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On operator equations and variational principles

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Abstract. A necessary and sufficient condition for operator equations $T\phi = M(u, \phi)$, $T^*u = N(u, \phi)$ to arise from a variational principle is presented.

1. *Introduction.* Let H_u and H_ϕ be real Hilbert spaces with inner products $(\,,\,)$ and $\langle\,,\rangle$ respectively. The adjoint of a linear operator T: $H_\phi \to H_u$ is defined to be the linear operator T^*: $H_u \to H_\phi$ such that

$$(u, T\phi) = \langle T^*u, \phi\rangle \tag{1·1}$$

for all $u \in H_u$, $\phi \in H_\phi$. The Cartesian product space $H = H_u \times H_\phi$ has elements $h = \begin{pmatrix} u \\ \phi \end{pmatrix}$ and is a Hilbert space for the inner product $\{\,,\}$ defined by

$$\{h_1, h_2\} = (u_1, u_2) + \langle \phi_1, \phi_2\rangle,$$

where

$$h_i = \begin{pmatrix} u_i \\ \phi_i \end{pmatrix} \quad (i = 1, 2).$$

Let $M(u, \phi) \in H_u$ and $N(u, \phi) \in H_\phi$ be given differentiable vector functions of u and ϕ. Then we consider the pair of equations

$$T\phi = M(u, \phi), \tag{1·2}$$

$$T^*u = N(u, \phi) \tag{1·3}$$

and ask when these equations arise from a variational principle. The answer is given by the

Theorem. *Equations* (1·2) *and* (1·3) *arise from a variational principle if and only if*

$$\left(\frac{\partial M}{\partial \phi}\right)^* = \frac{\partial N}{\partial u}. \tag{1·4}$$

Here derivatives denote Fréchet derivatives, and $\partial M/\partial\phi$: $H_\phi \to H_u$, $\partial N/\partial u$: $H_u \to H_\phi$.

2. *Proof of theorem.* First we rewrite equation (1·4) by noting that

$$M(u, \phi) = \frac{\partial}{\partial u}\int_0^1 (u - u_0, M\{u_0 + t(u - u_0), \phi\})\, dt, \tag{2·1}$$

using direct integration (3) where $u_0 \in H_u$ is an arbitrary function, and so

$$\left(\frac{\partial M}{\partial \phi}\right)^* = \frac{\partial}{\partial u}\frac{\partial}{\partial \phi}\int_0^1 (u - u_0, M\{u_0 + t(u - u_0), \phi\})\, dt. \tag{2·2}$$

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Hence (1·4) becomes

$$\frac{\partial}{\partial u}\left[N(u,\phi)-\frac{\partial}{\partial\phi}\int_0^1 (u-u_0, M\{u_0+t(u-u_0),\phi\})\,dt\right]=0. \tag{2·3}$$

2·1. Suppose that (1·4), that is (2·3), holds. Then

$$P(\phi)=N(u,\phi)-\frac{\partial}{\partial\phi}\int_0^1 (u-u_0, M\{u_0+t(u-u_0),\phi\})\,dt \tag{2·4}$$

is a function of ϕ only, and we can define on H the functional

$$W(u,\phi)=\int_0^1 (u-u_0, M\{u_0+t(u-u_0),\phi\})\,dt+\int_0^1 \langle \phi-\phi_0, P\{\phi_0+t(\phi-\phi_0)\}\rangle\,dt, \tag{2·5}$$

where $\phi_0\in H_\phi$ is an arbitrary function. It follows that

$$\frac{\partial W}{\partial u}=M(u,\phi), \tag{2·6}$$

and

$$\begin{aligned}\frac{\partial W}{\partial\phi}&=\frac{\partial}{\partial\phi}\int_0^1 (u-u_0, M\{u_0+t(u-u_0),\phi\})\,dt+P(\phi)\\&=N(u,\phi), \quad \text{by (2·4).}\end{aligned} \tag{2·7}$$

From (2·6) and (2·7), equations (1·2) and (1·3) may be written

$$T\phi=\frac{\partial W}{\partial u}, \tag{2·8}$$

$$T^*u=\frac{\partial W}{\partial\phi}, \tag{2·9}$$

and these arise from a variational principle for the potential

$$I(u,\phi)=(u,T\phi)-W(u,\phi), \tag{2·10}$$

$$=\langle T^*u,\phi\rangle-W(u,\phi). \tag{2·11}$$

2·2. Now suppose that (1·2) and (1·3) arise from a variational principle, which in turn means that there is a $W(u,\phi)$ such that

$$M(u,\phi)=\frac{\partial W}{\partial u}, \tag{2·12}$$

$$N(u,\phi)=\frac{\partial W}{\partial\phi}. \tag{2·13}$$

Then since second derivatives are symmetric (3), it follows that

$$\left(u_1,\frac{\partial^2 W}{\partial u\,\partial\phi}\phi_1\right)=\left\langle\frac{\partial^2 W}{\partial\phi\,\partial u}u_1,\phi_1\right\rangle, \quad \text{for all } u_1\in H_u,\ \phi_1\in H_\phi, \tag{2·14}$$

where $\partial^2 W/\partial u\,\partial\phi\equiv(\partial/\partial\phi)(\partial W/\partial u)$. Using (2·12) and (2·13) in (2·14) we obtain

$$\left(u_1,\frac{\partial M}{\partial\phi}\phi_1\right)=\left\langle\frac{\partial N}{\partial u}u_1,\phi_1\right\rangle, \tag{2·15}$$

or

$$\left\langle\left(\frac{\partial M}{\partial\phi}\right)^* u_1,\phi_1\right\rangle=\left\langle\frac{\partial N}{\partial u}u_1,\phi_1\right\rangle, \tag{2·16}$$

which gives

$$\left(\frac{\partial M}{\partial\phi}\right)^*=\frac{\partial N}{\partial u} \tag{2·17}$$

as required.

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3. *Applications.* 3.1. This theorem can be applied to the operator equation

$$T^*T\phi = E(\phi) \tag{3·1}$$

in H_ϕ. Following Rall (2), we decompose (3·1) into the pair of equations

$$T\phi = u + S(u, \phi), \tag{3·2}$$

$$T^*u = E(\phi) - T^*S(u, \phi), \tag{3·3}$$

which are of the form (1·2) and (1·3) with

$$M(u, \phi) = u + S(u, \phi), \tag{3·4}$$

$$N(u, \phi) = E(\phi) - T^*S(u, \phi). \tag{3·5}$$

It follows from the foregoing theorem that (3·2) and (3·3) will arise from a variational principle if and only if (1·4), or equivalently (2·3), holds with M and N given by (3·4) and (3·5). This means that

$$\frac{\partial}{\partial u}\left[T^*S(u, \phi) + \frac{\partial}{\partial \phi}\int_0^1 (u - u_0, S\{u_0 + t(u - u_0), \phi\})\, dt\right] = 0, \tag{3·6}$$

which for $u_0 = 0$ is the condition derived by Rall ((2), Theorem 8·1).

In many applications to equations of the form (3·1) it is possible to take $S(u, \phi) = 0$ (see (1)), in which case condition (1·4), or (3·6), is satisfied trivially.

3·2. As a second application, we consider the case

$$M(u, \phi) = f(u) + A\phi, \tag{3·7}$$

$$N(u, \phi) = Bu + g(\phi), \tag{3·8}$$

where $f(u) \in H_u$, $g(\phi) \in H_\phi$, $A: H_\phi \to H_u$, and $B: H_u \to H_\phi$. Then condition (1·4) becomes

$$A^* = B. \tag{3·9}$$

If this is satisfied, the equations $T\phi = M$, $T^*u = N$ arise from a variational principle, and the functional $W(u, \phi)$ in (2·10) and (2·11) is found from (2·5) to be

$$W(u, \phi) = \int_0^1 (u - u_0, f\{u_0 + t(u - u_0)\})\, dt + (u, A\phi) + \int_0^1 \langle \phi - \phi_0, g\{\phi_0 + t(\phi - \phi_0)\}\rangle\, dt, \tag{3·10}$$

where u_0 and ϕ_0 are arbitrary functions.

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