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Commutator length of annulus diffeomorphisms

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Abstract. We study the group $\mathrm{Diff}_0^r(\mathbb{A})$ of C^r-diffeomorphisms of the closed annulus that are isotopic to the identity. We show that, for $r \neq 2, 3$, the linear space of homogeneous quasi-morphisms on the group $\mathrm{Diff}_0^r(\mathbb{A})$ is one-dimensional. Therefore, the commutator length on this group is (stably) unbounded. In particular, this provides an example of a manifold whose diffeomorphism group is unbounded in the sense of Burago, Ivanov and Polterovich.

1. *Introduction*

Let M be a manifold. The *support* $\mathrm{supp}(f)$ of a homeomorphism f of M is defined to be the closure of the set:

$$\{x \in M,\ f(x) \neq x\}.$$

For r in $\mathbb{N} \cup \{\infty\}$, denote by $\mathrm{Diff}_0^r(M)$ the identity component of the group of compactly supported C^r-diffeomorphisms of M. We also write $\mathrm{Homeo}_0(M) = \mathrm{Diff}_0^0(M)$. We study algebraic properties of these groups.

A commutator in $\mathrm{Diff}_0^r(M)$ is an element of the form $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$. It is known that, for $r \neq \dim(M) = \dim(\partial M) + 1$ and $r \neq \dim(M) + 1$, every element in $\mathrm{Diff}_0^r(M)$ can be written as a product of commutators. Indeed, a result by Mather (see [**4, 10, 11**]) shows that if the manifold M is without boundary, the group $\mathrm{Diff}_0^r(M)$ is perfect if $r \neq \dim(M) + 1$, and the result for manifolds with a boundary follows from this one. A question naturally arises: how many commutators do we need to write a given diffeomorphism as a product of commutators? For an element f in $\mathrm{Diff}_0^r(M)$, the *commutator length* $cl_r(f)$ is the minimal number of commutators needed to write f as a product of commutators. Burago *et al* showed in [**5**] that when the manifold M is a sphere or compact and three-dimensional without boundary, the commutator length is bounded (by 4 in the case of the sphere and 10 in the case of three-dimensional manifolds). They also exhibited a wide class of open manifolds (which they call portable

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manifolds), including $\mathbb{R}^n$, for which the commutator length is bounded by 2. Tsuboi [**12**] generalized these results for odd-dimensional compact manifolds and gave a better bound; the commutator length is bounded by 6 in those cases.

A homeomorphism f is said to be *supported in a ball* if there exists a topological embedding $i : \mathbb{B} \to M$ of the closed unit ball $\mathbb{B}$ of the same dimension as M such that $\mathrm{supp}(f) \subset i(\mathbb{B})$ and $i(\mathbb{B}) \cap \partial M$ is homeomorphic to a closed cell. As in the case of commutators, every element f in $\mathrm{Diff}_0^r(M)$ can be written as a product of diffeomorphisms in $\mathrm{Diff}_0^r(M)$ supported in balls, and we define $\mathrm{Frag}_r(f)$ to be the minimal number of diffeomorphisms in such a decomposition. Burago, Ivanov, Polterovich and Tsuboi showed also that the fragmentation norm is bounded in the case of odd-dimensional manifolds and of open portable manifolds.

The commutator length and the fragmentation norm are two examples of the more general notion of a conjugation-invariant norm on a group introduced by Burago *et al* in [**5**]. They proved that the fragmentation norm plays a crucial role: every conjugation-invariant norm on $\mathrm{Diff}_0^r(M)$ is bounded if and only if the fragmentation norm is bounded.

In this paper, we consider the case of the closed annulus $\mathbb{A}$. We prove some estimates on the commutator length and the fragmentation norm, which imply, in particular, that they are unbounded, contrary to the above examples†. For a diffeomorphism f in $\mathrm{Diff}_0^r(\mathbb{A})$, let $\rho(f)$ be the difference of the translation numbers of f on the two boundary components (see Figure 1 and, for a precise definition of $\rho(f)$, see the next section). The following theorem shows that ρ is a quasi-isometry from $\mathrm{Diff}_0^r(M)$, endowed with the fragmentation norm or the commutator length, to $\mathbb{R}$.

THEOREM 1.1. *Let r be an integer different from 2 and 3. For any diffeomorphism f in* $\mathrm{Diff}_0^r(\mathbb{A})$,

$$\left| \frac{|\rho(f)|}{4} - cl_r(f) \right| \leq \frac{47}{4}$$

and

$$||\rho(f)| - \mathrm{Frag}_r(f)| \leq 42.$$

In fact, the map ρ is a quasi-morphism (this notion will be defined in the next section) on the group $\mathrm{Diff}_0^r(\mathbb{A})$, and is essentially the only quasi-morphism on this group.

THEOREM 1.2. *If $r \neq 2, 3$, every homogeneous quasi-morphism on* $\mathrm{Diff}_0^r(\mathbb{A})$ *is collinear to ρ.*

Remark. For $r = 2$ or $r = 3$, the result holds if we replace $\mathrm{Diff}_0^r(\mathbb{A})$ by $[\mathrm{Diff}_0^r(\mathbb{A}), \mathrm{Diff}_0^r(\mathbb{A})]$, the subgroup generated by commutators. Whether the equality $[\mathrm{Diff}_0^3(\mathbb{A}), \mathrm{Diff}_0^3(\mathbb{A})] = \mathrm{Diff}_0^3(\mathbb{A})$ holds or not is still an open question.

In the next section, we state a proposition that gives fine estimates on the commutator length. We also show that this proposition implies the first part of Theorems 1.1 and 1.2. The third section is devoted to the proof of this proposition. In the fourth section, we prove an analogous proposition on the fragmentation norm, which implies the second part

† Observe, in particular, that, for a manifold M with interior $\mathrm{int}(M)$, the boundedness of $\mathrm{Diff}_0^r(\partial M)$ and of $\mathrm{Diff}_0^r(\mathrm{int}(M))$ does not necessarily imply that $\mathrm{Diff}_0^r(M)$ is bounded (compare to [**1**]).

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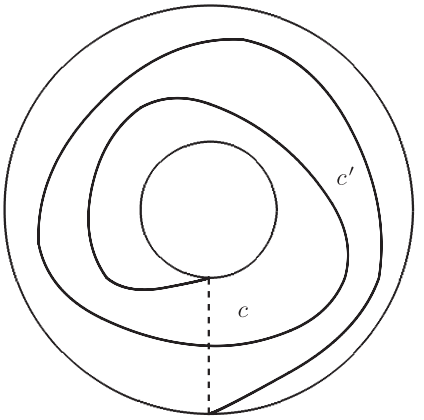


FIGURE 1. A diffeomorphism f which sends the curve c to the curve c' satisfies $\rho(f) = 2$.

of Theorem 1.1. Finally, in the last section, we discuss generalizations of these results to other surfaces. We will prove the following result.

THEOREM 1.3. *Let M be a surface with a non-empty boundary, which is different from the disk, and r be a number in $\mathbb{N} \cup \{\infty\}$. Then, the group $\mathrm{Diff}_0^r(M)$ admits non-trivial homogeneous quasi-morphisms. In particular, this group is stably unbounded.*

2. *Quasi-morphisms on* $\mathrm{Diff}_0^r(\mathbb{A})$

Definition 2.1. Given a group G, a homogeneous quasi-morphism on G is a map $q : G \to \mathbb{R}$, which satisfies the following.

(1) There exists $D > 0$, such that for all $a, b \in G$, $|q(ab) - q(a) - q(b)| \leq D$.

(2) For all $a \in G$, for all $n \in \mathbb{Z}$, $q(a^n) = nq(a)$.

The least constant $D(q)$ that satisfies (1) is called the defect of the quasi-morphism q. The trivial quasi-morphism is the quasi-morphism that maps every element of G to 0.

A first connection between commutator length and quasi-morphisms is given by the following classical lemma (see [**3, 6**]).

LEMMA 2.1. *Suppose G is perfect. If G admits a non-trivial homogeneous quasi-morphism q, then cl_G is unbounded. Moreover,*

$$\text{for all } g \in G, \quad |q(g)| \leq 2cl_G(g)D(q).$$

Quasi-morphisms on G are, in fact, more closely related to commutator length via the Bavard duality (see [**3, 6**]). A typical example is the *translation number* on the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of $\mathbb{R}$, which commute with integral translations (which is also the group of lifts of orientation-preserving homeomorphisms of the circle): for every homeomorphism f in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, the sequence $((f^n(x) - x)/n)_n$ converges for any x and the limit is independent of the chosen point x. This limit is called the translation number of f. It actually defines a quasi-morphism on the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ (see [**7**] for more information on the translation number).

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We now consider the closed annulus $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$. We build a quasi-morphism on the identity component $\mathrm{Homeo}_0(\mathbb{A})$ of the group of homeomorphisms of $\mathbb{A}$. The following construction is due to Frederic Le Roux. We denote by $\pi : \mathbb{R} \times [0, 1] \to \mathbb{A}$ the universal covering of $\mathbb{A}$. Given a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{A})$, consider a lift $F : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ of f, i.e., a homeomorphism of $\mathbb{R} \times [0, 1]$ that satisfies $\pi \circ F = f \circ \pi$. The maps $F_0 = F_{\mathbb{R}\times\{0\}}$ and $F_1 = F_{\mathbb{R}\times\{1\}}$ belong to the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$. If we replace F by some other lift $F + k$, then both translation numbers of F_0 and F_1 will increase by k. Thus, the difference of the translation numbers of these homeomorphisms is independent of the lift F chosen. We denote this number by $\rho(f)$, and call it the *torsion number*. As the translation number is a homogeneous quasi-morphism on the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, ρ is a (non-trivial) homogeneous quasi-morphism on $\mathrm{Homeo}_0(\mathbb{A})$, which restricts to a non-trivial quasi-morphism on $\mathrm{Diff}_0^r(\mathbb{A})$, for every r in $\mathbb{N} \cup \{\infty\}$.

We first reduce Theorem 1.2 to the following proposition, which gives estimates on the commutator length. We denote by E the lower integer part and, for an element f in $\mathrm{Diff}_0^r(\mathbb{A})$,

$$\alpha(f) = \min_{r\in\mathbb{R}} |F_1(r) - F_0(r)|,$$

where F is a lift of f (note that this quantity is independent of the choice of the lift).

PROPOSITION 2.2. *Let $f \neq \mathrm{Id}_{\mathbb{A}}$ be a homeomorphism in $\mathrm{Diff}_0^r(\mathbb{A})$. Then, for any $r \neq 2, 3$,*

$$E\left(\frac{\alpha(f)+3}{4}\right) + 9 \geq cl_r(f) \geq E\left(\frac{\alpha(f)}{4}\right) + 1.$$

If $r = 0$, then the 9 appearing in the upper bound may be improved to 5.

This proposition will be established in the next section. Now, we deduce the first part of Theorem 1.1 and Theorem 1.2 from Proposition 2.2.

Proof of the first part of Theorem 1.1. It suffices to show that, for every element f in $\mathrm{Diff}_0^r(\mathbb{A})$,

$$||\rho(f)| - \alpha(f)| \leq 2.$$

Take a real number r_0 in $\mathbb{R}$ such that $\alpha(f) = |F_1(r_0) - F_0(r_0)|$, where F is a lift of f. We will prove that

$$|\rho(f) - (F_1(r_0) - F_0(r_0))| \leq 2.$$

For any homeomorphisms G and H in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we have the following classical inequality:

$$\text{for all } r \in \mathbb{R}, \quad |(G(H(r)) - r) - (G(r) - r) - (H(r) - r)| \leq 1.$$

Using this formula, we obtain, by induction, for $i \in \{0, 1\}$,

$$\text{for all } n \in \mathbb{N}, \quad |F_i^n(r_0) - nF_i(r_0) + (n-1)r_0| \leq n - 1$$

and $|F_1^n(r_0) - F_0^n(r_0) - n(F_1(r_0) - F_0(r_0))| \leq 2n - 2$. Dividing by n and taking the limit $n \to +\infty$ allows us to conclude the proof. □

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Proof of Theorem 1.2. The strategy of the proof is the same as in the case of the translation number on $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ (see [**7**]). Let q be a homogeneous quasi-morphism on $\mathrm{Diff}_0^r(\mathbb{A})$. Let t be a C^∞-diffeomorphism in $\mathrm{Diff}_0^\infty(\mathbb{A})$, which is a 'twist': there exists a lift T of t which satisfies

$$\text{for all } x \in \mathbb{R}, \quad \begin{cases} T(x, 0) = (x, 0), \\ T(x, 1) = (x + 1, 1). \end{cases}$$

(For instance, one can take $t(x, r) = (x + r, r)$.) Then $q - q(t)\rho$ is a homogeneous quasi-morphism that vanishes on t, and the next lemma allows us to complete the proof of Theorem 1.2. □

Remark. Observe that t fixes the boundary, but there is no continuous path of diffeomorphisms fixing the boundary between the identity and t.

LEMMA 2.3. *Every homogeneous quasi-morphism on* $\mathrm{Diff}_0^r(\mathbb{A})$ *that vanishes on* t *is trivial, if* $r \neq 2, 3$.

Proof of Lemma 2.3. Let q be such a quasi-morphism. Then, for every integer $n \in \mathbb{Z}$, and every homeomorphism f in $\mathrm{Diff}_0^r(\mathbb{A})$, we have

$$|q(t^n f) - q(f)| \leq D(q).$$

Let us fix f in $\mathrm{Diff}_0^r(\mathbb{A})$ and choose an integer n_0 such that $\alpha(t^{n_0} f) < 1$ (if $\alpha(f) = \min_{r \in \mathbb{R}}(F_1(r) - F_0(r))$, then $\alpha(t^p f) = \alpha(f) + p$, as long as $p > -\alpha(f)$ and the other cases are similar or easier). Then, by Proposition 2.2, the homeomorphism $t^{n_0} f$ may be written as a product of at most 9 commutators. Therefore, using Lemma 2.1,

$$|q(t^{n_0} f)| \leq 18 D(q)$$

and $|q(f)| \leq 19 D(q)$. Hence, q is a bounded homogeneous quasi-morphism: it is trivial. □

3. *Estimation of the commutator length*

This section is devoted to the proof of Proposition 2.2. We will first make the proof in the case $r = 0$, and then an approximation argument will give the $r > 0$ case. We will need a theorem by Eisenbud, Hirsch and Neumann on the commutator length for lifts of circle homeomorphisms.

THEOREM 3.1. (Eisenbud–Hirsch–Neumann [**8**, Theorem 2.3]) *A homeomorphism* F *in* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ *can be written as a product of* n *commutators in* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ *if and only if*

$$\min_{x \in \mathbb{R}} |F(x) - x| < 2n - 1.$$

3.1. *Lower bound of the commutator length for* $r = 0$. Let f be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{A})$ with $cl_0(f) = n$. Let F denote a lift of f that can be written as a product of n commutators in the group of lifts of elements in $\mathrm{Homeo}_0(\mathbb{A})$, namely,

$$\{G \in \mathrm{Homeo}_0(\mathbb{R} \times [0, 1]) : \forall (x, t) \in \mathbb{R} \times [0, 1],\ G(x + 1, t) = G(x, t) + (1, 0)\}.$$

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Then F_0 and F_1, which belong to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, can be written as a product of n commutators. Using the Eisenbud–Hirsch–Neumann theorem, we get

$$\min_{x\in\mathbb{R}} |F_0(x) - x| = |F_0(x_0) - x_0| < 2n - 1$$

and

$$\min_{x\in\mathbb{R}} |F_1(x) - x| = |F_1(x_1) - x_1| < 2n - 1.$$

We may assume that $x_0 < x_1 < x_0 + 1$. These inequalities imply that

$$\begin{aligned}\alpha(f) &\leq |F_0(x_0) - F_1(x_0)| \\ &\leq |F_0(x_0) - x_0| + |(F_1(x_0) - x_0) - (F_1(x_1) - x_1)| + |F_1(x_1) - x_1| \\ &< (2n - 1) + 2 + (2n - 1) \\ &< 4n.\end{aligned}$$

The lower bound in Proposition 2.2 is therefore proved.

3.2. *Upper bound of the commutator length for $r = 0$.* To get an upper bound, we first compose a given homeomorphism by some number of commutators to get a homeomorphism that admits a lift that pointwise fixes the boundary. Then we compose by two commutators to get a homeomorphism with a lift that is the identity in a neighborhood of the boundary. Finally, such a homeomorphism can be written as a product of two commutators, by a result by Burago, Ivanov and Polterovich.

Let f be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{A})$. Let $k = E((\alpha(f) + 3)/4) + 1$.

3.2.1. *First step: getting a homeomorphism with a lift that fixes pointwise the boundary by composing with k commutators.* Up to conjugating by $(x, r) \to (x, 1 - r)$, we may assume that f is a 'positive twist': there exists a lift F of f and a real number x_0 that satisfy $\alpha(f) = F_1(x_0) - F_0(x_0)$.

By composing the lift F by an integral translation if necessary, we may suppose that

$$2k - 2 \leq F_1(x_0) - x_0 < 2k - 1.$$

Then, as $4k > \alpha(f) + 3$,

$$F_0(x_0) - x_0 = F_1(x_0) - x_0 - \alpha(f) \geq 2k - 2 - \alpha(f) > 1 - 2k.$$

By using the Eisenbud–Hirsch–Neumann theorem, we get that F_1 and F_0 can be written as products of k commutators in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$:

$$\begin{cases} F_1 = \prod_{i=1}^{k} [g_{i,1}, h_{i,1}], \\ F_0 = \prod_{i=1}^{k} [g_{i,0}, h_{i,0}], \end{cases}$$

where $g_{i,j}$ and $h_{i,j}$ belong to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

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For every index i, let us take $G_i : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ (respectively H_i), a lift of an element g_i (respectively h_i) of $\mathrm{Homeo}_0(\mathbb{A})$, which satisfies

$$\begin{cases} p_1 \circ G_i(\cdot, 1) = g_{i,1}, \\ p_1 \circ G_i(\cdot, 0) = g_{i,0} \end{cases}$$

$$\left(\text{respectively,} \quad \begin{cases} p_1 \circ H_i(\cdot, 1) = h_{i,1}, \\ p_1 \circ H_i(\cdot, 0) = h_{i,0} \end{cases}\right).$$

Note that the homeomorphisms H_i and G_i exist because $g_{i,j}$ and $h_{i,j}$ are isotopic to the identity. Then, the homeomorphism $[h_k, g_k] \circ [h_{k-1}, g_{k-1}] \circ \ldots \circ [h_1, g_1] \circ f$ admits a lift that pointwise fixes the boundary.

3.2.2. *Second step: a homeomorphism with a lift that pointwise fixes the boundary can be written as a product of four commutators.* Fix a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{A})$, with a lift F that is the identity on the boundary.

Consider a path $c : [0, 1] \to \mathbb{A}$ that satisfies the following conditions, where p_1 and p_2 are, respectively, the first and the second projection of $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$.

- $c(0) = (-\frac{1}{4}, 0)$ and $c(1) = (\frac{1}{4}, 0)$.
- For all $t \in (0, 1)$, $0 < p_2 \circ c(t) < \frac{1}{4}$.
- For all $t \in [0, 1]$, $p_2 \circ f \circ c(t) < \frac{1}{4}$.
- For all $t \in [0, 1]$, $p_1 \circ f \circ c(t) \in (-\frac{3}{8}, \frac{3}{8})$.
- For all $t \in [0, 1]$, $p_1 \circ c(t) \in (-\frac{3}{8}, \frac{3}{8})$.

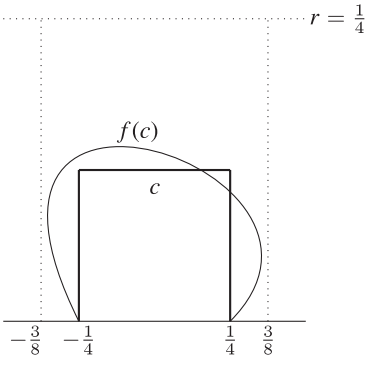


FIGURE 2. The path c.

Let D be the connected component of $\mathbb{A} - ([-\frac{1}{4}, \frac{1}{4}] \times \{0\} \cup c([0, 1]))$, whose boundary contains the point $(0, 0)$. According to the Schönfliess theorem, there exists a homeomorphism $f_{1,0}$ in $\mathrm{Homeo}_0(\mathbb{A})$ that coincides with f on the closure of D (which is a closed topological disk) and with support included in $(-\frac{3}{8}, \frac{3}{8}) \times [0, \frac{1}{2}) = D_{1,0}$. By taking a lace homotopic to the lower boundary component and close to it, and applying the Schönfliess theorem, we can find a homeomorphism $f_{2,0}$ that coincides with $f_{1,0}^{-1} \circ f$ (which is the identity on D) in a neighborhood of the lower boundary $\mathbb{R} \times \{0\}$ and which

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satisfies

$$\mathrm{supp}(f_{2,0}) \subset (\tfrac{1}{8}, \tfrac{7}{8}) \times [0, \tfrac{1}{2}) = D_{2,0}.$$

With the same technique, we can build homeomorphisms $f_{1,1}$ and $f_{2,1}$ in $\mathrm{Homeo}_0(\mathbb{A})$ that satisfy the following.

- The homeomorphism $f_{1,1} \circ f_{2,1}$ coincides with f in a neighborhood of the upper boundary $\mathbb{R} \times \{1\}$.
- The support of $f_{1,1}$ is included in $(-\frac{3}{8}, \frac{3}{8}) \times (\frac{1}{2}, 1] = D_{1,1}$.
- The support of $f_{2,1}$ is included in $(\frac{1}{8}, \frac{7}{8}) \times (\frac{1}{2}, 1] = D_{2,1}$.

Thus, we have $f = f_{1,1} \circ f_{2,1} \circ f_{1,0} \circ f_{2,0}$ near the boundary.

The next lemma is classical. The idea of the proof of this lemma seems to appear for the first time in [**2**].

LEMMA 3.2. *Let $\mathbb{H}$ denote the closed upper half-plane. Then, every element in* $\mathrm{Homeo}_0(\mathbb{H})$ *can be written as one commutator.*

Proof. Let h be an element in $\mathrm{Homeo}_0(\mathbb{H})$. Let U be a neighborhood of $\mathrm{supp}(h)$ and φ be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{H})$ that satisfies the following conditions.

– The open sets $\varphi^k(U)$, for k in $\mathbb{N}$, are pairwise disjoint.
– The sequence $(\varphi^k(U))_{k\in\mathbb{N}}$ converges to a singleton $\{p\}$, which lies on the boundary as k tends to $+\infty$.

Now, consider the homeomorphism g in $\mathrm{Homeo}_0(\mathbb{H})$ that satisfies the following.

– $g = \mathrm{Id}$ outside $\bigcup_{k\in\mathbb{N}} \varphi^k(U)$.
– For every non-negative integer k, $g_{|\varphi^k(U)} = \varphi^k \circ h \circ \varphi^{-k}$.

Then, $h = [g, \varphi]$. □

According to the lemma applied in the disks $D_{i,j}$, each $f_{i,j}$ is a commutator; thus, f coincides with a product of four commutators in a neighborhood of the boundary. We will see that we can improve this to two. Using this last lemma, we may consider homeomorphisms $g_{i,j}$ and $h_{i,j}$, supported in $D_{i,j}$, that satisfy $f_{i,j} = [g_{i,j}, h_{i,j}]$. Note that

$$\begin{aligned}
\mathrm{supp}(g_{i,0}) \cap \mathrm{supp}(g_{i,1}) &= \emptyset,\\
\mathrm{supp}(h_{i,0}) \cap \mathrm{supp}(h_{i,1}) &= \emptyset,\\
\mathrm{supp}(g_{i,0}) \cap \mathrm{supp}(h_{i,1}) &= \emptyset,\\
\mathrm{supp}(h_{i,0}) \cap \mathrm{supp}(g_{i,1}) &= \emptyset
\end{aligned}$$

and

$$\mathrm{supp}(f_{i,0}) \cap \mathrm{supp}(f_{i,1}) = \emptyset.$$

Thus, those pairs of homeomorphisms commute. Therefore,

$$\begin{aligned}
g &= f_{1,1} \circ f_{2,1} \circ f_{1,0} \circ f_{2,0} = f_{1,1} \circ f_{1,0} \circ f_{2,1} \circ f_{2,0}\\
&= [g_{1,1}, h_{1,1}] \circ [g_{1,0}, h_{1,0}] \circ [g_{2,1}, h_{2,1}] \circ [g_{2,0}, h_{2,0}]\\
&= [g_{1,1} \circ g_{1,0}, h_{1,1} \circ h_{1,0}] \circ [g_{2,1} \circ g_{2,0}, h_{2,1} \circ h_{2,0}].
\end{aligned}$$

Moreover, this homeomorphism coincides with f on a neighborhood of the boundary. Thus, $g^{-1} \circ f$ admits a lift that is the identity near the boundary. As the open annulus is a

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portable manifold (see [**5**, Theorem 1.18]), the homeomorphism $g^{-1} \circ f$ can be written as a product of two commutators. This ends the proof of Proposition 2.2 in the case $r = 0$.

3.3. *Case $r > 0$.* It follows directly from the result on homeomorphisms that

$$\text{for all } f \in \mathrm{Diff}_0^r(\mathbb{A}), \quad cl_r(f) \geq cl_0(f) \geq E\left(\frac{\alpha(f)}{4}\right) + 1.$$

To get an upper bound (for $r \neq 2, 3$), the process is the following. We first write a diffeomorphism f as a product of $E((\alpha(f) + 3)/4) + 5$ commutators of homeomorphisms, and we approximate every homeomorphism appearing in this product by a diffeomorphism. Hence, arbitrarily close to f in the C^0 topology, there is a product of $E((\alpha(f))/4) + 5$ commutators of diffeomorphisms. To conclude, it suffices to notice that, for $r \neq 2, 3$, a diffeomorphism sufficiently close to the identity can be written as a product of four commutators. Let us show this last fact. We need the following lemma, which is a consequence of a fragmentation lemma proved in the Appendix A.

LEMMA 3.3. *There exists a C^0 neighborhood Υ of the identity in* $\mathrm{Diff}_0^r(\mathbb{A})$ *such that,*

$$\text{for all } f \in \Upsilon, \text{ there exist } f_1,\ f_2 \in \mathrm{Diff}_0^r(\mathbb{A}), \text{ such that } \begin{cases} f = f_1 \circ f_2, \\ \mathrm{supp}(f_1) \subset (-\frac{3}{8}, \frac{3}{8}) \times [0, 1], \\ \mathrm{supp}(f_2) \subset (\frac{1}{8}, \frac{7}{8}) \times [0, 1]. \end{cases}$$

Now, the manifolds $(-\frac{3}{8}, \frac{3}{8}) \times [0, 1]$ and $(\frac{1}{8}, \frac{7}{8}) \times [0, 1]$ are portable, in the sense of [**5**]. It follows from [**5**, Theorem 1.18] that, if $r \neq 2, 3$, the diffeomorphisms f_1 and f_2 may each be written as a product of two commutators; the proof of this theorem remains valid for finite regularity, except that it uses the fact that the group $\mathrm{Diff}_0^r(\mathbb{A})$ is perfect, which is true only when $r \neq 2$ and $r \neq 3$.

This concludes the proof in the case $r > 0$.

4. *Estimation of the fragmentation norm*

The analog of Proposition 2.2 for the fragmentation norm is the following proposition, which implies the second part of Theorem 1.1.

PROPOSITION 4.1. *Let f be a homeomorphism in* $\mathrm{Diff}_0^r(\mathbb{A})$. *Suppose $f \neq \mathrm{Id}$. Then, for any r in $\mathbb{N} \cup \{\infty\}$,*

$$E(\alpha(f)) + 40 \geq \mathrm{Frag}_r(f) \geq E(\alpha(f)) + 1.$$

If $r = 0$, then the 40 *appearing in the upper bound may be improved to* 36.

Actually, we can have a far better upper bound in the preceding proposition by using the following result, which will be shown in a later paper.

PROPOSITION 4.2. *For any $r \in \mathbb{N} \cup \{\infty\}$, the fragmentation norm on* $\mathrm{Diff}_0^r(\mathrm{int}(\mathbb{A}))$ *is bounded by* 4.

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If we admit this last proposition, the constant 40 can be replaced by 16 in the preceding proposition.

To make estimates on the fragmentation norm for the closed annulus, we need an analog of the Eisenbud–Hirsch–Neumann theorem.

4.1. *Fragmentation norm on* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$. Let A be the subset of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ given by elements that fix pointwise a non-empty open interval. This subset generates $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ as a group. If F is a homeomorphism in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we denote by $\mathrm{Frag}(F)$ the minimal number of elements of A necessary to write F as a product of elements of A. The analog of the Eisenbud–Hirsch–Neumann theorem for the fragmentation norm is the following.

PROPOSITION 4.3. *Let F be a homeomorphism in* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$. *Then, for any $k \geq 0$,*

$$\mathrm{Frag}(F) \leq k+2 \Longleftrightarrow \min_{x\in\mathbb{R}} |F(x)-x| < k+1.$$

Proof. Let us start with the direct implication. Write

$$F = F_1 F_2 \dots F_{k+1} F_{k+2},$$

where each F_i belongs to A. Take a point x_0 such that $F_{k+2}(x_0) = x_0$. We have, for every integer $i \in [1, k+1]$,

$$|F_i F_{i+1} \dots F_{k+1} F_{k+2}(x_0) - F_{i+1} \dots F_{k+1} F_{k+2}(x_0)| < 1,$$

and, by summing these inequalities,

$$|F(x_0) - x_0| < k+1.$$

This proves the direct implication. Let us show the converse by induction on k. For $k=0$, up to replacing F by F^{-1}, we may assume that

$$0 \leq \min_{x\in\mathbb{R}} |F(x)-x| = F(x_0) - x_0 < 1.$$

Choose a point x_1 such that $F(x_0) < x_1 < x_0 + 1$. Then, $F(x_0) < F(x_1) < F(x_0)+1$, and we can find a homeomorphism h in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ that fixes a neighborhood of $F(x_0)$, and satisfies $h \circ F = \mathrm{Id}$ in a neighborhood of x_1. The decomposition $F = h^{-1} \circ (h \circ F)$ shows that $\mathrm{Frag}(F) \leq 2$.

Suppose the converse holds for an integer k. Let us prove it for the integer $k+1$. Suppose $\min_{x\in\mathbb{R}} |F(x)-x| < k+2$. We may also suppose that $\min_{x\in\mathbb{R}} |F(x)-x| \geq k+1$ (otherwise, we can conclude directly from the induction hypothesis). As usual, we may assume that

$$k+1 \leq \min_{x\in\mathbb{R}} (F(x)-x) < k+2.$$

Let x_0 be a point that satisfies

$$\min_{x\in\mathbb{R}} (F(x)-x) = F(x_0) - x_0.$$

In the same way as for the initialization, we can find a point $x_1 \in (x_0, x_0+1)$ such that $F(x_0) < x_1 + k + 1 < F(x_0) + 1$. Then, $F(x_0) < F(x_1) < F(x_0)+1$. Therefore, there exists a homeomorphism h that fixes a neighborhood of $F(x_0)$ and satisfies

$$h(F(x_1)) < x_1 + k + 1.$$

The induction hypothesis then allows us to finish the induction and the proof of Proposition 4.3. □

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4.2. *Proof of Proposition 4.1.* The technique is exactly the same as in the proof of Proposition 2.2, using Proposition 4.3 instead of the Eisenbud–Hirsch–Neumann theorem. Thus, the explanations will be briefer.

4.2.1. *Lower bound for the fragmentation norm.* Let f be an element in $\mathrm{Homeo}_0(\mathbb{A})$ with $\mathrm{Frag}_0(f) \geq 1$, and F be a lift of f. Fix a decomposition of f as a product of $\mathrm{Frag}_0(f)$ homeomorphisms supported in disks. Let k_0 (respectively k_1) be the number of homeomorphisms appearing in this decomposition whose support meets the lower boundary $\mathbb{R}/\mathbb{Z} \times \{0\}$ (respectively the upper boundary $\mathbb{R}/\mathbb{Z} \times \{1\}$) of $\mathbb{A}$. Note that $k_0 + k_1 \leq \mathrm{Frag}_0(f)$, as a disk, does not touch both components of the boundary, by definition. First, suppose $k_0 \geq 2$ and $k_1 \geq 2$. By Proposition 4.3, there exist points x_0 and x_1 in $\mathbb{R}$ such that

$$|F_0(x_0) - x_0| < k_0 - 1$$

and

$$|F_1(x_1) - x_1| < k_1 - 1.$$

From these inequalities, we get

$$\alpha(f) < k_0 - 1 + 1 + k_1 - 1 \leq \mathrm{Frag}_0(f),$$

which gives the lower bound in Proposition 4.1. In the cases $k_0 = 1$ and $k_1 \geq 2$ or $k_1 = 1$ and $k_0 \geq 2$, one of the inequalities

$$|F_0(x_0) - x_0| \leq k_0 - 1$$

and

$$|F_1(x_1) - x_1| \leq k_1 - 1$$

is indeed strict, which allows us to conclude.

In the case $k_0 = 0$ and $k_1 \geq 2$ (the case $k_1 = 0$ and $k_0 \geq 2$ is symmetric), we have, as $F_0 = \mathrm{Id}$,

$$\alpha(f) < |F_1(x_1) - x_1| + 1 \leq k_1 \leq \mathrm{Frag}_0(f).$$

In the case $(k_0, k_1) = (1, 1)$, we have $\alpha(f) < 1$ and $\mathrm{Frag}_0(f) \geq 2$, so the inequality $\alpha(f) < \mathrm{Frag}_0(f)$ holds.

In the cases $(k_0, k_1) \in \{(0, 0), (0, 1), (1, 0)\}$, we have $\alpha(f) = 0$, so the inequality $\alpha(f) < \mathrm{Frag}_0(f)$ holds.

4.2.2. *Upper bound for the fragmentation norm.* Let f be an element in $\mathrm{Homeo}_0(\mathbb{A})$ with lift F. As usual, we may assume that $\alpha(f) = p_1 \circ F(x_0, 1) - p_1 \circ F(x_0, 0)$. Consider a lift of f that satisfies

$$-1 < F_0(x_0) - x_0 \leq 0.$$

Then, $F_1(x_0) - x_0 = F_0(x_0) - x_0 + \alpha(f) < E(\alpha(f)) + 1$. Using Proposition 4.3, we can see that, after composing by at most $E(\alpha(f)) + 2$ homeomorphisms supported in disks that touch the upper boundary, we get a homeomorphism with a lift that fixes the upper

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boundary and, by composing by two more homeomorphisms supported in disks we get a homeomorphism with a lift that fixes both boundary components. Then, by composing by four homeomorphisms supported in disks, we get a homeomorphism with a lift that fixes a neighborhood of the boundary. Finally, by the result by Burago, Ivanov and Polterovich (see the proof of Theorem 1.17 in [**5**]), such a homeomorphism may be written as a product of 28 homeomorphisms supported in disks. Using Proposition 4.2, we see that this might be improved to four.

Then, an approximation argument using Proposition A.1 combined with the fragmentation lemma yields the case $r > 0$.

5. *Generalization to other surfaces*

A similar construction as the one made in §2 can be carried out on every compact surface M with a boundary, to obtain quasi-morphisms. However, in those cases, we do not know the dimension of the space of quasi-morphisms, we just have a lower bound of it.

5.1. *Case of open surfaces.* Suppose M is a non-compact surface with p boundary components that are circles. Let us fix such a boundary component of M: C_1. Take a path homeomorphic to the half-line, which begins on C_1, goes to infinity, and touches the boundary component C_1 only at the endpoint. Consider the cyclic covering $p : \tilde{M} \to M$ associated with this path. Now, a homeomorphism f in $\mathrm{Homeo}_0(M)$ admits a unique lift $F : \tilde{M} \to \tilde{M}$, which is supported in $p^{-1}(\mathrm{supp}(f))$. The translation number of the restriction of F to $p^{-1}(C_1)$ gives rise to a quasi-morphism. With this method, p independent quasi-morphisms can be built. Moreover, if the commutator length is bounded on the group $\mathrm{Diff}_0^r(\mathrm{int}(M))$, where $\mathrm{int}(M)$ denotes $M - \partial M$, then it can be shown, with the same techniques as in the case of the annulus, that the vector space of homogeneous quasi-morphisms is p-dimensional. As in the case of the closed annulus, this is a consequence of an estimation of the commutator length. Take a homeomorphism f. First use the Eisenbud–Hirsch–Neumann theorem to get a homeomorphism with a lift that fixes the boundary components after a composition by a number of commutators proportional to the sum of the values of these quasi-morphisms at f. Then, by composing this homeomorphism by a bounded number of commutators (depending only on the number of boundary components), we can get a homeomorphism in $\mathrm{Diff}_0^r(\mathrm{int}(M))$. Finally, we have to use the fact that the commutator length is bounded on this latter group, to conclude. In particular, the vector space of homogeneous quasi-morphism is one-dimensional in the case of the half-open annulus $\mathbb{R}/\mathbb{Z} \times [0, 1)$. However, it is not known whether or not the commutator length is bounded on the group $\mathrm{Diff}_0^r(\mathrm{int}(M))$, when $\mathrm{int}(M)$ is different from the open annulus or from $\mathbb{R}^2$.

5.2. *Case of other compact oriented surfaces with a boundary.* When M is a compact oriented surface with a boundary that is different from the closed disk or the closed annulus, its universal covering $\tilde{M}$ may be seen as a subspace of the Poincaré disk $\mathbb{D}$. We endow M and $\mathbb{D}$ with Riemannian metrics of constant curvature -1, such that M has a geodesic boundary, and the projection $\tilde{M} \to M$ is a Riemannian covering. Denote by d

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the associated distance. Given two points x and y on $\mathbb{D}$, and an oriented geodesic g that contains both x and y, we define $a(x, y, g) = d(x, y)$ if the geodesic from x to y has the same orientation as g, and $a(x, y, g) = -d(x, y)$ otherwise.

Let us take a homeomorphism f in $\mathrm{Homeo}_0(M)$. The homeomorphism f admits a canonical lift $F : \tilde{M} \to \tilde{M}$, which is the identity on the limit set of $\tilde{M}$ at infinity. Given a point x on the boundary component C of $\tilde{M}$, which is a geodesic, define

$$q(x) = \lim_{n \to +\infty} \left(\frac{a(x, F^n(x), C)}{n} \right).$$

This number is the translation number of F, restricted to C, and does not depend on the point x chosen on C. With this construction, we can build p independent quasi-morphisms, where p is equal to the number of boundary components of M. Here also, the vector space of quasi-morphisms on $\mathrm{Diff}_0^r(M)$ is p-dimensional if the commutator length on $\mathrm{Diff}_0^r(\mathrm{int}(M))$ is bounded.

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A. *Appendix. A C^0-fragmentation lemma for diffeomorphisms*

In this section, we will prove Lemma 3.3. This lemma follows directly from the following proposition, which is more general.

PROPOSITION A.1. *Let M be a compact surface, possibly with a boundary, and $(U_i)_{1 \le i \le n}$ be a finite open covering of M. Denote by d a distance on* $\mathrm{Homeo}_0(M)$ *compatible with the C^0-topology. Then, for every $\epsilon > 0$, there exists $\alpha > 0$ such that, given a diffeomorphism f in* $\mathrm{Diff}_0^r(M)$ *satisfying $d(f, \mathrm{Id}_M) < \alpha$, there exist diffeomorphisms $f_1, f_2, \ldots, f_n$ such that:*

- $f = f_1 \circ f_2 \circ \cdots \circ f_n$;
- *for every index i,* $\mathrm{supp}(f_i) \subset U_i$; *and*
- $d(f_i, \mathrm{Id}_M) < \epsilon$.

To prove this proposition, we will need the following extension lemma.

The half-closed disk is the subset of $\mathbb{R}^2$:

$$\{(x, y) \in \mathbb{R}^2, x \ge 0 \text{ and } x^2 + y^2 \le 1\}.$$

LEMMA A.2. *Let U be an open set, F be a closed set in M, and N be an open neighborhood of F. Let K be a subset of U, which is C^∞-diffeomorphic to a closed disk if $K \cap \partial M = \emptyset$, or to a half-closed disk if $K \cap \partial M \neq \emptyset$. Then, for every $\epsilon > 0$, there exists $\beta > 0$ such that, given a diffeomorphism f in* $\mathrm{Diff}_0^r(M)$ *satisfying $d(f, \mathrm{Id}_M) < \beta$, and $f = \mathrm{Id}_M$ on N, there exists a diffeomorphism g in* $\mathrm{Diff}_0^r(M)$ *such that:*

- $\mathrm{supp}(g) \subset U$;
- *$f = g$ in a neighborhood of K;*
- *$g = \mathrm{Id}_M$ in a neighborhood of F; and*
- $d(g, \mathrm{Id}_M) < \epsilon$.

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Let us see first how this lemma implies Proposition A.1.

Proof of Proposition A.1. First, we claim that it suffices to prove the proposition when each open set U_i is homeomorphic to an open disk. Indeed, suppose the proposition holds for a covering by open disks. Take an arbitrary finite covering $(U_i)_{1\leq i\leq n}$. Then, we can find a covering by open disks $D_{i,j}$ with $1\leq i\leq n$ and $1\leq j\leq k_i$ such that $D_{i,j}$ is included in U_i. Then, if f is a diffeomorphism of M sufficiently close to the identity, there exist diffeomorphisms $f_{i,j}$ close to the identity such that

$$\begin{cases}\operatorname{supp}(f_{i,j})\subset D_{i,j},\\ f=f_{1,1}\circ f_{1,2}\circ\cdots\circ f_{1,k_1}\circ f_{2,1}\circ\cdots\circ f_{n,k_n}.\end{cases}$$

It suffices to take $f_i=f_{i,1}\circ f_{i,2}\circ\cdots\circ f_{i,k_i}$ to conclude the proof for an arbitrary finite covering.

Suppose now that each open set U_i is homeomorphic to an open disk. For each index i, take a subset K_i of U_i diffeomorphic to a closed disk, in such a way that

$$\bigcup_{1\leq i\leq n} K_i = M.$$

We show by induction on i the following statement (the case $i=n$ proves Proposition A.1): for every $\epsilon>0$, there exists $\alpha>0$ such that, given a diffeomorphism f in $\mathrm{Diff}_0^r(M)$ satisfying $d(f,\mathrm{Id}_M)<\alpha$, there exist diffeomorphisms $f_1, f_2, \ldots, f_i$ in $\mathrm{Diff}_0^r(M)$ such that:

(1) $f=f_1\circ f_2\circ\cdots\circ f_i$ in a neighborhood of $\bigcup_{j\leq i} K_j$;
(2) for all $j\leq i$, $\operatorname{supp}(f_j)\subset U_j$; and
(3) $d(f_j,\mathrm{Id}_M)<\epsilon$.

The above statement is true for $i=1$, by Lemma A.2. Suppose the above statement holds for an integer i. Fix $\epsilon>0$. Let β be given by Lemma A.2, applied with $F=\bigcup_{j\leq i} K_j$, $K=K_{i+1}$ and $U=U_{i+1}$. Using the induction hypothesis for small enough ϵ', there exists $\alpha>0$ such that if we take a diffeomorphism f α-close to the identity, we can get a family of diffeomorphisms $(f_j)_{1\leq j\leq i}$ that satisfies (1), (2) and (3), and such that $f_i^{-1}\circ f_{i-1}^{-1}\circ\cdots\circ f_1^{-1}\circ f$ is β-close to the identity. By Lemma A.2, we can find a diffeomorphism f_{i+1} in $\mathrm{Diff}_0^r(M)$ such that:

(1) $\operatorname{supp}(f_{i+1})\subset U_{i+1}$;
(2) $f_{i+1}=f_i^{-1}\circ f_{i-1}^{-1}\circ\cdots\circ f_1^{-1}\circ f$ in a neighborhood of K_{i+1};
(3) $f_{i+1}=\mathrm{Id}_M$ in a neighborhood of $\bigcup_{j\leq i} K_j$; and
(4) $d(f_{i+1},\mathrm{Id}_M)<\epsilon$.

Properties (2) and (3), together with the induction hypothesis, imply then that $f=f_1\circ f_2\circ\cdots\circ f_{i+1}$ on a neighborhood of $\bigcup_{j\leq i+1} K_j$. This concludes the proof. □

To prove Lemma A.2, we need some specific extension lemmas. The first lemma is proved in the appendix of [**9**] by Khanevsky.

LEMMA A.3. *Let $\epsilon>0$, $\epsilon_1\in(0,1)$, $r\in\mathbb{N}\cup\{\infty\}$. Consider a finite union of closed intervals $I\subset\mathbb{S}^1$. Then, there exists $\alpha>0$ such that, for every C^r-embedding $f:\mathbb{S}^1\times(-\epsilon_1,\epsilon_1)\to\mathbb{S}^1\times(-1,1)$, which satisfies*

$$\begin{cases}d(f,\mathrm{Id})<\alpha,\\ f_{|I\times(-\epsilon_1,\epsilon_1)}=\mathrm{Id},\end{cases}$$

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there exists a diffeomorphism g *in* $\mathrm{Diff}_0^r(\mathbb{S}^1 \times (-1, 1))$ *such that:*
(1) $d(g, \mathrm{Id}) < \epsilon$;
(2) $g_{|\mathbb{S}^1 \times 0} = f_{|\mathbb{S}^1 \times 0}$; *and*
(3) $g_{|I \times (-1,1)} = \mathrm{Id}$.

The following lemma deals with embeddings of the annulus that preserve a circle.

LEMMA A.4. *Let* $\epsilon_1 \in [0, 1)$, $r \in \mathbb{N} \cup \{\infty\}$ *and* I *denote a finite union of closed intervals in* $\mathbb{S}^1$. *Denote by* f *a* C^r*-embedding of* $\mathbb{S}^1 \times (-\epsilon_1, \epsilon_1)$ *in* $\mathbb{S}^1 \times (-1, 1)$, *which fixes pointwise* $\mathbb{S}^1 \times \{0\}$ *and* $I \times (-\epsilon_1, \epsilon_1)$. *Then, for any* $\epsilon > 0$, *there exists a diffeomorphism* g *in* $\mathrm{Diff}_0^r(\mathbb{S}^1 \times (-1, 1))$ *fixing pointwise* $\mathbb{S}^1 \times \{0\}$, *which satisfies the following:*
(1) $g = f$ *in a neighborhood of* $\mathbb{S}^1 \times \{0\}$;
(2) $d(g, \mathrm{Id}) < \epsilon$; *and*
(3) $g_{|I \times (-1,1)} = \mathrm{Id}$.

Proof. Consider $\epsilon' > 0$ such that the product of two diffeomorphisms in $\mathrm{Diff}_0^r(\mathbb{R} \times (-1, 1))$ ϵ'-close to the identity is ϵ-close to the identity. We will first find a diffeomorphism g_1 ϵ'-close to the identity that preserves the foliation with leaves $\mathbb{S}^1 \times \{y\}$ and for which, for $\epsilon_2 > 0$ small enough, $g_1^{-1} \circ f$ preserves the germ at $\mathbb{S}^1 \times \{0\}$ of the foliation with leaves $\{x\} \times (-1, 1)$. Then, we will find a diffeomorphism g_2 ϵ'-close to the identity that equals $g_1^{-1} \circ f$ on a neighborhood of $\mathbb{S}^1 \times \{0\}$ and take $g = g_1 \circ g_2$.

Take $\delta_1 < \epsilon_1$ such that f^{-1} is well defined on $\mathbb{S}^1 \times (-\delta_1, \delta_1)$ and, if the expression of f^{-1} in coordinates is given by

$$\begin{aligned} f^{-1} : \mathbb{S}^1 \times (-\delta_1, \delta_1) &\to \mathbb{S}^1 \times (-1, 1), \\ (x, y) &\mapsto (u(x, y), v(x, y)), \end{aligned}$$

then

$$\text{for all } (x, y) \in \mathbb{S}^1 \times (-\delta_1, \delta_1), \quad \begin{cases} |u(x, y) - x| < \epsilon', \\ \dfrac{\partial u}{\partial x}(x, y) > 0. \end{cases}$$

Denote by u_y^{-1} the inverse of $u(\cdot, y)$. Denote by $\lambda : (-1, 1) \to [0, 1]$ a C^∞ map that is supported in $(-\delta_1, \delta_1)$, and is equal to one on a neighborhood of 0. Now, define

$$\begin{aligned} g_1 : \mathbb{S}^1 \times (-1, 1) &\to \mathbb{S}^1 \times (-1, 1), \\ (x, y) &\mapsto (\lambda(y) u_y^{-1}(x) + (1 - \lambda(y))x, y). \end{aligned}$$

In other words, when y is close to 0, $g_1(x, y)$ is the point with ordinate y that meets $f(\{x\} \times (-\delta_1, \delta_1))$. For $\epsilon_2 > 0$ small enough, the map g_1 then satisfies the following properties:
(1) for all $x \in \mathbb{S}^1$, $g_1^{-1} \circ f(\{x\} \times (-\epsilon_2, \epsilon_2)) \subset \{x\} \times (-1, 1)$;
(2) $d(g_1, \mathrm{Id}) < \epsilon$; and
(3) $g_{1\,|I \times (-1,1)} = \mathrm{Id}$.

Now, the expression of the embedding $f_1 = g_1^{-1} \circ f$ in coordinates is of the form

$$\begin{aligned} f_1 : \mathbb{S}^1 \times (-\epsilon_2, \epsilon_2) &\to \mathbb{S}^1 \times (-1, 1), \\ (x, y) &\mapsto (x, w(x, y)). \end{aligned}$$

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Consider the isotopy, defined for t in $[0, 1]$,

$$f_t : \mathbb{S}^1 \times (-\epsilon_1, \epsilon_1) \to \mathbb{S}^1 \times (-1, 1),$$
$$(x, y) \mapsto (x, (1-t)y + tw(x, y)).$$

The vector field associated to this isotopy is

$$X(t, x, y) = \frac{d}{dt}(f_t)(f_t^{-1}(x, y)).$$

It is defined for all t in a time-independent neighborhood of $\mathbb{S}^1 \times \{0\}$, which we denote by $\mathbb{S}^1 \times (-\delta_2, \delta_2)$. Observe that this vector fields vanishes on $\mathbb{S}^1 \times \{0\}$ and on $I \times (-\delta_2, \delta_2)$. Take $0 < \delta_3 < \delta_2$ such that

$$\sup_{(t,x,y)\in[0,1]\times\mathbb{S}^1\times(-\delta_3,\delta_3)} \|X(t, x, y)\| < \epsilon'.$$

Now, let $\lambda : \mathbb{R} \to [0, 1]$ be a C^∞ function supported in $(-\delta_3, \delta_3)$, which is equal to 1 on a neighborhood of 0. Define a vector field Y on $\mathbb{S}^1 \times (-1, 1)$ for t in $[0, 1]$ by

- $Y(t, x, y) = \lambda(y)X(t, x, y)$ if $(x, y) \in \mathbb{S}^1 \times (-\delta_3, \delta_3)$; and
- $Y(t, x, y) = 0$ otherwise.

Then, if we denote by g_2 the time 1 of the flow of Y, the diffeomorphism g_2 satisfies the following properties:

(1) $g_2 = f_1$ in a neighborhood of $\mathbb{S}^1 \times \{0\}$;
(2) $d(g_2, \mathrm{Id}) < \epsilon'$; and
(3) $g_{2\,|I\times(-1,1)} = \mathrm{Id}$.

Now, the diffeomorphism $g = g_1 \circ g_2$ satisfies the required properties. □

Now, we are ready to prove Lemma A.2 in the case where $K \cap \partial M = \emptyset$.

Proof of Lemma A.2 in the case where $K \cap \partial M = \emptyset$. Consider a tubular neighborhood of ∂K sufficiently small so that, after identification of this neighborhood with $\partial K \times (-1, 1) \subset U$, there exists I, a finite union of intervals of ∂K, such that $F \cap \partial K \times (-1, 1) \subset I \times (-1, 1) \subset N$. Choose $\epsilon' > 0$ so that the product of two diffeomorphisms in $\mathrm{Diff}_0^r(M)$, ϵ'-close to the identity for the C^0 distance d, is ϵ-close to the identity. By applying successively Lemmas A.3 and A.4 to a diffeomorphism f sufficiently close to the identity, we find a diffeomorphism g_1 with support included in U, which satisfies

$$\begin{cases} d(g_1, \mathrm{Id}) < \epsilon', \\ d(g_1^{-1} \circ f, \mathrm{Id}) < \epsilon', \end{cases}$$

which coincides with f in a neighborhood of ∂K, and which fixes a neighborhood of F. Denote by g_2 the diffeomorphism that is equal to $g_1^{-1} \circ f$ on K, and to the identity elsewhere. The diffeomorphism $g = g_1 \circ g_2$ then satisfies the required properties. □

To prove the case $K \cap \partial M \neq \emptyset$ in Lemma A.2, we need a few extra lemmas.

LEMMA A.5. *Let J be an open interval in $\mathbb{R}$, J' be a closed subinterval of J, and I be a finite union of closed intervals in $\mathbb{R}$. For any $\epsilon > 0$, there exists $\alpha > 0$ such that, given a C^r increasing embedding $f : J \to \mathbb{R} \times \{0\} \subset \mathbb{R} \times [0, 1)$ with:*

- $d(f, \mathrm{Id}) < \alpha$*; and*
- $f_{|I\cap J} = \mathrm{Id}$,

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there exists a diffeomorphism g in $\mathrm{Diff}_0^r(\mathbb{R}\times[0,1))$ *that satisfies:*

(1) $g_{|J'\times\{0\}} = f_{|J'\times\{0\}}$;
(2) $d(g, \mathrm{Id}) < \epsilon$; *and*
(3) $g_{|I\times[0,1)} = \mathrm{Id}$.

Proof. Let us take $\alpha < \min(\epsilon, d(J', J^c))$, where J^c is the complement of J in $\mathbb{R}$. Take an embedding f as in the hypothesis of the lemma. Consider the isotopy

$$h : [0,1]\times J\times[0,1)\to\mathbb{R}\times[0,1),\qquad (x,y)\mapsto((1-t)x+tf(x),y)$$

and denote by h_t the embedding $h(t,\cdot)$. The vector field associated with this isotopy is

$$X(t,x,y) = \frac{d(h_t)}{dt}(h_t^{-1}(x,y)).$$

Notice that $X(t,\cdot)$ is defined on $h_t(J\times[0,1))$, which contains a neighborhood of $J'\times[0,1)$ by the choice of α. Then, by multiplying X by a C^∞ function supported in a neighborhood of $J'\times[0,\frac{1}{2}]$, and which is equal to one on a smaller neighborhood of $J'\times[0,\frac{1}{2}]$, we obtain a compactly supported vector field on $\mathbb{R}\times[0,1)$, whose time one flow satisfies the required properties. □

The following lemma is analogous to Lemma A.4.

LEMMA A.6. *Let $\epsilon_1\in[0,1)$, $r\in\mathbb{N}\cup\{\infty\}$, J be an open interval in $\mathbb{R}$, J' be a closed interval included in J, and I denote a finite union of closed intervals in $\mathbb{R}$. Denote by f a C^r-embedding of $J\times[0,\epsilon_1)$ in $\mathbb{R}\times[0,1)$ that fixes pointwise $J\times\{0\}$ and $I\cap J\times[0,\epsilon_1)$. Then, for any $\epsilon>0$, there exists a diffeomorphism g in* $\mathrm{Diff}_0^r(\mathbb{R}\times[0,1))$ *fixing pointwise $\mathbb{R}\times\{0\}$, which satisfies:*

(1) $g = f$ *in a neighborhood of* $J'\times\{0\}$;
(2) $d(g,\mathrm{Id}) < \epsilon$; *and*
(3) $g_{|I\times[0,1)} = \mathrm{Id}$.

Proof. This proof is almost the same as the proof of Lemma A.4. Consider $\epsilon'>0$ such that the product of two diffeomorphisms in $\mathrm{Diff}_0^r(\mathbb{R}\times|0,1))$ ϵ'-close to the identity is ϵ-close to the identity. We will first find a diffeomorphism g_1 ϵ'-close to the identity that preserves the foliation with leaves $\mathbb{R}\times\{y\}$, and for which, for $\epsilon_2>0$ small enough, and x in a neighborhood of J', $g_1^{-1}\circ f(\{x\}\times[0,\epsilon_2))\subset\{x\}\times(-1,1)$. Then, we will find a diffeomorphism g_2 ϵ'-close to the identity that equals $g_1^{-1}\circ f$ on a neighborhood of $J'\times\{0\}$ and take $g = g_1\circ g_2$. Suppose the expression of f^{-1} in coordinates is given by

$$f^{-1} : J\times[0,\epsilon_1)\to\mathbb{R}\times(-1,1),\qquad (x,y)\mapsto(u(x,y),v(x,y)).$$

Take $\delta<\epsilon_1$ and an open interval J'', with $J'\subset J''\subset\overline{J''}\subset J$, such that

$$\text{for all } (x,y)\in J''\times[0,\delta),\ \begin{cases}|u(x,y)-x|<\epsilon',\\ \dfrac{\partial u}{\partial x}(x,y)>0.\end{cases}$$

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Denote by u_y the function $u(\cdot, y)$. Let $\lambda : \mathbb{R} \to [0, 1]$ be a C^∞ function supported in $(-\delta, \delta)$, and which is equal to one in a neighborhood of 0. Consider the embedding

$$\begin{array}{rcl} h : J'' \times [0, 1) & \to & \mathbb{R} \times [0, 1), \\ (x, y) & \mapsto & \begin{cases} ((1 - \lambda(y))y + \lambda(y)u_y^{-1}(x), y) & \text{if } y \leq \delta, \\ (x, y) & \text{otherwise.} \end{cases} \end{array}$$

It coincides with f on a neighborhood of $J'' \times \{0\}$. We now have to extend it in the horizontal direction in order to have a diffeomorphism. To do this, we consider the isotopy (for t in [0,1])

$$\begin{array}{l} h_t : J'' \times [0, 1) \to J'' \times [0, 1), \\ \qquad (x, y) \mapsto ((1 - t)x + tu'(x, y), y), \end{array}$$

where u' is the first coordinate of h. It suffices then to cut the vector field generating this isotopy by a C^∞ function to obtain a compactly supported vector field defined on $\mathbb{R} \times [0, 1)$. Then, the time one of the flow of this vector field gives a diffeomorphism g_1 of $\mathbb{R} \times [0, 1)$ that satisfies the following properties:

(1) for x in a neighborhood of J', $g_1^{-1} \circ f(\{x\} \times [0, \epsilon_2)) \subset \{x\} \times [0, 1)$;
(2) $d(g_1, \mathrm{Id}) < \epsilon'$; and
(3) $g_{1\,|I \times [0,1)} = \mathrm{Id}$.

The construction of g_2 is done exactly the same way as in Lemma A.4: consider a well-chosen isotopy that joins the identity to $f_1 = g_1^{-1} \circ f$, the vector field generating it, and cut this vector field outside a neighborhood of $J' \times \{0\}$ to get a global compactly supported vector field. Take the time one of the flow of this vector field. Then, it suffices to take $g = g_1 \circ g_2$. □

Now, we can finish the proof of Lemma A.2 by treating the case where $K \cap \partial M \neq \emptyset$.

Proof of Lemma A.2: case where $K \cap \partial M \neq \emptyset$. We will just explain what has to be taken care of in this proof, which is analogous to the proof of the case where $\partial K \cap M = \emptyset$. Take the diffeomorphism f sufficiently close to the identity. By using Lemma A.5, we may find a diffeomorphism g_1 C^0-close to the identity that coincides with f on a neighborhood in ∂M of $\partial K \cap \partial M$. Then, we use Lemma A.6 to get a diffeomorphism g_2 C^0-close to the identity that coincides with $g_1^{-1} \circ f$ on a neighborhood V in M of $\partial K \cap \partial M$. Let W be a neighborhood of $\partial K \cap \partial M$ whose closure is included in V. Consider then a simple closed path $\gamma : \mathbb{S}^1 = [0, 1]/\mathbb{Z} \to M$, which satisfies

$$\begin{cases} \gamma([-\frac{1}{4}, \frac{1}{4}]) \supset \partial K - W, \\ \gamma(\mathbb{S}^1) \subset (\partial K \cup W) - \partial M. \end{cases}$$

By applying the first case of Lemma A.2 with K' the interior of the Jordan curve γ, and F' the union of F with the closure of W, we may find a C^r-diffeomorphism g_3 C^0-close to the identity that is equal to $g_1^{-1} \circ f$ in a neighborhood of K. Then, we take $g = g_1 \circ g_2 \circ g_3$ to end the proof. □

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