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THE SQUARE TERMS IN GENERALIZED LUCAS SEQUENCES

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Abstract. Let P and Q be non-zero integers. The generalized Fibonacci sequence $\{U_n\}$ and Lucas sequence $\{V_n\}$ are defined by $U_0 = 0$, $U_1 = 1$ and $U_{n+1} = PU_n + QU_{n-1}$ for $n \geqslant 1$ and $V_0 = 2$, $V_1 = P$ and $V_{n+1} = PV_n + QV_{n-1}$ for $n \geqslant 1$, respectively. In this paper, we assume that $Q = 1$. Firstly, we determine indices n such that $V_n = kx^2$ when $k|P$ and P is odd. Then, when P is odd, we show that there are no solutions of the equation $V_n = 3\square$ for $n > 2$. Moreover, we show that the equation $V_n = 6\square$ has no solution when P is odd. Lastly, we consider the equations $V_n = 3V_m\square$ and $V_n = 6V_m\square$. It has been shown that the equation $V_n = 3V_m\square$ has a solution when $n = 3$, $m = 1$, and P is odd. It has also been shown that the equation $V_n = 6V_m\square$ has a solution only when $n = 6$. We also solve the equations $V_n = 3\square$ and $V_n = 3V_m\square$ under some assumptions when P is even.

§1. *Introduction.* Let P and Q be non-zero integers. The generalized Fibonacci sequence $\{U_n\}$ and Lucas sequence $\{V_n\}$ are defined as follows:

$$U_0 = 0,\ U_1 = 1 \text{ and } U_{n+1} = PU_n + QU_{n-1} \text{ for } n \geqslant 1$$

and

$$V_0 = 2,\ V_1 = P \text{ and } V_{n+1} = PV_n + QV_{n-1} \text{ for } n \geqslant 1,$$

respectively. U_n and V_n are called the nth generalized Fibonacci number and nth generalized Lucas number, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = \frac{-U_n}{(-Q)^n} \quad \text{and} \quad V_{-n} = \frac{V_n}{(-Q)^n}, \tag{1.1}$$

respectively.

Now assume that $P^2 + 4Q \neq 0$. Then it is well known that

$$U_n = U_n(P, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = V_n(P, Q) = \alpha^n + \beta^n, \tag{1.2}$$

where $\alpha = (P + \sqrt{P^2 + 4Q})/2$ and $\beta = (P - \sqrt{P^2 + 4Q})/2$, which are the roots of the characteristic equation $x^2 - Px - Q = 0$.

For $P = Q = 1$, we have the classical Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$. For $P = 2$ and $Q = 1$, we have the Pell and Pell–Lucas sequences $\{P_n\}$

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and $\{Q_n\}$. For more information about the generalized Fibonacci and Lucas sequences one can consult [7, 12, 14, 17].

There has been much interest in when the terms of such sequences are perfect squares ($=\square$) or $k\square$. We give a summary here of results on this problem. Firstly, in [9], Ljunggren showed that for $n \geqslant 2$, P_n is a perfect square precisely for $P_7 = 13^2$, and $P_n = 2\square$ precisely for $P_2 = 2$. In [2], Cohn solved the Diophantine equations $F_n = \square, 2\square$ and $L_n = \square, 2\square$. In [3, 4], same author solved the same problems with odd P and $Q = \pm 1$. Moreover, in [15], Ribenboim and McDaniel determined all indices n such that for all odd relatively prime integers P and Q, $U_n, 2U_n, V_n$ or $2V_n$ is a square. Besides, in [11], Mignotte and Pethő showed that if $P \geqslant 3$ and $Q = -1$, then the equation $U_n = \square$ has the solutions $(P, n) = (338, 4)$ or $(3, 6)$ for $n \geqslant 3$, and that if $P \geqslant 4$ and $Q = -1$, then the equation $U_n = w\square$, $w \in \{2, 3, 6\}$, has no solutions for $n \geqslant 4$. In [13], Nakamula and Pethő gave the solutions of the equations $U_n = w\square$ for $Q = 1$ with $w \in \{1, 2, 3, 6\}$. Moreover, in [16], Ribenboim and McDaniel showed that if P is even, $Q \equiv 3 \pmod 4$, and $U_n = \square$, then n is a square or twice an odd square, and all prime factors of n divide $P^2 + 4Q$. Also, in [18], they solved the equation $V_n = 3\square$ for $P \equiv 1, 3 \pmod 8$, $Q \equiv 3 \pmod 4$, $(P, Q) = 1$, and the equation $U_n = 3\square$ for $(P, Q) = 1$. In [6], Kagawa and Terai showed that if $P = 2s$ with even s and $Q = 1$, then $U_n, 2U_n, V_n$ or $2V_n = \square$ implies $n \leqslant 3$ under the following assumptions: for prime q and an integer $s \neq 0$, let $e_q(s)$ be the integer such that $q^{e_q(s)}$ exactly divides s, and such that $e_q(s)$ is odd for $q = 3, 5$ or 7. Moreover, in [8], Keskin and Yosma give the solutions of the equations $F_n = 2F_m\square$, $L_n = 2L_m\square$, $F_n = 3F_m\square$, $F_n = 6F_m\square$, and $L_n = 6L_m\square$.

More generally, we can recall the following theorem proved by Shorey and Stewart in [19].

Let $A > 0$ be an integer; then there exists an effectively computable number $C > 0$, which depends on A, such that if $n > 0$ and $U_n = A\square$ or $V_n = A\square$ then $n < C$.

In this paper, we assume that $Q = 1$ and $P > 1$. We will use the Jacobi symbol $(\frac{a}{m})$ throughout this study. Our method is elementary and used by Cohn *et al* in [3, 18].

§2. *Preliminaries.* In this section, we give some theorems, lemmas, and identities about generalized Fibonacci and Lucas numbers, which will be needed later.

Firstly, we give some theorems and lemmas. Since the proof of the following lemma can be proved by induction, we omit it.

LEMMA 1. *If* $3 \nmid P$, *then*

$$V_{2^r} \equiv \begin{cases} 2 \pmod 3 & \text{if } r \geqslant 3, \\ 1 \pmod 3 & \text{if } r = 2, \\ 0 \pmod 3 & \text{if } r = 1, \end{cases}$$

and if $3 | P$, *then* $V_{2^r} \equiv 2 \pmod 3$ *for all positive integers* r.

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The proofs of the following two theorems are given in [20].

THEOREM 2.1. *Let $n \in \mathbb{N}\cup\{0\}$, $m, r \in \mathbb{Z}$ and m be a non-zero integer. Then*

$$U_{2mn+r} \equiv (-1)^{mn} U_r \pmod{U_m} \tag{2.1}$$

and

$$V_{2mn+r} \equiv (-1)^{mn} V_r \pmod{U_m}. \tag{2.2}$$

THEOREM 2.2. *Let $n \in \mathbb{N}\cup\{0\}$ and $m, r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^{(m+1)n} U_r \pmod{V_m} \tag{2.3}$$

and

$$V_{2mn+r} \equiv (-1)^{(m+1)n} V_r \pmod{V_m}. \tag{2.4}$$

When P is odd, using (2.1) and (2.2) we get

$$U_{12q+r} \equiv U_r \pmod 8 \tag{2.5}$$

and

$$V_{12q+r} \equiv V_r \pmod 8. \tag{2.6}$$

Thus

$$8 \nmid V_n. \tag{2.7}$$

The following lemma can be found in [18].

LEMMA 2. *Let P be odd, m be an odd positive integer, and $r \geqslant 1$. Then*

$$V_{2^r m} \equiv \begin{cases} 2 \pmod 8 & \text{if } 3|m, \\ 3 \pmod 8 & \text{if } 3\nmid m \text{ and } r = 1, \\ 7 \pmod 8 & \text{if } 3\nmid m \text{ and } r > 1. \end{cases}$$

By the above lemma, when P is odd, it can be seen that

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \tag{2.8}$$

for $r \geqslant 1$.

Secondly, we give some identities concerning generalized Fibonacci and Lucas numbers.

$$U_{2n} = U_n V_n, \tag{2.9}$$

$$V_{2n} = V_n^2 - 2(-1)^n, \tag{2.10}$$

$$U_{3n} = U_n((P^2+4)U_n^2 + 3(-1)^n) = U_n(V_n^2 - (-1)^n), \tag{2.11}$$

$$V_{3n} = V_n(V_n^2 - 3(-1)^n), \tag{2.12}$$

$$\text{if } P \text{ is odd, then } 2|V_n \Leftrightarrow 2|U_n \Leftrightarrow 3|n, \tag{2.13}$$

$$V_n^2 - (P^2+4)U_n^2 = 4(-1)^n. \tag{2.14}$$

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Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geqslant 0$, and $d = (m, n)$. Then

$$(U_n, U_m) = U_d, \tag{2.15}$$

$$(U_m, V_n) = \begin{cases} V_d & \text{if } a > b, \\ 1 \text{ or } 2 & \text{if } a \leqslant b, \end{cases} \tag{2.16}$$

$$(V_m, V_n) = \begin{cases} V_d & \text{if } a = b, \\ 2 & \text{if } a \neq b \text{ and } P \text{ is even}, \\ 1 \text{ or } 2 & \text{if } a \neq b \text{ and } P \text{ is odd}, \end{cases} \tag{2.17}$$

$$\text{if } U_m \neq 1, \text{ then } U_m | U_n \Leftrightarrow m | n, \tag{2.18}$$

$$\text{if } V_m \neq 1, \text{ then } V_m | V_n \Leftrightarrow m | n \text{ and } \frac{n}{m} \text{ is odd}, \tag{2.19}$$

$$V_m | U_n \Leftrightarrow m | n \text{ and } \frac{n}{m} \text{ is even}, \tag{2.20}$$

$$\text{if } P \text{ is odd, } 2 | m, \text{ and } 3 \nmid m, \text{ then } \left(\frac{V_3}{V_m}\right) = \left(\frac{-2}{P}\right), \tag{2.21}$$

$$\text{if } r > 1, \text{ then } \left(\frac{U_3}{V_{2^r}}\right) = 1, \tag{2.22}$$

$$\text{if } r \geqslant 3, \text{ then } V_{2^r} \equiv 2 \pmod{V_2}. \tag{2.23}$$

Moreover, when P is even, it can be easily seen that

$$U_n \text{ is odd } \Leftrightarrow n \text{ is odd},$$
$$U_n \text{ is even } \Leftrightarrow n \text{ is even},$$
$$V_n \text{ is even for all } n \in \mathbb{N}.$$

Identities between (2.9)–(2.14) and (2.15)–(2.20) can be found in [**17, 18, 20**] and [**10, 16, 18**], respectively. Identities (2.21) and (2.22) are given in [**3**] and [**18**], respectively. The proofs of the others are easy and will be omitted.

From now on, we assume that $n \geqslant 1$.

§3. *Main theorems.* In [**5**], Cohn solved the equations $V_n = V_m x^2$, $V_n = 2V_m x^2$ when P is odd. Here we will solve the equation $V_n = 2V_m x^2$ when P is even.

We begin with two theorems which can be obtained from Theorems 11 and 12 given in [**5**].

THEOREM 3.1. *Let P be an odd integer, $m \geqslant 1$ be an integer, and $V_n = V_m x^2$ for some integer x. Then $n = m$.*

THEOREM 3.2. *If P is odd, then there is no integer x such that $V_n = 2V_m x^2$ for $m \geqslant 1$.*

The following lemma can be proved by induction.

LEMMA 3. *If n is even, then $V_n \equiv 2 \pmod{P^2}$, and if n is odd, then $V_n \equiv nP \pmod{P^2}$.*

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THEOREM 3.3. *If P is even, then there is no integer x such that $V_n = 2V_m x^2$ for $m \geqslant 1$.*

Proof. Assume that P is even and $V_n = 2V_m x^2$. Since V_m is even, we get $4|V_n$. Then it is seen that n is odd by Lemma 3. Moreover, since $V_m|V_n$, there exists an odd integer t such that $n = mt$ by (2.19). Thus m is odd. Therefore we have $V_n \equiv nP \pmod{P^2}$ and $V_m \equiv mP \pmod{P^2}$ by Lemma 3. Thus $nP \equiv 2mPx^2 \pmod{P^2}$, i.e., $n \equiv 2mx^2 \pmod{P}$. This is a contradiction since n is odd. This completes the proof. □

The proof of the following lemma can be seen from identity (2.21).

LEMMA 4. *If P is odd and $r \geqslant 1$, then $\left(\frac{P^2+3}{V_{2^r}}\right) = 1$.*

THEOREM 3.4. *Let $k > 1$ be a square-free positive divisor of P and P be an odd integer. If $V_n = kx^2$ for some integer x, then $n = 1$ or $n = 3$.*

Proof. Assume that $k|P$ and $V_n = kx^2$. Then, it is seen that n is odd by Lemma 3. Assume that $n > 3$. Then we can write $n = 4q + 1$ or $n = 4q + 3$ for some $q > 0$. Also since $k|P$, we can write $P = kM$ for some positive integer M. We divide the proof into two cases.

Case 1. Assume that $n = 4q + 1$. Then $n = 4q + 1 = 2(2^r z) + 1$ for some odd integer z with $r \geqslant 1$. Thus we get

$$V_n \equiv -P \pmod{V_{2^r}},$$

i.e.,

$$kx^2 \equiv -P \pmod{V_{2^r}}$$

by (2.4). This shows that $J = \left(\frac{-kP}{V_{2^r}}\right) = 1$. Since $P = kM$, it follows that

$$J = \left(\frac{-kP}{V_{2^r}}\right) = \left(\frac{-k^2M}{V_{2^r}}\right) = \left(\frac{-M}{V_{2^r}}\right).$$

Since $V_{2^r} \equiv 2 \pmod{P}$ by Lemma 3, it is seen that $V_{2^r} \equiv 2 \pmod{M}$ and thus $(M, V_{2^r}) = 1$. Also we have $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.8). Assume that $M \equiv 1, 3 \pmod 8$. Hence we get

$$\begin{aligned} J &= \left(\frac{-M}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = (-1)\left(\frac{V_{2^r}}{M}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{M-1}{2}\right)} \\ &= (-1)\left(\frac{2}{M}\right)(-1)^{\left(\frac{M-1}{2}\right)} = (-1)(-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = -1, \end{aligned}$$

since $M \equiv 1, 3 \pmod 8$. This contradicts the fact that $J = 1$. Assume that $M \equiv 5, 7 \pmod 8$. If we write $n = 4q + 1 = 4(q + 1) - 3 = 2(2^r z) - 3$ for some odd integer z with $r \geqslant 1$, then we get

$$V_n \equiv V_3 \pmod{V_{2^r}},$$

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i.e.,

$$kx^2 \equiv V_3 \pmod{V_{2^r}}$$

by (2.4). This shows that

$$1 = J = \left(\frac{kV_3}{V_{2^r}}\right) = \left(\frac{k^2M(P^2+3)}{V_{2^r}}\right) = \left(\frac{M(P^2+3)}{V_{2^r}}\right).$$

Since $M \equiv 5, 7 \pmod 8$, $V_{2^r} \equiv 3, 7 \pmod 8$, and $V_{2^r} \equiv 2 \pmod M$ by Lemma 3, it follows that

$$\begin{aligned}\left(\frac{M}{V_{2^r}}\right) &= \left(\frac{V_{2^r}}{M}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{M-1}{2}\right)} \\ &= \left(\frac{2}{M}\right)(-1)^{\left(\frac{M-1}{2}\right)} = (-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = -1.\end{aligned}$$

Thus, since $\left(\frac{P^2+3}{V_{2^r}}\right) = 1$ by Lemma 4, we get

$$\begin{aligned}J &= \left(\frac{kV_3}{V_{2^r}}\right) = \left(\frac{k^2M(P^2+3)}{V_{2^r}}\right) \\ &= \left(\frac{M(P^2+3)}{V_{2^r}}\right) = \left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3}{V_{2^r}}\right) = -1,\end{aligned}$$

which is a contradiction.

Case 2. Assume that $n = 4q + 3$. Then we can write $n = 2(2^r z) + 3$ for some odd integer z with $r \geqslant 1$. Thus

$$V_n \equiv -V_3 \pmod{V_{2^r}},$$

i.e.,

$$kx^2 \equiv -V_3 \pmod{V_{2^r}}$$

by (2.4). This shows that

$$1 = J = \left(\frac{-kV_3}{V_{2^r}}\right) = \left(\frac{-k^2M(P^2+3)}{V_{2^r}}\right) = \left(\frac{-M(P^2+3)}{V_{2^r}}\right).$$

Assume that $M \equiv 1, 3 \pmod 8$. Then since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.8), it follows that

$$\left(\frac{M}{V_{2^r}}\right) = (-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = 1.$$

Thus, since $\left(\frac{P^2+3}{V_{2^r}}\right) = 1$ by Lemma 4, we get

$$J = \left(\frac{kV_3}{V_{2^r}}\right) = \left(\frac{-M(P^2+3)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3}{V_{2^r}}\right) = -1,$$

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which is a contradiction. Assume that $M \equiv 5, 7 \pmod 8$. If we write $n = 4q+3 = 4(q+1)-1 = 2(2^r z)-1$ for some odd integer z with $r \geqslant 1$, then we get

$$V_n \equiv P \pmod{V_{2^r}},$$

i.e.,

$$kx^2 \equiv P \pmod{V_{2^r}}$$

by (2.4). This shows that

$$1 = J = \left(\frac{kP}{V_{2^r}}\right) = \left(\frac{k^2 M}{V_{2^r}}\right) = \left(\frac{M}{V_{2^r}}\right).$$

On the other hand, since $M \equiv 5, 7 \pmod 8$, it is seen that

$$\left(\frac{M}{V_{2^r}}\right) = (-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = -1,$$

which contradicts the fact that $J = 1$. This completes the proof. □

The following lemma can be proved by using Theorems 2.1 and 2.2.

LEMMA 5.

$$3|V_n \Leftrightarrow \begin{cases} n \equiv 1 \pmod 2 & \text{if } 3|P, \\ n \equiv 2 \pmod 4 & \text{if } 3\nmid P, \end{cases}$$

and

$$3|U_n \Leftrightarrow \begin{cases} n \equiv 0 \pmod 2 & \text{if } 3|P, \\ n \equiv 0 \pmod 4 & \text{if } 3\nmid P. \end{cases}$$

In [**18**], Ribenboim and McDaniel solved the equation $U_n = 3x^2$ for $(P, Q) = 1$, and the equation $V_n = 3x^2$ for $P \equiv 1, 3 \pmod 8$, $Q \equiv 3 \pmod 4$ and $(P, Q) = 1$. In [**15**], the same authors showed that $V_n \neq 3x^2$ and $V_n \neq 6x^2$ for all odd relatively prime integers P and Q with $Q \equiv 3 \pmod 4$ and $n \equiv \pm 3 \pmod 8$. Besides, in [**1**], Antoniadis solved the equation $V_n = 3x^2$ for $3\nmid P$ with P odd and $Q = 1$. We now give the proof of the following theorem in a different way for the sake of completeness.

THEOREM 3.5. *If P is odd, then the equation $V_n = 3x^2$ has the solution $n = 1$ or $n = 2$, and if P is even and $3\nmid P$, then there is no integer x such that $V_n = 3x^2$.*

Proof. Assume that $3\nmid P$. Since $3|V_n$, it follows that $n \equiv 2 \pmod 4$ by Lemma 5. If P is even, then we get $3x^2 = V_n \equiv 2 \pmod 4$ by Lemma 3 since n is even. This is impossible. Therefore, we assume that P is odd. If $n = 2$, then $V_n = V_2 = P^2 + 2 = 3x^2$. That is, $P^2 - 3x^2 = -2$. Since $(u_1, v_1) = (1, 1)$ is the fundamental solution of the equation $u^2 - 3v^2 = -2$, then all positive integer solutions of the equation $u^2 - 3v^2 = -2$ are given by

$$(u, v) = (5U_m(4, -1) - U_{m-1}(4, -1), 3U_m(4, -1) - U_{m-1}(4, -1))$$

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with $m \geqslant 0$. Therefore when $n = 2$, the equation $V_n = 3x^2$ has a solution for $P = 5U_m(4, -1) - U_{m-1}(4, -1)$. If $n = 6$, then $3x^2 = V_6 = V_3^2 + 2$ by (2.10). Since P is odd, V_3 is even by (2.13) and therefore

$$3x^2 = V_3^2 + 2 \equiv 2 \pmod 4,$$

which is impossible. Then we can write $n = 16c \pm 2$ or $n = 16c \pm 6$ for some positive integer c. Assume that $n = 16c \pm 6$. Thus

$$V_n \equiv V_6 \pmod{V_4},$$

i.e.,

$$3x^2 \equiv V_6 \pmod{V_4} \tag{3.1}$$

by (2.4). Moreover, it can be easily shown that $V_6 \equiv -V_2 \pmod{V_4}$. Hence we get $3x^2 \equiv -V_2 \pmod{V_4}$. Then $J = \left(\frac{-3V_2}{V_4}\right) = 1$. On the other hand, since $\left(\frac{-1}{V_{2^r}}\right) = -1$ for $r \geqslant 1$ by (2.8), $V_4 \equiv 1 \pmod 3$ by Lemma 1 and $V_4 \equiv -2 \pmod{V_2}$ by (2.10), it is seen that

$$\left(\frac{-1}{V_4}\right) = (-1)^{\left(\frac{V_4-1}{2}\right)} = -1,$$
$$\left(\frac{3}{V_4}\right) = \left(\frac{V_4}{3}\right)(-1)^{\left(\frac{V_4-1}{2}\right)} = -1,$$

and

$$\left(\frac{V_2}{V_4}\right) = \left(\frac{V_4}{V_2}\right)(-1)^{\left(\frac{V_4-1}{2}\right)\left(\frac{V_2-1}{2}\right)} = \left(\frac{-2}{V_2}\right)(-1)$$
$$= \left(\frac{-1}{V_2}\right)\left(\frac{2}{V_2}\right)(-1) = -1.$$

Thus it follows that

$$J = \left(\frac{-3V_2}{V_4}\right) = \left(\frac{-1}{V_4}\right)\left(\frac{3}{V_4}\right)\left(\frac{V_2}{V_4}\right) = (-1)(-1)(-1) = -1.$$

This contradicts the fact that $J = 1$. Assume that $n = 16c \pm 2$. If we write n as $n = 2(2^r z) \pm 2$ for some odd z with $r \geqslant 3$, then it is seen that

$$V_n \equiv -V_2 \pmod{V_{2^r}},$$

i.e.,

$$3x^2 \equiv -V_2 \pmod{V_{2^r}}$$

by (2.4). This shows that $J = \left(\frac{-3V_2}{V_{2^r}}\right) = 1$. On the other hand, we have $V_{2^r} \equiv 2 \pmod{V_2}$ for $r \geqslant 3$ by (2.23) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.8). Thus we get

$$\left(\frac{V_2}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{V_2}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{V_2-1}{2}\right)} = \left(\frac{2}{V_2}\right)(-1) = 1.$$

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Moreover, $V_{2^r} \equiv 2 \pmod 3$ by Lemma 1. Then

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{3}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{3-1}{2}\right)} = \left(\frac{2}{3}\right)(-1) = 1.$$

Thus we get

$$J = \left(\frac{-3V_2}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{V_2}{V_{2^r}}\right) = -1,$$

which is a contradiction.

Now assume that $3|P$. Then $n = 1$ or $n = 3$ by Theorem 3.4. If $n = 1$, then $V_1 = P = 3x^2$. It is obvious that this is a solution. If $n = 3$, then it follows that $V_3 = P(P^2 + 3) = 3x^2$. Moreover, $(P, (P^2 + 3)/3) = 1$ since $3|P$. Therefore $P = a^2$ and $P^2 + 3 = 3b^2$ for some positive integers a and b. Since $3|P$, $P = 3c$ for some positive integer c. Hence we obtain the Pell equation $b^2 - 3c^2 = 1$. It is well known that this equation has the solution $(b, c) = ((V_m(4, -1))/2, U_m(4, -1))$ for $m \geqslant 0$. It follows that $P = a^2 = 3c = 3U_m$, i.e., $U_m = 3\square$. It is seen that the equation $U_m(4, -1) = 3\square$ has no solutions by Theorem 2 given in [**13**]. Therefore $n = 3$ is impossible. This completes the proof. □

THEOREM 3.6. *If P is odd, then there is no integer x such that $V_n = 6x^2$.*

Proof. Assume that $3\nmid P$. If x is even, it follows that $8|V_n$, which is impossible by (2.7). Therefore x is odd. Since $3|V_n$ and $2|V_n$, it is seen that $n/2$ is odd by Lemma 5 and $3|n$ by (2.13). Then we can write $n = 12q + 6$ for some positive integer q. Thus

$$V_n \equiv V_6 \pmod 8$$

by (2.6). That is,

$$6x^2 \equiv 2 \pmod 8,$$

which is impossible since x is odd.

Assume that $3|P$. Since $3|V_n$, n is odd by Lemma 5. Also since $3|n$, it follows that $n \equiv \pm 3 \pmod{12}$. Thus we get

$$V_n \equiv V_{\pm 3} \equiv \pm 4P \equiv 4 \pmod 8$$

by (2.6). That is,

$$6x^2 \equiv 4 \pmod 8,$$

which is impossible since x is odd. This completes the proof. □

THEOREM 3.7. *If P is odd, $m \geqslant 1$, and $V_n = 3V_m x^2$, then $m = 1$ and $n = 3$.*

Proof. Assume that $3\nmid P$. Since $V_m|V_n$ and $3|V_n$, it follows that $n = mt$ for some odd positive integer t by (2.19) and $n/2$ is odd by Lemma 5. Therefore m is even and $m/2$ is odd. Then we can write $m = 12q \pm 2$ or $m = 12q + 6$ for some positive integer q. Thus we get

$$V_m \equiv V_2,\ V_6 \equiv 2, 3 \pmod 8$$

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by (2.6). Similarly, we see that $V_n \equiv 2, 3 \pmod 8$. Also we get $3x^2 \equiv 0, 3, 4 \pmod 8$. Thus $V_n = 3V_m x^2 \equiv 0, 1, 4, 6 \pmod 8$, which contradicts the fact that $V_n \equiv 2, 3 \pmod 8$.

Now assume that $3|P$. Since $3|V_n$, it is seen that n is odd by Lemma 5. Therefore m is also odd. We divide the proof into two cases.

Case 1. Assume that $3|t$. Then $t = 3s$ for some odd integer s. Thus $n = mt = 3ms$. By (2.12), we get $V_n = V_{3ms} = V_{ms}(V_{ms}^2 + 3) = 3V_m x^2$. Since ms is odd and $3|P$, it follows that $(V_{ms}/V_m)((V_{ms}^2 + 3)/3) = x^2$. It can be seen that $(V_{ms}/V_m, (V_{ms}^2 + 3)/3) = 1$. Therefore $V_{ms} = V_m a^2$ and $V_{ms}^2 + 3 = 3b^2$ for some positive integers a and b. Then $ms = m$, i.e., $s = 1$ by Theorem 3.1. Thus $V_m^2 + 3 = 3b^2$. Since $V_{2m} = V_m^2 + 2$ by (2.10), we get $V_{2m} = 3b^2 - 1$. Assume that $m > 1$. Then we can write $2m = 2(4q \pm 1) = 2(2^r z) \pm 2$, for some odd z with $r \geqslant 2$. Hence we obtain

$$V_{2m} \equiv -V_2 \equiv -(P^2 + 2) \pmod{V_{2^r}}$$

by (2.4). That is,

$$3b^2 \equiv -(P^2 + 2 - 1) \equiv -U_3 \pmod{V_{2^r}}.$$

This shows that $J = \left(\frac{-3U_3}{V_{2^r}}\right) = 1$. On the other hand, $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.8). Moreover, since $3|P$, it follows that $V_{2^r} \equiv 2 \pmod 3$ by Lemma 1 and therefore

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{3}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)} = \left(\frac{2}{3}\right)(-1) = 1.$$

Since $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by (2.22), we obtain

$$J = \left(\frac{-3U_3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right) = -1,$$

which is a contradiction. Thus $m = 1$, i.e., $n = 3$. Then we get $P^2 + 3 = 3x^2$ from the equation $V_n = 3V_m x^2$. Since $3|P$, $P = 3c$ for some positive integer c. Hence we obtain the Pell equation $x^2 - 3c^2 = 1$. It is well known that this equation has the solution $(x, c) = ((V_m(4, -1))/2, U_m(4, -1))$. Therefore $m = 1$ and $n = 3$ is a solution.

Case 2. Assume that $3\nmid t$. It is obvious that $t > 1$ and therefore we can write $t = 6q \pm 1$ for some positive integer q. Then $n = mt = 2(3mq) \pm m$. Hence we get

$$V_n \equiv \pm V_m \pmod{V_{3m}},$$

i.e.,

$$3V_m x^2 \equiv \pm V_m \pmod{V_{3m}}$$

by (2.4). It follows that

$$3V_m x^2 \equiv \pm V_m \pmod{V_m(V_m^2 + 3)},$$

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by (2.12) and therefore

$$3x^2 \equiv \pm 1 \pmod{V_m^2 + 3}.$$

Since $3|P$ and m is odd, it is seen that $3|V_m$ by Lemma 5. Therefore

$$3x^2 \equiv \pm 1 \pmod 3,$$

which is impossible. This completes the proof. □

THEOREM 3.8. *If P is even, $3\nmid P$ and $m \geqslant 1$, then there is no integer x such that $V_n = 3V_m x^2$.*

Proof. Assume that P is even, $3\nmid P$ and $V_n = 3V_m x^2$. Since $V_m|V_n$ and $3|V_n$, there exist two positive odd integers t and z such that $n = mt$ and $n = 2z$ by (2.19) and Lemma 5, respectively. Therefore $m = 2r$ for some odd positive integer r. It is obvious that $t > 1$. Then we can write $t = 4q \pm 1$ for some positive integer q. Thus, since $n = mt = 4mq \pm m$, we get

$$V_n \equiv \pm V_m \pmod{V_{2m}}$$

by (2.4), i.e.,

$$3V_m x^2 \equiv \pm V_m \pmod{V_{2m}}.$$

It is seen that $(V_m, V_{2m}) = 2$ by (2.17). Hence we obtain

$$3x^2 \equiv \pm 1 \pmod{V_{2m}/2}.$$

Since $2m/4 = 4r/4 = r$ is odd, it is seen that $V_4|V_{2m}$ by (2.19) and thus

$$3x^2 \equiv \pm 1 \pmod{V_4/2}.$$

This shows that $J = \left(\frac{\pm 3}{V_4/2}\right) = 1$. On the other hand, since $V_4 = P^4 + 4P^2 + 2$ and P is even, it is seen that $V_4 \equiv 2 \pmod 8$ and therefore

$$\left(\frac{-1}{V_4/2}\right) = (-1)^{\left(\frac{V_4-2}{4}\right)} = 1.$$

Moreover, since $V_4 \equiv 1 \pmod 3$ by Lemma 1, it is seen that $V_4/2 \equiv 2 \pmod 3$. Therefore,

$$\left(\frac{3}{V_4/2}\right) = \left(\frac{V_4/2}{3}\right)(-1)^{\left(\frac{V_4-2}{4}\right)} = \left(\frac{2}{3}\right) = -1.$$

Thus we get $J = -1$, which is a contradiction. This completes the proof. □

THEOREM 3.9. *If P is even, $3|P$, $2\|P$ and $m \geqslant 1$, then there is no integer x such that $V_n = 3V_m x^2$.*

Proof. Assume that P is even, $3|P$, $2\|P$ and $V_n = 3V_m x^2$. Since $V_m|V_n$ and $3|V_n$, there exists an odd positive integer t such that $n = mt$ and n is odd by (2.19) and Lemma 5, respectively. Therefore m is also odd. It is obvious

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that $t > 1$. Assume that $3|t$. Then $t = 3s$ for some positive integer s. Thus, since $n = mt = 3ms$, we get

$$3V_m x^2 = V_n = V_{3ms} = V_{ms}(V_{ms}^2 + 3)$$

by (2.12). Since $3|P$ and ms is odd, it follows that $3|V_{ms}$ by Lemma 5. Thus

$$(V_{ms}/V_m)((V_{ms}^2 + 3)/3) = x^2.$$

It can be easily seen that $(V_{ms}/V_m, (V_{ms}^2 + 3)/3) = 1$. This shows that

$$V_{ms} = V_m u^2 \quad \text{and} \quad V_{ms}^2 + 3 = 3v^2$$

for some positive integers u and v. Hence we get $3v^2 = V_{ms}^2 + 3 = V_{2ms} + 1$ by (2.10). Thus

$$3v^2 \equiv 1 \pmod{V_{2ms}}.$$

Since $V_2|V_{2ms}$, i.e., $(P^2 + 2)|V_{2ms}$ by (2.19), it follows that

$$3v^2 \equiv 1 \pmod{P^2 + 2}.$$

This shows that

$$J = \left(\frac{3}{(P^2 + 2)/2}\right) = 1.$$

On the other hand, since $3|P$, it is seen that $(P^2 + 2)/2 \equiv 1 \pmod 3$. Therefore

$$J = \left(\frac{3}{(P^2 + 2)/2}\right) = \left(\frac{(P^2 + 2)/2}{3}\right)(-1)^{\frac{P^2}{4}} = (-1)^{\frac{P^2}{4}}.$$

Since $2\|P$, we get $J = -1$, which is a contradiction. Now assume that $3\nmid t$. Then since $t > 1$, we can write $t = 6q \pm 1$ for some positive integer q. Thus $n = mt = 2(3mq) \pm m$. Hence we get

$$V_n \equiv \pm V_m \pmod{V_{3m}},$$

i.e.,

$$3V_m x^2 \equiv \pm V_m \pmod{V_{3m}}$$

by (2.4). It follows that

$$3V_m x^2 \equiv \pm V_m \pmod{V_m(V_m^2 + 3)},$$

i.e.,

$$3x^2 \equiv \pm 1 \pmod{V_m^2 + 3}$$

by (2.12). Since $3|P$ and m is odd, it is seen that $3|V_m$ by Lemma 5. Therefore

$$3x^2 \equiv \pm 1 \pmod 3,$$

which is impossible. This completes the proof. □

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The following lemma can be proved by induction.

LEMMA 6. *If P is odd and $k \geqslant 2$, then $V_{2^k} \equiv -1 \pmod{V_4 + 1}$.*

THEOREM 3.10. *If $V_n = 6V_m x^2$ and $m \geqslant 1$, then $m = 2$ and $n = 6$.*

Proof. Firstly, assume that P is odd. Now assume that $3 \nmid P$. Then since $3|V_n$ and $2|V_n$, it follows that $n = 2z$ for some odd integer z by Lemma 5 and $3|n$ by (2.13), respectively. This shows that $3|z$ and therefore $z = 3a$ for some odd positive integer a. Thus $n = 2z = 6a = 12q + 6$ for some positive integer q. Hence we get

$$V_n \equiv V_6 \equiv 2 \pmod 8$$

by (2.6). It is obvious that x is odd. Therefore $6V_m \equiv 6V_m x^2 = V_n \equiv 2 \pmod 8$. Besides, since $V_m|V_n$, there exists an odd positive integer t such that $n = mt$ by (2.19). Since $n = 2z$ with odd z, it follows that $m = 2r$ for some odd positive integer r. Assume that $3|r$. Then $V_m \equiv 2 \pmod 8$ by Lemma 2 and therefore $6V_m \equiv 4 \pmod 8$, which contradicts the fact that $6V_m \equiv 2 \pmod 8$. Assume that $3 \nmid r$. Then since $3|n$ and $n = mt$, it is seen that $3|t$, i.e., $t = 3s$ for some odd positive integer s. Thus $n = mt = 3ms$. Therefore we get $V_n = V_{3ms} = V_{ms}(V_{ms}^2 - 3) = 6V_m x^2$ by (2.12). Since $3 \nmid P$ and $ms = 2rs \equiv 2 \pmod 4$, it follows that $3|V_{ms}$ by Lemma 5. Thus

$$(V_{ms}/V_m)((V_{ms}^2 - 3)/3) = 2x^2.$$

It can be easily seen that $(V_{ms}/V_m, (V_{ms}^2 - 3)/3) = 1$. This shows that

$$V_{ms} = 2V_m u^2 \quad \text{and} \quad V_{ms}^2 - 3 = 3v^2 \tag{3.2}$$

or

$$V_{ms} = V_m u^2 \quad \text{and} \quad V_{ms}^2 - 3 = 6v^2 \tag{3.3}$$

for some integers u and v. Identity (3.2) is impossible by Theorem 3.2. Assume that (3.3) is satisfied. By Theorem 3.1, it follows that $s = 1$. Therefore $V_m^2 - 3 = 6v^2$. Since $V_{2m} = V_m^2 - 2$ by (2.10), we get $V_{2m} = V_{4r} = 6v^2 + 1$. Assume that $r > 1$. Then $2m = 4r = 4(4q \pm 1) = 2(2^k z) \pm 4$ for some odd integer z with $k \geqslant 3$. Hence we obtain

$$V_{2m} \equiv -V_4 \pmod{V_{2^k}}$$

by (2.4). That is,

$$6v^2 \equiv -(V_4 + 1) \pmod{V_{2^k}}.$$

Since V_4 and V_{2^k} are odd by (2.13), it follows that

$$3v^2 \equiv -(V_4 + 1)/2 \pmod{V_{2^k}}.$$

This shows that

$$J = \left(\frac{-3(V_4 + 1)/2}{V_{2^k}}\right) = 1.$$

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On the other hand, since $\left(\frac{-1}{V_{2^k}}\right) = -1$ by (2.8) and

$$\left(\frac{V_{2^k}}{3}\right) = \left(\frac{2}{3}\right) = -1$$

by Lemma 1, it is seen that

$$\left(\frac{3}{V_{2^k}}\right) = (-1)^{\left(\frac{V_{2^k}-1}{2}\right)}\left(\frac{V_{2^k}}{3}\right) = (-1)(-1) = 1.$$

Besides, it is seen that $V_4 \equiv 7 \pmod{16}$ and therefore $(V_4+1)/8$ is odd. Thus, we have

$$\left(\frac{V_{2^k}}{(V_4+1)/8}\right) = \left(\frac{-1}{(V_4+1)/8}\right)$$

by Lemma 6. Hence we obtain

$$\begin{aligned} J &= \left(\frac{-3(V_4+1)/2}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{3}{V_{2^k}}\right)\left(\frac{4}{V_{2^k}}\right)\left(\frac{(V_4+1)/8}{V_{2^k}}\right) \\ &= (-1)\left(\frac{(V_4+1)/8}{V_{2^k}}\right). \end{aligned}$$

Let $y = (V_4+1)/8$. Then

$$\begin{aligned} J &= (-1)\left(\frac{(V_4+1)/8}{V_{2^k}}\right) = (-1)\left(\frac{y}{V_{2^k}}\right) = (-1)(-1)^{\left(\frac{y-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)}\left(\frac{V_{2^k}}{y}\right) \\ &= (-1)(-1)^{\left(\frac{y-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)}\left(\frac{-1}{y}\right) = (-1)(-1)^{\left(\frac{y-1}{2}\right)}(-1)^{\left(\frac{y-1}{2}\right)} = -1, \end{aligned}$$

which is a contradiction. Therefore $r = 1$. Thus $m = 2$ and $n = 6$. Since $V_2 = P^2+2$ and $V_6 = P^6+6P^4+9P^2+2$, we get $P^4+4P^2+1 = 6x^2$ from the equation $V_n = 6V_m x^2$. Completing the square gives $(P^2+2)^2 - 3 = 6x^2$. It follows that $3|P^2+2$ and therefore $P^2+2 = 3c$ for some positive integer c. Thus we get the equation $3c^2 - 2x^2 = 1$. By Lemma 1 given in [21], if $k\sqrt{A}+t\sqrt{B}$ is a solution of $AX^2 - BY^2 = 1$ and $r+s\sqrt{AB}$ is a solution of $X^2 - ABY^2 = 1$, then the product $(kr+Bts)\sqrt{A} + (tr+Aks)\sqrt{B}$ is a solution of $AX^2 - BY^2 = 1$. Thus, since $\sqrt{3}+\sqrt{2}$ is a solution of $3c^2 - 2x^2 = 1$ and $5+2\sqrt{6}$ is a solution of $c^2 - 6x^2 = 1$, the equation $3c^2 - 2x^2 = 1$ has infinitely many solutions. Therefore $m = 2$ and $n = 6$ is a solution.

Now assume that $3|P$. Then since $3|V_n$ and $2|V_n$, it follows that n is odd by Lemma 5 and $3|n$ by (2.13). Therefore $n = 12q \pm 3$ for some positive integer q. Thus we get

$$V_n \equiv V_{\pm 3} \equiv \pm 4P \equiv 4 \pmod{8}$$

by (2.6). That is,

$$6V_m x^2 \equiv 4 \pmod{8}.$$

Also since x is odd, it follows that $6V_m \equiv 4 \pmod{8}$. Thus V_m is even and therefore $3|m$ by (2.13). Moreover, since n is odd and $V_m|V_n$, it is seen that m

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is odd by (2.19). Then $V_3|V_m$ by (2.19). Besides, since P is odd, it follows that $4|V_3$ and therefore $4|V_m$. This shows that $8|V_n$, which is impossible by (2.7).

Secondly, assume that P is even. If n is odd, then m is also odd. Hence we have $V_n \equiv nP \pmod{P^2}$ and $V_m \equiv mP \pmod{P^2}$ by Lemma 3. This shows that

$$nP \equiv 6mPx^2 \pmod{P^2},$$

i.e.,

$$n \equiv 6mx^2 \pmod{P},$$

which is impossible since P is even and n is odd. If n is even, then m is also even. Hence we have $V_n \equiv 2 \pmod 4$ and $V_m \equiv 2 \pmod 4$ by Lemma 3. This shows that

$$6V_m x^2 \equiv 0 \pmod 4,$$

i.e.,

$$V_n \equiv 0 \pmod 4,$$

which contradicts the fact that $V_n \equiv 2 \pmod 4$. This completes the proof. □

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