$$E_i^*(-\omega) = E_i(\omega). \tag{10.42}$$

$E_i(\omega)$ is assumed to have a Gaussian (normal) distribution; this is the simplest and most natural proposal, given the vast number of sources contributing to it. The average over all its possible realizations is of course zero; denoting this ensemble average by $\langle\rangle_0$, and considering (10.40),

$$\langle E_i(t)\rangle_0 = 0, \quad \langle E_i(\omega)\rangle_0 = 0. \tag{10.43}$$

To fully characterize the Gaussian distribution, it is sufficient to specify some other statistical property, such as the autocorrelation of the field. This can be done using Eq. (10.38) for the spectral energy density $\rho_0(\omega)$. The total energy density can be written in one of the following alternative forms,

$$\begin{aligned} u = \frac{\langle \boldsymbol{E}^2(t)\rangle_0}{4\pi} &= \frac{3}{4\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \langle E_i(\omega) E_i^*(\omega')\rangle_0 e^{i(\omega-\omega')t} \\ &= \int_0^{\infty} \rho_0(\omega) d\omega = \tfrac{1}{2}\int_{-\infty}^{\infty} \rho_0(\omega) d\omega, \end{aligned} \tag{10.44}$$

where $\rho(-\omega) = \rho(\omega)$, $\omega > 0$. For the second and last equalities to hold, it is required that

$$\langle E_i(\omega) E_j^*(\omega')\rangle_0 = \frac{2\pi}{3}\rho_0(\omega)\delta(\omega - \omega')\delta_{ij}. \tag{10.45}$$

This expression means that the different modes of the field are uncorrelated, and shows that the spectral energy density is the Fourier transform of the autocorrelation function of the field up to a numerical factor,

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$$\langle E_i(t')E_j(t)\rangle_0 = \varphi(t'-t)\delta_{ij} = \frac{2\pi}{3}\delta_{ij}\int \rho_0(\omega)e^{i\omega(t'-t)}d\omega. \tag{10.46}$$

The integration results in an autocorrelation that decreases with time as t^{-4}; for a white (ω-independent) noise one gets instead the much slower decrease t^{-1}. The ω^3-dependence of the spectral energy density means that the particles are subject to a highly colored noise.

10.2.1 The Casimir forces, direct manifestation of the ZPF

In 1948, Casimir[4] predicted that the presence of two parallel, neutral conducting plates would modify the distribution of vacuum field modes in the vicinity of the plates, particularly in the space between them, resulting in an effective force of attraction given by

$$F = -\frac{\pi^2\hbar c A}{240R^4}, \tag{10.47}$$

where A is the area of the plates and R is the distance separating them. Half a century later the effect was experimentally confirmed. Casimir's analysis of idealized metal plates has been extended to arbitrary dielectrics and realistic boundaries, giving rise to a new branch of physics. In most of the literature on the subject, the calculations are carried out in the context of quantum field theory, where the vacuum is taken to be the ground state of the quantized electromagnetic field, conventionally called *vacuum fluctuations* (see Section 17.3).

The SED approach successfully reproduces the quantum predictions under a wide range of conditions, with the advantage of greater simplicity and physical transparency. The following exercise illustrates this for the Casimir force between two parallel metal plates. These calculations are so convincing that they are often taken as a demonstration of the reality of the ZPF and its action on quantum matter.

Exercise E10.3. Casimir force due to the ZPF

Calculate the Casimir force between two parallel metal plates, taking into account the modification of the ZPF by the presence of the plates.

Solution. Let us take two square plates of side L, separated by a distance R, which is considered small compared to L. To calculate the effect of the plates on the field, we place a vertical perfectly conducting wall at a large distance D from the left plate ($D \to \infty$ at the end of the calculation), and enclose the array within conducting walls as shown in Fig. 10.2.

[4] Hendrik Brugt Gerhard Casimir (1909–2000) was a Dutch physicist who made significant contributions to the field of QM and QED. He is best known for his work on the Casimir effect and for his research on the two-fluid model of superconductors. For 30 years from 1942, Casimir worked and (co-)directed the Philips Physics Laboratory in Eindhoven, the Netherlands, where many fundamental steps in the development and applications of electronics were made.

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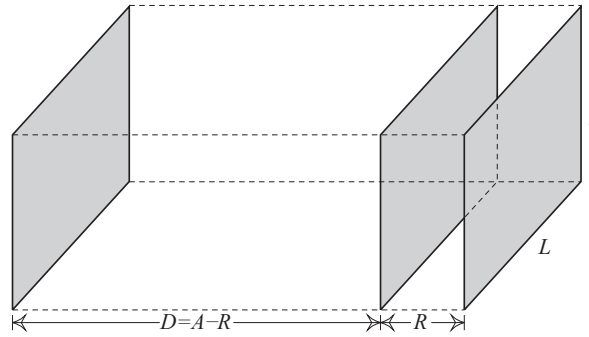


Figure 10.2 A simple model for calculating the Casimir force between two plates separated by a distance R. To complete the calculation, a conducting wall is placed at a large distance D from the left plate and the array is surrounded by conducting walls to form a parallelopiped.

The field trapped between the plates can be expanded in normal modes as usual. Given the boundary conditions on the electric and magnetic components, it is easy to see that the allowed frequencies between the plates are given by the formula

$$\omega_{lmn} = ck_{lmn} = c\pi \left(\frac{l^2 + m^2}{L^2} + \frac{n^2}{R^2} \right)^{\frac{1}{2}}, \tag{10.48}$$

where l, m, n are positive integers or zero. The energy of this field is given by the triple sum over modes,

$$E(R) = 2{\sum_{lmn}}' \frac{\hbar\omega_{lmn}}{2} = \pi\hbar c {\sum_{lmn}}' \left(\frac{l^2 + m^2}{L^2} + \frac{n^2}{R^2} \right)^{\frac{1}{2}}. \tag{10.49}$$

The factor 2 arises from the two independent polarizations for $l, m, n \neq 0$. Only one of these numbers can be zero at a time. When this happens a factor $1/2$ must be inserted because there is only one polarization; this is the meaning of the prime in the summation sign. For large values of L, the double sum over l and m transforms into a double integral over k_x and k_y, respectively,

$$\sum_{lm} \longrightarrow_{L\to\infty} \left(\frac{L}{2\pi} \right)^2 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y, \tag{10.50}$$

whence

$$E(R) = \frac{\hbar c L^2}{\pi^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y {\sum_{n}}' \left(k_x^2 + k_y^2 + \frac{\pi^2 n^2}{R^2} \right)^{\frac{1}{2}}. \tag{10.51}$$

Setting $k_x^2 + k_y^2 = k_\rho^2$ we perform the angular integration on the xy-plane, to get

$$E(R) = \frac{\hbar c L^2}{4\pi} \int_0^{\infty} dk_\rho^2 {\sum_{n}}' k_n, \tag{10.52}$$

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with $k_n = \left(k_\rho^2 + \frac{\pi^2 n^2}{R^2}\right)^{\frac{1}{2}}$. Similarly, the energy of the ZPF between the conducting wall and the left plate is given by $E(D)$, so the energy of the whole configuration is $E(R) + E(D)$.

The Casimir energy $E_C(R)$, i.e., the energy change produced by the central plate, is obtained by subtracting the energy in the absence of this plate and making the distance D of the wall tend to infinity,

$$E_C(R) = \lim_{D\to\infty} E(R) + E(D) - E(D+R). \tag{10.53}$$

This expression represents the work performed to bring the central plate adiabatically to its position at a distance R from the right plate.

From Eq. (10.52) it is clear that $E(R)$ is infinite and the same applies to the remaining terms in (10.53). However, since even well conducting materials become poor conductors at very short wavelengths, a cut-off in the wavenumber k_n should be introduced to obtain a finite value for the integrals. A convenient cut-off function is $\exp(-k_n/k_c)$, where k_c is assumed to be very large. Equation (10.52) gives after a change of variable

$$E(R;k_c) = \frac{\hbar c L^2 k_c^3}{2\pi} \sum_n{}' \int_{n/x}^{\infty} ds\, s^2 e^{-s}, \tag{10.54}$$

with $s = k_n/k_c$ and $x = Rk_c/\pi$. Integrating gives

$$E(R;k_c) = \frac{\hbar c L^2 k_c^3}{2\pi} \frac{d^2}{dx^2} x^2 \sum_n{}' e^{-n/x} = \frac{\hbar c L^2 k_c^3}{2\pi} \left(-1 + 2\Delta + \frac{2\varsigma\Delta^2}{x} - \frac{\varsigma\Delta^2}{x^2} + \frac{2\varsigma\Delta^3}{x^2}\right), \tag{10.55}$$

with $\varsigma = \exp(-1/x)$ and $\Delta = (1-\varsigma)^{-1}$. By introducing this expression into (10.53), we obtain after some algebra the exact result

$$E_C(R;k_c) = \frac{\hbar c L^2 k_c^3}{2\pi} \left(-1 - 6x + 2\Delta + \frac{2\varsigma\Delta^2}{x} - \frac{\varsigma\Delta^2}{x^2} + \frac{2\varsigma\Delta^3}{x^2}\right). \tag{10.56}$$

For macroscopic objects, $x \gg 1$ and the above expression takes the simple form (with $L^2 = A$)

$$E_C(R;k_c) = -\frac{\pi^2 \hbar c A}{720 R^3}, \tag{10.57}$$

which gives

$$F = -\frac{\partial E_C}{\partial R} = -\frac{\pi^2 \hbar c A}{240 R^4}, \tag{10.58}$$

in agreement with Casimir's formula.

Note that this result is independent of k_c for large values of the cut-off frequency, i.e., for good metallic conductors. In the case of dielectric materials or very small objects, the Casimir energy E_C has a more complicated behavior as a function of distance, as can be seen from Eq. (10.56).

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10.2.2 Initial SED analysis of the harmonic oscillator

The early SED treatment (around 1960) of the HO already showed that the ZPF is somehow responsible for its quantum behavior. The following presentation is necessarily schematic; the interested reader is invited to consult the bibliography at the end of the chapter for details.

10.2.2.1 Dynamics of the linear oscillator

The usual starting point is the equation of motion for the nonrelativistic 1D HO in the long-wavelength or electric-dipole approximation, Eq. ((2.3)),

$$\ddot{x} + \omega_0^2 x - \tau \dddot{x} = \frac{e}{m} E(t), \tag{10.59}$$

where

$$\tau = \frac{2e^2}{3mc^3}. \tag{10.60}$$

We recall that the term $m\tau\dddot{x}$ is the self-force due to the radiation reaction; for an electron, $\tau \approx 10^{-23}$ s.

$E(t)$ is an electric component of the random ZPF, which means that (10.59) is a stochastic differential equation, i.e., each realization of $E(t)$ leads to a different trajectory, even under the same initial conditions.[5] Since one cannot know in advance which $E(t)$ will be realized in a particular instance, one can only make statistical predictions about trajectories from the statistical properties of the field E.

The stationary solution of Eq. (10.59) is obtained by taking the Fourier transform of $x(t)$ and computing the inverse transform. Since our interest centers on the *stationary motions generated and maintained by the field*, we neglect the transient part of the solution that decays exponentially with time, at a rate given by $\tau\omega_0^2$. This gives

$$x(t) = \frac{e}{m} \int_{-\infty}^{\infty} d\omega \frac{E(\omega) e^{i\omega t}}{\omega_0^2 - \omega^2 + i\tau\omega^3} + \text{c. c.}, \tag{10.61}$$

which means that for general initial conditions, the system will reach an equilibrium state – which, as we shall see, corresponds to the ground state of QM. It is important to note that Eq. (10.61) expresses the effect of the direct action of the ZPF on the particle in a stationary state, suggesting that a similar procedure should be applied to the study of stationary states more generally. This observation is the starting point for the SED approach to be presented in later sections.

10.2.2.2 Minimal fluctuations and minimal energy of the oscillator

By averaging (10.61) over the ensemble using Eq. (10.43), it becomes clear that at equilibrium, the oscillators remain on average centered at the origin,

$$\langle x(t) \rangle_0 = 0. \tag{10.62}$$

[5] Equations of motion of this type are given the generic name *Langevin equation*; more specifically, the Langevin equation of SED is often called the *Braffort–Marshall equation*.

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Further, Eqs. (10.61) and (10.45) give for the average of x^2

$$\left\langle x^2 \right\rangle_0 = \frac{2\pi c^3 \tau}{m} \int_0^\infty d\omega \frac{\rho_0(\omega)}{\Delta(\omega;\omega_0)}, \tag{10.63}$$

where

$$\Delta = (\omega_0^2 - \omega^2)^2 + \tau^2 \omega^6. \tag{10.64}$$

To carry out the integration we take into account that for the relevant (atomic) frequencies $\omega_0 \leq 10^{15}\, s^{-1}$,

$$\tau\omega_0 \leq 10^{-8} \ll 1, \tag{10.65}$$

and the denominator Δ becomes very small for $\omega \sim \omega_0$. Introducing $y = \omega - \omega_0$, noting that the main contribution to the integral comes from y close to zero, and using

$$\lim_{\gamma \to 0} \frac{\gamma}{\gamma^2 + y^2} = \pi \delta(y), \tag{10.66}$$

we obtain

$$\left\langle x^2 \right\rangle_0 = \frac{\pi^2 c^3}{m\omega_0^4} \rho_0(\omega_0) = \frac{\hbar}{2m\omega_0}, \tag{10.67}$$

and the dispersion of x is given by

$$\sigma_x^2 \equiv \langle (\Delta x)^2 \rangle_0 = \langle x^2(t) \rangle_0 - \langle x(t) \rangle_0^2 = \frac{\hbar}{2m\omega_0}. \tag{10.68}$$

Similarly, the higher moments $\langle x^n(t) \rangle_0 \geq 2$ can be shown to correspond to a Gaussian distribution with zero mean value and variance σ_x^2. Therefore, the spatial density normalized to unity is

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-x^2/2\sigma_x^2} = \sqrt{\frac{m\omega_0}{\pi\hbar}} e^{-m\omega_0 x^2/\hbar}, \tag{10.69}$$

in agreement with the quantum result.

The similar calculation of the moments of p gives

$$\langle p(t) \rangle_0 = 0, \tag{10.70}$$

$$\sigma_p^2 \equiv \langle (\Delta p)^2 \rangle_0 = \langle p^2 \rangle_0 - \langle p \rangle_0^2 = \tfrac{1}{2} m\hbar\omega_0, \tag{10.71}$$

and the mean energy of the ground state is therefore

$$E_0 = \langle \widehat{H} \rangle_0 = \left\langle \frac{p^2}{2m} + \tfrac{1}{2} m\omega_0^2 x^2 \right\rangle_0 = \frac{\sigma_p^2}{2m} + \tfrac{1}{2} m\omega_0^2 \sigma_x^2, \tag{10.72}$$

i.e.,

$$E_0 = \tfrac{1}{2}\hbar\omega_0, \tag{10.73}$$

again in accord with QM. Note that this mechanical energy *is due to the fluctuations induced by the ZPF on the particle*. The equality $\sigma_x^2 \sigma_p^2 = \frac{1}{4}\hbar^2$ – the minimum possible value of the Heisenberg inequality $\sigma_x^2 \sigma_p^2 \geq \frac{1}{4}\hbar^2$ – is an indication of the irreducible nature of the fluctuations of the oscillator: they will linger there as long as the electron interacts with the ZPF.

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10.2.2.3 Canonical ensemble of harmonic oscillators

The previous procedure can be extended to construct the spectral energy density of material oscillators in equilibrium with the field oscillators at a temperature $T > 0$, given by the complete Planck distribution (including the zero-point energy). In terms of the dimensionless variable $\beta E_0 = \hbar\omega/2kT$ it is written as follows (see Eq. (2.83) of Chapter 2),

$$\rho_T(\omega) = \rho_0(\omega)\frac{e^{2\beta E_0}+1}{e^{2\beta E_0}-1} = \rho_0(\omega)\coth\beta E_0. \tag{10.74}$$

This equilibrium distribution of the mechanical oscillators in configuration space is again a Gaussian function, with $\langle x^2\rangle_T$ calculated as in Eq. (10.63), with $\rho_0(\omega)$ replaced by $\rho_T(\omega)$, whence

$$\rho(x;T) = \sqrt{\frac{m\omega_0}{\pi\hbar\coth\beta E_0}}e^{-\frac{m\omega_0}{\hbar}x^2\tanh\beta E_0}, \tag{10.75}$$

and

$$E = E_0\coth\beta E_0. \tag{10.76}$$

By expanding the results in a power series with respect to the variable $e^{-2\beta E_0}$, we obtain

$$E = E_0\frac{1+e^{-2\beta E_0}}{1-e^{-2\beta E_0}} = \frac{1}{Z}\sum_n E_n e^{-2\beta E_0 n}, \tag{10.77}$$

with the partition function Z given by

$$Z = \sum_n e^{-2\beta E_0 n} = \frac{1}{1-e^{-2\beta E_0}} \tag{10.78}$$

and $E_n = E_0(2n+1)$. By expanding (10.75) with respect to the same variable,

$$\rho(x,T) = \frac{1}{Z}\sum_n \rho_n(x)e^{-(2n+1)\beta E_0}, \tag{10.79}$$

the quantum expression involving the Hermite polynomial $H_n(x)$

$$\rho_n(x) = |\psi_n(x)|^2 = c_n^2 e^{-x^2/2\sigma_x^2}H_n^2(x) \tag{10.80}$$

for the spatial density of state n is obtained as in Section 9.2 – but without the need to use the Schrödinger equation.

The early SED results, particularly those on the HO, were instrumental in opening up SED as a legitimate field of research. Still over time it became clear that the HO could be deceptive; what you learn from it could not automatically be extended to other systems. Something fundamental was not taken into account, namely that the action of the ZPF on the particle is not just a perturbation of classical Newtonian motion, but leads to a *qualitative* change in its behavior, which is even reflected in the kinematics, as will be seen in Section 4.

In preparation, we will show how the general theory of stochastic processes provides interesting insights into the statistical content of QM.

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10.3 Statistical Approach: The Fokker–Planck Equation

In the equation of motion (10.59)

$$m\ddot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}) - m\tau\dddot{\boldsymbol{x}} = e\boldsymbol{E}(t), \tag{10.81}$$

the right-hand side is the electric force due to the random ZPF, and therefore its solution requires a statistical treatment. The general procedure, well known from the theory of stochastic processes (see the bibliography at the end of the chapter), goes as follows. As a first step, starting from Eq. (10.81), one derives an equation for the probability density in the whole (particle+field) phase space. Then, through a smoothing process, one gets rid of the field variables and obtains a generalized Fokker-Planck equation (GFPE) for the particle phase-space probability density $Q(\boldsymbol{x}, \boldsymbol{p}, t)$. This is an integro-differential equation, or equivalently, a differential equation with an infinite number of time-dependent terms, which in principle gives a statistical description of the particle dynamics for all times since the initial time when the particle and field begin to interact. It is a highly complicated differential equation that is impossible to solve exactly; however, some general observations can be drawn from it.

At the beginning, the system is far from equilibrium; in this regime, the main effect of the zpf on the particle is due to the modes of very high frequency, close to $10^{21}\mathrm{s}^{-1}$, which produce violent accelerations and randomize the motion. Eventually, the interplay between the electric field force and radiation reaction drives the system to equilibrium; in this regime, the initial conditions have become irrelevant and the Markovian approximation applies. The time averaging implicit in the Markov description which, as discussed in Section 1 above, is estimated to be of the order of $10^{-17}\mathrm{s}$, prevents the quantum description from capturing the finer details of the motion.

In the Markov approximation, describing the slow dynamics, the GFPE is simplified by retaining terms up to second order. The ensuing Fokker-Planck equation (x_i, p_i are the Cartesian coordinates and corresponding momenta),

$$\frac{\partial Q}{\partial t} + \frac{1}{m}\frac{\partial}{\partial x_i}p_i Q + \frac{\partial}{\partial p_i}f_i Q = -m\tau\frac{\partial}{\partial p_i}\dddot{x}_i Q + D_{ij}^{px}\frac{\partial^2 Q}{\partial p_i \partial x_j} + D_{ij}^{pp}\frac{\partial^2 Q}{\partial p_i \partial p_j}, \tag{10.82}$$

contains remnants of the memory build-up, expressed through the diffusion tensors

$$D_{ij}^{px}(t) = e\left\langle x_i E_j\right\rangle_0 = e^2\int_{-\infty}^{t} ds\ \left.\frac{\partial x_i(t)}{\partial p_k(s)}\right|_{x^{(0)}}\left\langle E_k(s)E_j(t)\right\rangle_0, \tag{10.83}$$

$$D_{ij}^{pp}(t) = e\left\langle p_i E_j\right\rangle_0 = e^2\int_{-\infty}^{t} ds\ \left.\frac{\partial p_i(t)}{\partial p_k(s)}\right|_{x^{(0)}}\left\langle E_k(s)E_j(t)\right\rangle_0, \tag{10.84}$$

where the derivatives are calculated to zero order in the coupling constant e, and the field correlation $\left\langle E_k(s)E_j(t)\right\rangle_0$ is given by Eq. (10.46),

$$\left\langle E_k(s)E_j(t)\right\rangle_0 = \varphi(s-t)\delta_{kj}. \tag{10.85}$$

Equation (10.82) provides key statistical information about the evolution of the system in the quantum regime. Multiplying it from the left by a dynamical function

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$\mathcal{G}(\boldsymbol{x}, \boldsymbol{p}, t)$ and integrating over the phase space (assuming in the integration by parts that the system is bounded so that Q vanishes at infinity), one gets

$$\frac{d}{dt}\langle \mathcal{G} \rangle = \left\langle \frac{d\mathcal{G}}{dt} \right\rangle_{\text{nr}} + m\tau \left\langle \dddot{x}_i \frac{\partial \mathcal{G}}{\partial p_i} \right\rangle - e^2 \left\langle \frac{\partial \mathcal{G}}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle, \tag{10.86}$$

where

$$\left\langle \frac{d\mathcal{G}}{dt} \right\rangle_{\text{nr}} = \left\langle \frac{\partial \mathcal{G}}{\partial t} \right\rangle + \left\langle \dot{x}_i \frac{\partial \mathcal{G}}{\partial x_i} + f_i \frac{\partial \mathcal{G}}{\partial p_i} \right\rangle \tag{10.87}$$

corresponds to the (Liouvillian) nonradiative contribution to $\langle d\mathcal{G}/dt \rangle$. This equation describes the evolution of the mean value of $\mathcal{G}$ in line with quantum mechanics, i.e., in the radiationless approximation, as will be shown below.

The remaining terms in Eq. (10.86), with the diffusion operator $\hat{\mathcal{D}}_i$ given by

$$e^2 \hat{\mathcal{D}}_i = D_{ij}^{px} \frac{\partial}{\partial x_j} + D_{ij}^{pp} \frac{\partial}{\partial p_j}, \tag{10.88}$$

represent the radiative corrections; they contain information that is not part of QM and pertains to the realm of (nonrelativistic) QED. In Chapter 17 this information will be used to obtain the general formulas for the atomic transition probabilities and the Lamb shift.

Note that for $\mathcal{G} = p_i$ in Eq. (10.86), one gets

$$m\langle \ddot{x}_i \rangle = \langle f_i \rangle + m\tau \langle \dddot{x}_i \rangle - e\left\langle \hat{\mathcal{D}}_i \right\rangle, \tag{10.89}$$

which is the Ehrenfest version of Newton's second law including the radiative terms: $m\tau \langle \dddot{x}_i \rangle$ is the average radiation reaction, and $-e\langle \hat{\mathcal{D}}_i \rangle$ represents an effective force originating from the electric component of the ZPF; this *diffusive force* plays a role analogous to that played by the osmotic force in Brownian-motion theory.

Exercise E10.4. Time inversion of the evolution equation

Analyze the behavior of the terms appearing in Eq. (10.86) with respect to the time inversion operator, for a function $\mathcal{G}$ that is invariant under time inversion. Assume that enough time has elapsed so that the time integrals in Eqs. (10.83) and (10.84) can be extended to $t \to \infty$.

Solution. The Liouvillian contribution to Eq. (10.86), given by (10.87), contains only terms that change sign under time inversion. In contrast, the two remaining terms in (10.86), which represent the radiative contribution, are invariant under time inversion. Therefore, for long times, assuming that the system has reached a stationary state, $\frac{d}{dt}\langle \mathcal{G} \rangle = 0$, the equation can be separated into two equations,

$$\left\langle \frac{d\mathcal{G}}{dt} \right\rangle_{\text{nr}} = 0, \quad m\tau \left\langle \dddot{x}_i \frac{\partial \mathcal{G}}{\partial p_i} \right\rangle - e^2 \left\langle \frac{\partial \mathcal{G}}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle = 0. \tag{10.90}$$

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10.3.1 Mean evolution of a constant of motion: A fluctuation-dissipation relation

When $\mathcal{G} = \xi(\boldsymbol{x}, \boldsymbol{p})$ represents an integral of motion, or more generally a function of integrals of motion, (10.87) gives

$$\left\langle \dot{x}_i \frac{\partial \xi}{\partial x_i} + f_i \frac{\partial \xi}{\partial p_i} \right\rangle = 0. \tag{10.91}$$

as the student may verify using Hamilton's equations. Equation (10.86) reduces then to

$$\frac{d}{dt} \langle \xi \rangle = m\tau \left\langle \dddot{x}_i \frac{\partial \xi}{\partial p_i} \right\rangle - e^2 \left\langle \frac{\partial \xi}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle, \tag{10.92}$$

which shows that only the radiative terms contribute to the evolution of $\langle \xi \rangle$. The system reaches a state of equilibrium when

$$m\tau \left\langle \dddot{x}_i \frac{\partial \xi}{\partial p_i} \right\rangle = e^2 \left\langle \frac{\partial \xi}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle. \tag{10.93}$$

This equation is a general diffusion- (or fluctuation-) dissipation relation: the left side represents the average loss of ξ per unit time due to the radiation reaction and the right term represents the average exchange of ξ per unit time with the background field. In particular, when ξ is the Hamiltonian, the condition (10.93) implies that the system has reached a stationary state (an energy eigenstate). It is precisely the combined effect of radiation (dissipation) and diffusion that allows the system to achieve the equilibrium necessary to remain in a stationary quantum state. In Chapter 17, the above equations will be used to calculate the probability of transition between stationary states when the equilibrium is perturbed.

10.4 Transition to the Quantum Description in Terms of Operators

This section is devoted to discussing the change of role of the dynamical variables in the passage to the quantum regime and to explaining the emergence of the corresponding matrix formalism.

We start by recalling the HO case presented in Section 10.2.2, which showed that in the stationary state, the particle responds to the electric component of the radiation field, which acts as a driving force. As shown in Eqs. (10.63)–(10.66), the oscillator's response is a sharp resonance at the frequency ω_0. This is confirmed by the selection rules for dipolar transitions obtained in Section 9.3. A similar situation occurs in the more general case of a conservative (binding) force, when the system can respond resonantly to field modes of more than one frequency. Our purpose in the following is to study the response of the system in such case.

A central observation is that in the transition to the quantum description, the Hamiltonian (symplectic) structure of classical physics is preserved. Therefore, the approach will be Hamiltonian.

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10.4.1 Hamiltonian dynamics

Consider an (otherwise) classical charged particle subject to an electromagnetic radiation field. Due to the electrodynamic nature of the problem, we start with the Hamiltonian of the complete system (particle+field),

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}) + H_{\mathrm{R}}, \tag{10.94}$$

where $\boldsymbol{A}$ is the vector potential of the radiation field—the ZPF, at a minimum—and H_{R} is its Hamiltonian. The interaction Hamiltonian H_{I} has been included in this equation by means of the *minimal coupling*,[6]

$$H_{\mathrm{I}} = -\frac{e}{mc}\boldsymbol{A}\cdot\boldsymbol{p} + \frac{e^2}{2mc^2}\boldsymbol{A}^2. \tag{10.95}$$

(By choosing the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, one avoids commutation problems that can arise in the quantum case, since $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{p}}$ then commute.) The radiation field is expressed as usual in terms of Fourier plane waves,

$$\boldsymbol{A} = \sum_{n,\sigma}\left(\frac{\pi\hbar c^2}{V\omega_n}\right)^{1/2}\boldsymbol{\epsilon}_{n\sigma}a_{n\sigma}\exp\left[-i(\omega_n t - \boldsymbol{k}_n\cdot\boldsymbol{r})\right] + \text{c.c.} \tag{10.96}$$

assuming that the system is confined in a (normalization) volume V, usually a parallelopiped, which must be made to tend to infinity at the end of the calculations. The field component $(\boldsymbol{n},\sigma)$ propagates in the direction $\boldsymbol{k}_n$ with $\mathbf{n} = (n_1, n_2, n_3), n_i$ integer; $\omega_n = ck_n$ is its frequency and $\boldsymbol{\epsilon}_{n\sigma}$ $(\sigma = 1,2)$[7] are the two polarization vectors, orthogonal to each other and to $\boldsymbol{k}_n$,

$$\boldsymbol{\epsilon}_{n\sigma}\cdot\boldsymbol{\epsilon}_{n\sigma'} = \delta_{\sigma\sigma'}, \quad \boldsymbol{k}_n\cdot\boldsymbol{\epsilon}_{n\sigma} = 0, \tag{10.97}$$

$$\sum_{\sigma}\epsilon_{n\sigma i}\epsilon_{n\sigma j} + (k_{ni}k_{nj}/k_n^2) = \delta_{ij}, \tag{10.98}$$

where $\epsilon_{n\sigma i}$ is the i-Cartesian component.

The mode amplitudes $a_{n\sigma}$ are complex random variables, such that $\boldsymbol{A}$ has the correct statistical properties. With the coefficients introduced in the expansion (10.96), the $a_{n\sigma}$ are dimensionless and have the following properties, where $\langle g\rangle_0$ as before denotes the average of g,

$$\langle a_{n\sigma}\rangle_0 = 0, \quad \langle a^*_{n\sigma}\rangle_0 = 0, \tag{10.99}$$

$$\langle a_{n\sigma}a_{n'\sigma'}\rangle_0 = 0, \quad \langle a^*_{n\sigma}a^*_{n'\sigma'}\rangle_0 = 0, \tag{10.100}$$

$$\langle a_{n\sigma}a^*_{n'\sigma'}\rangle_0 = \delta_{nn'}\delta_{\sigma\sigma'}, \tag{10.101}$$

which implies that the field amplitudes corresponding to different modes are statistically independent, i.e., uncorrelated.

[6] It is considered in general that the minimal coupling of matter to the electromagnetic field is sufficient and that any further terms required should emerge from the minimal theory; see Section 12.7.1

[7] In principle, a vector field has three degrees of polarization. However, the longitudinal polarization is lost in this case because the electromagnetic field is massless.

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For the Hamiltonian H_R of the ZPF one obtains, using (10.96),

$$H_{\mathrm{R}} = \frac{1}{8\pi}\int \left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) d^3x = \tfrac{1}{2}\sum_{n,\sigma} \hbar\omega_n a^*_{n\sigma} a_{n\sigma}\,, \tag{10.102}$$

so that the average energy per mode is

$$\langle H_{\mathrm{R}}(\omega_n,\sigma)\rangle_0 = \tfrac{1}{2}\hbar\omega_n \langle a^*_{n\sigma} a_{n\sigma}\rangle = \tfrac{1}{2}\hbar\omega_n. \tag{10.103}$$

Using the Hamiltonian (10.94) one can derive the Hamilton equations for the particle in the usual way,

$$\dot{x}_i = \{x_i, H\}\,, \tag{10.104}$$

$$\dot{p}_i = \{p_i, H\}\,, \tag{10.105}$$

where the Poisson brackets are calculated not only with respect to the canonical variables of the particle, as in a mechanical system, but with respect to the entire set of canonical variables of both the particle and the field.[8] With H given by (10.94), the Hamilton equations lead to an equation of motion containing a time integral; if this is expanded as a series of time derivatives and truncated to the third order as usual, Eq. (10.81) is obtained in the dipole approximation.

10.4.2 The new kinematics

To analyze the change in the kinematics, we start from the Poisson bracket of the particle's canonical variables,

$$\left\{x_j, p_i\right\}_{xp} = \sum_k \left(\frac{\partial x_j}{\partial x_k}\frac{\partial p_i}{\partial p_k} - \frac{\partial p_i}{\partial x_k}\frac{\partial x_j}{\partial p_k}\right), \tag{10.106}$$

which satisfies

$$\left\{x_j, p_i\right\}_{xp} = \delta_{ij}, \tag{10.107}$$

where $i, j, k = 1, 2, 3$, and the variables and the derivatives are taken at the same time t.

The *full* set of canonical variables at any time comprises both those of the particle, $\{x_i;p_i\}$, and those of the field modes, $\{\mathrm{q}_\alpha;\mathrm{p}_\alpha\}$ (roman typography is used for the field's canonical variables to distinguish them from those of the particle, and a discrete set of field modes is considered for reasons that will become clear later), i.e.,

$$\{q;p\} = \{x_i, \mathrm{q}_\alpha; p_i, \mathrm{p}_\alpha\}\,. \tag{10.108}$$

[8] In classical theory, the phase space of a physical system consists of all the possible values of the (independent) position and momentum variables allowed by the system. It is naturally endowed with a Poisson bracket or symplectic form (see (10.107)), which allows one to formulate the Hamilton equations and describe the dynamics of the system through phase space in time. If the complete set of canonical variables is $\{q;p\}$, the Poisson bracket of any pair of functions $f(q,p,t), g(q,p,t)$ is

$$\{f,g\} = \sum_{q,p}\left[(\partial f/\partial q)\,(\partial g/\partial p) - (\partial f/\partial p)\,(\partial g/\partial q)\right]$$

where the sum extends over all possible values of (q,p). See also footnote 4.

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Note that a semicolon is used to distinguish a set of variables from a Poisson bracket. At the initial time t_o, when particle and field begin to interact, the canonical variables are

$$\{q_o;p_o\} = \{x_{io}, \mathrm{q}_{\alpha o};p_{io}, \mathrm{p}_{\alpha o}\}\,, \tag{10.109}$$

where x_{io}, p_{io} are the initial values of the particle's variables, and $\mathrm{q}_{\alpha o}, \mathrm{p}_{\alpha o}$ are those of the field. Since the whole system is Hamiltonian, the variables at times t_0 and t are related by a canonical transformation, and the particle's Poisson bracket at time t can be taken with respect to either set of variables,

$$\{x, p\}_{xp} = \{x, p\}_{x_o p_o} + \{x, p\}_{\mathrm{q}_{\alpha o}\mathrm{p}_{\alpha o}}\,. \tag{10.110}$$

Therefore, according to Eq. (10.107),

$$\big\{x_i(t), p_j(t)\big\}_{x_o p_o} + \big\{x_i(t), p_j(t)\big\}_{\mathrm{q}_{\alpha o}\mathrm{p}_{\alpha o}} = \delta_{ij}. \tag{10.111}$$

Now according to the discussion in Section 10.3, when in a stationary state, the particle has lost track of its initial conditions $\boldsymbol{x}_o, \boldsymbol{p}_o$ and responds resonantly to the action of the field. Then the first term in Eq. (10.111) vanishes,

$$\big\{x_i(t), p_j(t)\big\}_{q_o p_o} \to \big\{x_i(t), p_j(t)\big\}_{\mathrm{q}_{\alpha o}\mathrm{p}_{\alpha o}}\,, \tag{10.112}$$

and the particle's Poisson bracket becomes defined by the canonical variables of the field modes α with which it interacts,

$$\big\{x_i(t), p_j(t)\big\}_{\mathrm{q}_{\alpha o}\mathrm{p}_{\alpha o}} = \delta_{ij}. \tag{10.113}$$

Using the normal field amplitudes introduced in Eq. (10.96), related to $\mathrm{q}_{\alpha o}, \mathrm{p}_{\alpha o}$ by the transformation

$$\omega_\alpha \mathrm{q}_{\alpha o} = \sqrt{\hbar\omega_\alpha/2}(a_\alpha + a_\alpha^*),\ \mathrm{p}_{\alpha o} = -i\sqrt{\hbar\omega_\alpha/2}(a_\alpha - a_\alpha^*), \tag{10.114}$$

we write the Poisson bracket of two functions f, g with respect to a_α, a_α^*,

$$\{f, g\}_{aa^*} = \sum_\alpha \left(\frac{\partial f}{\partial a_\alpha}\frac{\partial g}{\partial a_\alpha^*} - \frac{\partial g}{\partial a_\alpha}\frac{\partial f}{\partial a_\alpha^*}\right) = i\hbar\{f, g\}_{q_{\alpha o}p_{\alpha o}}\,. \tag{10.115}$$

Applied to the particle's canonical variables, (10.115) reads

$$i\hbar\big\{x_i, p_j\big\}_{\mathrm{q}_{\alpha o}\mathrm{p}_{\alpha o}} = \big\{x_i, p_j\big\}_{aa^*}\,, \tag{10.116}$$

and therefore, according to Eq. (10.113), for the particle in a stationary state we get

$$\big\{x_i, p_j\big\}_{aa^*} = i\hbar\delta_{ij}. \tag{10.117}$$

This result indicates that the relation between the particle variables x_i and p_j becomes determined by their functional dependence on the normal field amplitudes $\{a_\alpha;a_\alpha^*\}$, with the scale given by Planck's constant. The symplectic structure is inherited from classical mechanics, but the meaning of the quantities x_i, p_j has changed, as we will see below.

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10.4.3 From dynamical variables to quantum operators

When the particle is subject solely to the ZPF, the only possible stationary state is the ground state $n = 0$; when there is an additional (external) field, e.g., a thermal field, the particle can reach an excited state $n > 0$. Given the general character of (10.117), it can be applied to any stationary state n. We therefore label the variables x_i, p_j with the subindex n, and write using (10.115) (limiting the discussion to the 1D case, for simplicity, and henceforth omitting the subindices aa^* from the Poisson bracket),

$$\{x, p\}_{nn} = \sum_{\alpha} \left(\frac{\partial x_n}{\partial a_\alpha} \frac{\partial p_n}{\partial a_\alpha^*} - \frac{\partial p_n}{\partial a_\alpha} \frac{\partial x_n}{\partial a_\alpha^*} \right) = i\hbar. \tag{10.118}$$

The constant value $i\hbar$ of this bilinear form implies that x_n and $p_n = m\dot{x}_n$ are linear functions of the set $\{a_\alpha; a_\alpha^*\}$. This means that the (electric component of the) field has taken control of the particle's behavior. The field modes involved are those to which the particle responds resonantly, i.e., those that can take it from state n to another state, say k, i.e., $\{a_\alpha; a_\alpha^*\} \to \{a_{nk}; a_{nk}^*\}$,

$$x_n(t) = \sum_k x_{nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.}, \qquad p_n(t) = \sum_k p_{nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.}, \tag{10.119}$$

where a_{nk} is associated with the field mode connecting state n with state k, and x_{nk}, $p_{nk} = -im\omega_{kn} x_{nk}$ are the respective dipolar response coefficients. Substituting these expressions into Eq. (10.118) gives

$$\{x, p\}_{nn} = 2im \sum_k \omega_{kn} |x_{nk}|^2 = i\hbar \tag{10.120}$$

for any n. Further, since the $a_{nk}, a_{n'k}$ connecting different states n, n' with state k are independent random variables, as mentioned in Section 10.4.1, by combining Eqs. (10.115) and (10.119) one obtains

$$\{x, p\}_{nn'} = i\hbar\delta_{nn'}. \tag{10.121}$$

The quantities x_{nk} and a_{nk} refer to the transition $n \to k$, while x_{kn} and a_{kn} refer to the inverse transition, with $\omega_{nk} = -\omega_{kn}$; therefore, from (10.119), $x_{nk}^*(\omega_{nk}) = x_{kn}(\omega_{kn})$, $p_{nk}^*(\omega_{nk}) = p_{kn}(\omega_{kn})$, $a_{nk}^*(\omega_{nk}) = a_{kn}(\omega_{kn})$, whence (10.121) takes the form

$$\sum_k (x_{nk} p_{kn'} - p_{n'k} x_{kn}) = i\hbar\delta_{nn'}. \tag{10.122}$$

The coefficients x_{nk} and p_{nk} can thus be identified as the elements of two matrices $\hat{x}$ and $\hat{p}$, respectively, with as many rows and columns as there are different states, and the Poisson bracket (10.121) transforms into the commutator

$$[\hat{x}, \hat{p}]_{nn'} = i\hbar\delta_{nn'}, \tag{10.123}$$

i.e., $[\hat{x}, \hat{p}] = i\hbar\mathbb{I}$, where $\mathbb{I}$ is the $n \times n$ unit matrix, or

$$[\hat{x}, \hat{p}] = i\hbar \tag{10.124}$$

for short. Equation (9.3) is thus identified with the Thomas-Reiche-Kuhn sum rule, Eq. (3.82).

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In conclusion, the classical phase-space variables x, p are replaced by the corresponding operators $\hat{x}, \hat{p}$, and the Poisson brackets are replaced by commutators,

$$\{x_i, p_j\} \Rightarrow \frac{1}{i\hbar}\left[\hat{x}_i, \hat{p}_j\right]. \tag{10.125}$$

The relationship between Poisson brackets and commutators, formally established by Dirac as a matter of fact, finds its physical basis here. The commutator (10.124) is no longer a postulate, as presented in Chapter 3, but an expression of the response of the particle to the field when the stationary regime is reached. Although the ZPF has explicitly disappeared from the picture – from the description of QM! – it has left Planck's constant $\hbar$ as an indelible mark. The vanishing of the ZPF means that although we are dealing with a causal phenomenon, the theory becomes non-causal, with unexplained quantum fluctuations. This is the origin of some of the QM conceptual problems mentioned in Chapter 1.

The matrix element x_{ln} represents the amplitude (or coefficient) of the *linear resonant response* of the particle to the field mode (ln) that takes it from state n to state l. This result is consistent with the quantum description and, as will be shown in Sections 12.2 and 17.2, is expressed in the selection rules for atomic dipole transitions, which depend directly on the coefficients x_{ln}.

It is important to note that while there is a formal one-to-one correspondence between variables and operators, they differ substantially in nature *and* physical meaning. Since $\hat{x}, \hat{p}$ do not describe particle trajectories, there is no phase space associated with them. Of course, this does not mean that trajectories have ceased to exist; they are simply not part of the quantum description. At the cost of a detailed dynamics in space-time, the Hilbert space formalism provides a compact and elegant description in terms of the elements connecting the quantum states.

10.4.4 Applying the quantum formalism

Equation (10.124) is used as the basis for the general transformation of dynamical variables into operators. In particular, it allows one to formulate the Heisenberg equations that govern the quantum dynamics in Hilbert space. Once x and p become operators, all dynamical variables $G(x, p; t)$ become operators $\widehat{G}(\widehat{x}, \widehat{p}; t)$ that act on the states, which are represented by column matrices.

Therefore, when applying Eq. (10.86) and the following to study the dynamics in the quantum regime, one must express them in terms of operators instead of c-variables, and replace the averages by expectation values. In particular, this means that the Liouvillian (radiationless) equation (10.87), which can be written as the average of Hamilton's equation (with $H = (\boldsymbol{p}^2/2m) + V$),

$$\langle \dot{G} \rangle = \left\langle \frac{\partial \mathcal{G}}{\partial t} \right\rangle + \langle \{G, H\} \rangle, \tag{10.126}$$

takes the form

$$\left\langle \dot{\hat{G}} \right\rangle = \left\langle \frac{\partial \hat{\mathcal{G}}}{\partial t} \right\rangle + i\hbar \left\langle \left[\hat{G}, \hat{H}\right] \right\rangle, \tag{10.127}$$

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where the angular bracket is to be interpreted as the expectation value of the Heisenberg equation. Experience shows that this replacement is usually straightforward, although occasionally some additional analysis is required to take account of the order of the (noncommuting) operators. A representative example is the calculation of the hydrogen Lamb shift in Section 17.4.4.

To recapitulate, quantum mechanics – whether in the Heisenberg or any other representation – is obtained by taking the radiationless approximation of the SED statistical equations in the Markovian limit, which implies a coarse graining over time. For very short times there is no guarantee that the quantum equations hold. The stationary quantum regime is ensured by a general fluctuation-dissipation relation, which expresses the balanced effect of the radiation reaction and the zero-point-field-induced stochasticity. In this regime the operators $\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}$ represent the resonant response of the particle to the field, which induces transitions between states, and all dynamical variables become functions of these operators. The classical Hamiltonian structure is preserved by the quantum formalism in terms of operators. The radiative terms in the SED equations contain valuable additional information that takes us beyond QM.

10.5 Hydrodynamic Quantum Analogs

We have seen in Section 10.1 that SQM leads to a differential equation that is parabolic (and dissipative) in the classical case of Brownian motion and hyperbolic (wave-like and conservative) in the quantum case. Furthermore, we have just seen that quantization can be understood as the result of the balancing action of the background field, which compensates for the particle's radiation. The fact that this underlying physical field, which acts on the particles and mediates between them, is hidden from the quantum description introduces *apparent* nonlocalities that permeate the whole of quantum theory.

It turns out that a similar transition from a classical dynamics to one that reproduces characteristic quantum features, has been observed on a macroscopic scale in experiments with oil droplets bouncing on a vibrating bath of the same oil. Under certain conditions the droplet can self-propel due to its interaction with the traveling waves it generates. As the walking droplet bounces repeatedly off the surface of the oil bath, the sequence of waves generated is superimposed. In the high-memory regime (close to the Faraday threshold, beyond which standing waves form even in the absence of the drop), the resulting waves decay very slowly in time, so the trajectory of the droplet is significantly influenced by waves accumulated since the distant past, resulting in memory-induced wave-like particle statistics. From the perspective of SED, the system has a suggestive quantum similarity, with the advantage that the droplet trajectories can be visualized, and the origin of the apparent nonlocalities can be understood as arising from the (local) wave-particle interaction.

This topic has given rise to an attractive and vibrant new field of research known as *hydrodynamic quantum analogs* (HQA). Some relevant references are given at the end of the chapter. To illustrate the similarities, below is a brief description of an experiment with circularly orbiting droplets carried out by Fort et al. (2010).

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In this experiment the oil is contained in a small cylindrical cell (150 mm diameter) which is vibrated vertically and rotated about a central vertical axis at an angular velocity Ω. When the cell is set in rotation, the walker experiences a Coriolis force. Seen from the rotating frame, the droplet trajectory is a circular orbit. For long-term memory, as Ω is increased, the radius of the orbit undergoes transitions between successive values and a slight increase in velocity is observed as the orbits become smaller. The result is that each droplet describes a circular trajectory with a radius that can have any of the values given by

$$R_n \simeq \frac{1}{2}\left(n+\frac{1}{2}\right)\lambda_F, \tag{10.128}$$

where $n = 0, 1, 2, 3, \ldots$ and λ_F corresponds to the Faraday wavelength of the oil bath.

In QM, the motion of a charge moving in a plane and subject to a homogeneous magnetic field perpendicular to the plane is quantified in Landau levels (see Exercise 9.33). In the case of circular orbits, the charge can orbit in equidistant rings where the radius takes discrete values given by

$$r_n = \frac{1}{\pi}\left(n+\frac{1}{2}\right)\lambda, \tag{10.129}$$

and λ is the de Broglie wavelength.

While the classical Faraday and the quantum de Broglie pilot-wave systems are of completely different scales, their theoretical descriptions differ only in minor details and the mathematical formulae (10.128), (10.129) have similar structure and the same meaning.

How far analogies like this one can be taken remains to be seen. However, to the more or less "obvious" analogy we must add a profound fundamental kinship. As discussed in several places in this book, with the introduction of the ZPF, QM becomes a causal local theory of nature; it becomes a true matter-field theory, just like the theory of the walking droplets, where quantum – or "apparent" quantum – behavior arises through the action of the immediate field on the particle, as modulated by the particle's own motion.

Problems

P10.1 Two very thin metal plates, 10 Å on each side and initially neutral, are 20 Å apart. Calculate the number of electrons that need to be added to each plate for the Coulomb repulsion to counteract the Casimir attraction.

P10.2 Construct the general solution $x(t)$ for the HO immersed in the ZPF with arbitrary initial conditions, and show that the stationary solution (for $t \to \infty$) is given approximately by Eq. (10.56).

P10.3 Derive in detail the autocorrelation function and the variance of $p(t)$ for the stationary HO.

P10.4 Show that the definitions used in early SED give for the dispersion of the ground state energy of the HO $\sigma_H^2 = E_0^2$.

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P10.5 Stationary white noise is so named because its spectral energy density is a constant, the same for all frequencies. Show that the autocorrelation function of a white noise is $\langle E(t')\,E(t)\rangle = A\delta(t'-t)$ and express the value of the constant A in terms of the power density.

P10.6 Consider the 1D infinite square well of width a. a) Using the eigenfunctions obtained in Exercise 4.4, determine the transition frequencies ω_{nk} and the corresponding response coefficients (matrix elements) x_{kn} for $n=1$. b) Carry out the same calculations for $n=2$. c) Make a drawing of the spectral absorption bands obtained for the two cases.

Bibliographical Notes

The standard reference on SQM is Nelson (1966). See also de la Peña (1969) and Nelson (2012).

For the distinction between Brownian motion and QM as two stochastic processes, see de la Peña and Cetto (1975, 1982).

For a simulation of Brownian motion, see https://en.wikipedia.org/wiki/Brownian_motion#/media/File:Brownian_motion_large.gif.

For the introduction of the quantum potential see Bohm (1951).

For a detailed discussion of causality and nonlocality in the context of Bohm's theory, see de la Peña, Cetto, and Valdés-Hernández (2015), Chapter 8.

To familiarize the reader with the theory of stochastic processes, in particular with the Fokker-Planck equation, we recommend van Kampen (1981) and Risken (1996).

To learn about classical electrodynamics, we recommend Zangwill (2012), a comprehensive textbook that emphasizes physics without sacrificing mathematical rigor.

For pioneering work on SED, see Marshall (1963, 1965), Santos (1968), and Boyer (1979). For a more complete exposition, which includes some aspects of stochastic optics, see de la Peña, and Cetto (1996).

On more recent developments in SED and its connection with SQM and QM, see de la Peña, Cetto, and Valdés-Hernández (2015). In Cetto, de la Peña, and Valdés-Hernández (2021) and de la Peña and Cetto (2024) the transition to quantum operators is explained in detail.

For the hydrodynamic quantum analogs and the connection with de Broglie's pilot wave, see Bush (2015) and Bush and Oza (2020). For the experiment described in Section 5, see Fort et al. (2010).

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