Teaching Notes

Studying $\dfrac{A}{x+B}+\dfrac{C}{x+D}$ without derivatives

How do we 'sell' derivative functions to a calculus student? Probably, for many of us, it would be as a tool to analyse the variations of a function, although we do not need them in some elementary cases. The aim of this Note is to propose an idea to study the functions of the form

$$\frac{A}{x+B}+\frac{C}{x+D}$$

without using derivatives. It can be used to

- illustrate a calculus course,
- warn against a systematic use of derivatives,
- promote the search for nice algebraic tricks, **far from any formal theory**, when solving a specific problem.

If A and C have the same sign, the problem is trivial as both terms $\dfrac{1}{x+B}$ and $\dfrac{1}{x+D}$ are decreasing functions. In the case where A and C do not have the same sign, as an example, let us study the function

$$F(x)=\frac{3}{x-1}-\frac{2}{x}.$$

The same ideas can be used with any other value of A, B, C, D.

First, let us study the auxiliary function

$$G_\alpha(x)=x+\frac{\alpha}{x}.$$

This problem is trivial if $\alpha \leqslant 0$, as G_α is then a sum of increasing functions. When $\alpha > 0$, we just have to see that, for $x > 0$

$$G_\alpha(x)=\left(\sqrt{x}-\frac{\sqrt{\alpha}}{\sqrt{x}}\right)^2+2\sqrt{\alpha}.$$

In Figure 1, we plotted the graph of the increasing function

$$\sqrt{x}-\frac{\sqrt{\alpha}}{\sqrt{x}}.$$

From this graph, we can deduce the graph of G_α for $x > 0$ and, consequently, for $x < 0$ (as G_α is odd).

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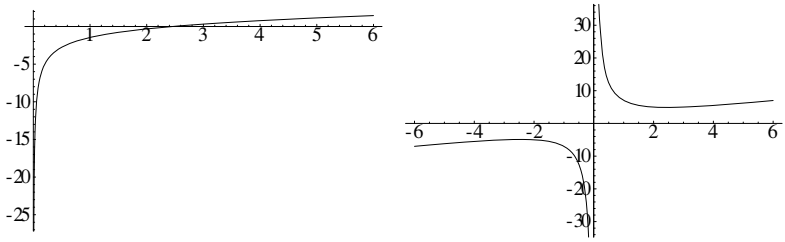


FIGURE 1: the function $\sqrt{x} - \frac{\sqrt{\alpha}}{\sqrt{x}}$ (on the left) and the function G_α (on the right), $\alpha = 6$

In particular, we can see that $G_\alpha(x)$ has a minimum value of $2\sqrt{\alpha}$ on $(0, \infty)$ (which is achieved when $x = \sqrt{\alpha}$) and a maximum value of $-2\sqrt{\alpha}$ on $(-\infty, 0)$ (which is achieved when $x = -\sqrt{\alpha}$). This follows directly from our last expression of this function on $(0, \infty)$. It is also a consequence of the AM–GM inequality applied to x and α/x, namely

$$\frac{x + \frac{\alpha}{x}}{2} \geqslant \sqrt{x \cdot \frac{\alpha}{x}} = \sqrt{\alpha},$$

with equality when $x = \alpha/x$. It should be noticed that the standard calculus method for this sort of function is also very quick and straightforward.

Moreover, using this result, we can obtain the variations of the function

$$\cosh x = \frac{G_1(e^x)}{2},$$

without computing a derivative. But let us go back to F. We have

$$F(x) = \frac{3}{x-1} - \frac{2}{x} = \frac{\boldsymbol{x+2}}{x^2 - x} = \frac{x+2}{((\boldsymbol{x+2}) - 2)^2 - ((\boldsymbol{x+2}) - 2)}$$
$$= \frac{x+2}{(x+2)^2 - 5(x+2) + 6} = \frac{1}{(x+2) + \frac{6}{x+2} - 5} = \frac{1}{G_6(x+2) - 5}.$$

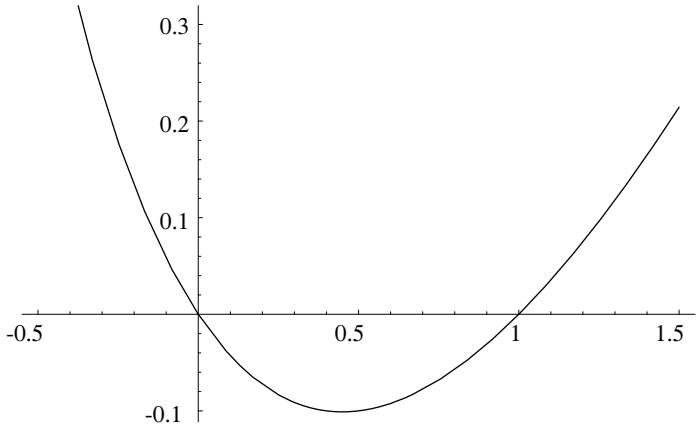


FIGURE 2: the function $1/F$

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Applying the translation by vector $(-2, -5)$ to the graph of G_6, we obtain the graph of $1/F$, which vanishes at 0 and 1 and tends to infinity at -2 (Figure 2). From this graph, we easily deduce the graph of F (Figure 3).

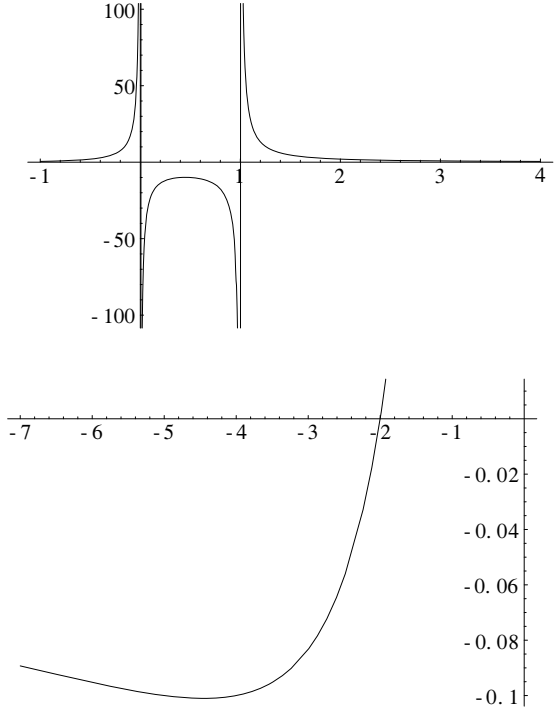


FIGURE 3: the function F

Using the same technique, we can study more general functions

$$\frac{Ax^2 + Bx + C}{Dx^2 + Ex + F}.$$

By long division they can be expressed in the form

$$P + \frac{u}{Qu^2 + Ru + S} = P + \frac{1}{Qu + R + \frac{S}{u}},$$

where u is the linear numerator after the long division.

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MARTIN CELLI

Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Av. San Rafael Atlixco, 186. Col. Vicentina. Del. Iztapalapa. CP 09340. México, D.F.

e-mail: *cell@xanum.uam.mx*

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