Combinatorics, Probability and Computing (2004) 13, 911–912. © 2004 Cambridge University Press
DOI: 10.1017/S0963548304006509 Printed in the United Kingdom

PROBLEM SECTION

Ramanujan Signing of Regular Graphs

YONATAN BILU AND NATHAN LINIAL

Institute of Computer Science, Hebrew University of Jerusalem, Jerusalem 91904, Israel
(e-mail: {johnblue,nati}@cs.huji.ac.il)

Received 10 June 2004

A *signing* of a graph $G = (V, E)$ is a function $s : E \to \{-1, 1\}$. A signing defines a graph $\widehat{G}$, called a *2-lift of* G, with vertex set $V(G) \times \{-1, 1\}$. The vertices (u, x) and (v, y) are adjacent iff $(u, v) \in E(G)$, and $x \cdot y = s(u, v)$. The corresponding *signed adjacency matrix* $A_{G,s}$ is a symmetric $\{-1, 0, 1\}$-matrix, with $(A_{G,s})_{u,v} = s(u, v)$ if $(u, v) \in E$, and 0 otherwise.

Let G be a graph and let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of G's adjacency matrix A_G. We also refer to $\lambda_1, \ldots, \lambda_n$ as the *spectrum* of G or the spectrum of A_G. It is easy to see that if G is d-regular, then $\lambda_1 = d$. The *spectral radius* of G is $\lambda(G) = \max\{\lambda_2, |\lambda_n|\}$. We recall the Alon–Boppana bound [5] that $\lambda(G) \geqslant 2\sqrt{d-1} - o(1)$. Here the $o(1)$ goes to 0 as the number of vertices grows. *Expanders* are graphs for which $\lambda(G)$ is small. For a survey of these fascinating graphs and their role in theoretical computer science, see [2].

It is a celebrated result of Margulis and Lubotzky, Phillips and Sarnak [3, 4], that for d a prime power plus one, arbitrarily large d-regular graphs can be constructed with $\lambda(G) \leqslant 2\sqrt{d-1}$. The above constructions are based on deep results concerning certain number-theoretic conjectures by Ramanujan. Therefore, such graphs are called *Ramanujan*.

Our hope is to find an elementary construction of Ramanujan graphs. In addition, it remains open whether large d-regular Ramanujan graphs exist for every $d \geqslant 3$, and an elementary approach may be able to resolve this question. In a recent paper [1] we have suggested an approach to this problem, using 2-lifts. A key observation is the following easy exercise.

Lemma 1. *Let s be a signing of a d-regular graph, G, and let $\widehat{G}$ the resulting 2-lift. Then $\widehat{G}$ is also d-regular, and its spectrum is the union of the spectrum of A_G and that of $A_{G,s}$.*

Consider the following scheme for constructing Ramanujan graphs. Start with $G_0 = K_{d+1}$, the complete graph on $d + 1$ vertices, or any other small d-regular Ramanujan

https://doi.org/10.1017/S0963548304006509 Published online by Cambridge University Press

graph. (Note that small Ramanujan graphs are easy to come by.) We make the following conjecture.

Conjecture 2. *Every d-regular Ramanujan graph G has a signing s with spectral radius $\rho(A_{G,s}) \leqslant 2\sqrt{d-1}$.*

Assuming this to be true, define successively G_i as a 2-lift of G_{i-1}, obtained from a signing s of G_{i-1} that satisfies the conjecture. Then these G_i would constitute an infinite family of d-regular Ramanujan graphs.

In fact, we make the even more daring:

Conjecture 3. **Every** *d-regular graph G has a signing s with spectral radius $\rho(A_{G,s}) \leqslant 2\sqrt{d-1}$.*

We note that by the Alon–Boppana bound, the conjecture cannot in general be improved. In [1] we establish the somewhat weaker result:

Theorem 4. *Every graph G of maximal degree d has a signing s with spectral radius $\rho(A_{G,s}) = O(\sqrt{d \cdot \log^3 d})$.*

If indeed the conjectures hold, then it would be very interesting to develop efficient algorithms that find a good signing s for a given G.

References

[1] Bilu, Y. and Linial, N. Constructing expander graphs by 2-lifts and discrepancy vs. spectral gap. Submitted.
[2] Linial, N. and Wigderson, A. Expander graphs and their applications. Lecture notes available at http://www.math.ias.edu/~boaz/ExpanderCourse/
[3] Lubotzky, A. Phillips, R. and Sarnak, P. (1988) Ramanujan graphs. *Combinatorica* **8** 261–277.
[4] Margulis, G. A. (1988) Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii* **24** 51–60.
[5] Nilli, A. (1991) On the second eigenvalue of a graph. *Discrete Math.* **91** 207–210.

https://doi.org/10.1017/S0963548304006509 Published online by Cambridge University Press