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Piecewise expanding maps on the plane with singular ergodic properties

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Abstract. For $1 \leq r < \infty$, we construct a piecewise C^r expanding map $F : D \to D$ on the domain $D = (0, 1) \times (-1, 1) \subset \mathbb{R}^2$ with the following property: there exists an open set B in D such that the diameter of $F^n(B)$ converges to 0 as $n \to \infty$ and the empirical measure $n^{-1} \sum_{k=0}^{n-1} \delta_{F^k(x)}$ converges to the point measure δ_p at $p = (0, 0)$ as $n \to \infty$ for any point $x \in B$.

1. *Introduction*

In this paper, we construct examples of piecewise C^r expanding maps that have singular ergodic properties. Our examples give the conclusion that the analogue of Lasota–Yorke theory [**3**] for piecewise expanding maps on intervals is not valid for general piecewise C^r expanding maps in higher dimensions.

In the famous paper [**3**], Lasota and Yorke studied ergodic properties of piecewise C^2 expanding maps on intervals. They considered Perron–Frobenius operators associated to such maps, which operate on the space of integrable functions on the domains. Their discovery was that these operators decrease the variation of functions if the original functions are of bounded variation and their variation is sufficiently large compared to their L^1 norm. In particular, they proved that there are invariant compact subsets in the space of integrable functions, which consist of functions of bounded variation. From these facts and general theory, the existence of absolutely continuous invariant measures and some other ergodic properties were derived in [**3**].

After their work, it was quite natural to expect similar results in higher dimensions. However, it has turned out that the higher-dimensional case is not so simple as the one-dimensional case. The difficulty comes from the fact that the shapes of maximal regions where some iteration of a piecewise C^r expanding map is continuous can be quite complicated.

Piecewise expanding maps in higher dimensions have been studied in many works. In each of [**1–9**], an analogue of the argument of Lasota and Yorke for higher-dimensional

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maps was developed. However, some additional conditions on maps were always assumed. Thus there has arisen the question of whether the additional conditions are really necessary or not. For piecewise C^r expanding maps constructed in this paper, any analogue of the argument of Lasota and Yorke is clearly not valid. Hence, the additional conditions turn out to be necessary.

In order to state the properties of our examples precisely, let us give some definitions. A curve $c : [a, b] \to \mathbb{R}^2$ in the plane $\mathbb{R}^2$ is said to be a C^r curve if it can be extended to a neighborhood of the interval $[a, b]$ as a C^r map. A curve $c : [a, b] \to \mathbb{R}^2$ is said to be a piecewise C^r curve if there exists a sequence $a = a_0 < a_1 < \cdots < a_n = b$ such that the restrictions $c|_{[a_i, a_{i+1}]}$ of c to the intervals $[a_i, a_{i+1}]$, $0 \le i \le m-1$, are C^r curves in the sense above. We say that a bounded region in the plane has a piecewise C^r boundary if the boundary of the region consists of finitely many simple closed piecewise C^r curves.

Definition 1. Let $D \subset \mathbb{R}^2$ be a bounded region that has piecewise C^r boundary. A map $F : D \to D$ is said to be a piecewise C^r expanding map if there exists a family of finitely many regions $D_i \subset D$, $1 \le i \le k$, with the following properties:

(i) each region D_i, $1 \le i \le k$, has piecewise C^r boundary;
(ii) regions D_i, $1 \le i \le k$, are mutually disjoint and the union of the closures of them equals the closure of D;
(iii) each restriction $F|_{D_i}$ of the map F to D_i can be extended to a neighborhood of the closure of D_i as a C^r diffeomorphism; and
(iv) there exists a number $\rho > 1$ such that $\|DF(x)v\| \ge \rho\|v\|$ for any $x \in \cup_{i=1}^k D_i$ and any $v \in T_x\mathbb{R}^2$.

Let $1 \le r < \infty$. In this paper we construct examples of piecewise C^r expanding maps F on an open rectangle $D = (0, 1) \times (-1, 1)$ such that there exists a non-empty open subset $B \subset D$ with the following properties:

(A) the diameter of the set $F^n(B)$ converges to 0 as $n \to \infty$; and
(B) the empirical measure $n^{-1} \sum_{i=0}^{n-1} \delta_{F^i(x)}$ for $x \in B$ converges to the point measure δ_p at $p = (0, 0)$ as $n \to \infty$.

We emphasize that we do not know whether such examples exist in the C^∞ case. For piecewise real-analytic expanding maps on the plane, Buzzi [**2**] and the current author [**8**] recently showed the existence of absolutely continuous invariant measures by an argument parallel to that of Lasota and Yorke. Hence, no such examples as above exist for the case $r = \omega$.

2. *Construction of the examples*

For $1 \le r < \infty$, we construct a map F defined on an open rectangle

$$D = (0, 1) \times (-1, 1) = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, -1 < y < 1\}.$$

Let us fix an integer $\gamma > r$ and a small number $\varepsilon > 0$ satisfying

$$\varepsilon < \gamma^{-3}(1 - r^{-1})^2 \log 2. \tag{1}$$

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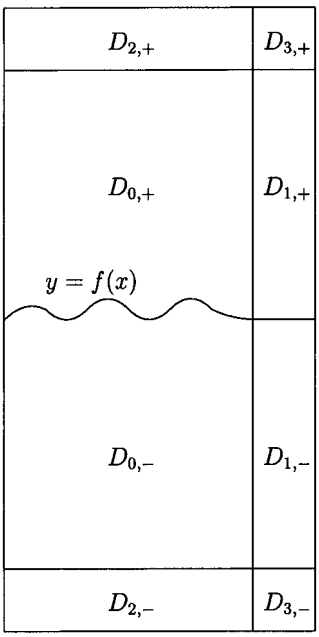


FIGURE 1. The partition of D.

We put $\lambda = \exp(\varepsilon)$. Let $f : [0, \exp(-\varepsilon)] \to [0, \exp(-\varepsilon))$ be a function of class C^r. We divide the rectangle D into the following eight regions with piecewise C^r boundaries (see Figure 1):

$$D_{0,+} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \exp(-\varepsilon),\ f(x) < y < \exp(-\varepsilon)\};$$
$$D_{0,-} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \exp(-\varepsilon),\ -\exp(-\varepsilon) < y < f(x)\};$$
$$D_{1,+} = \{(x, y) \in \mathbb{R}^2 \mid \exp(-\varepsilon) < x < 1,\ 0 < y < \exp(-\varepsilon)\};$$
$$D_{1,-} = \{(x, y) \in \mathbb{R}^2 \mid \exp(-\varepsilon) < x < 1,\ -\exp(-\varepsilon) < y < 0\};$$
$$D_{2,+} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \exp(-\varepsilon),\ \exp(-\varepsilon) < y < 1\};$$
$$D_{2,-} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \exp(-\varepsilon),\ -1 < y < -\exp(-\varepsilon)\};$$
$$D_{3,+} = \{(x, y) \in \mathbb{R}^2 \mid \exp(-\varepsilon) < x < 1,\ \exp(-\varepsilon) < y < 1\};$$
$$D_{3,-} = \{(x, y) \in \mathbb{R}^2 \mid \exp(-\varepsilon) < x < 1,\ -1 < y < -\exp(-\varepsilon)\}.$$

Let $\eta : [0, \exp(-\varepsilon)] \to [0, 1]$ be a function of class C^r satisfying

$$f(x) \geq \eta(x) \quad \text{for } x \in [0, \exp(-\varepsilon)].$$

On the regions $D_{0,\pm}$ and $D_{1,+}$, we define

$$F(x, y) = \begin{cases} (\lambda x, \lambda(y - \eta(x))), & \text{for } (x, y) \in D_{0,+}; \\ (\lambda x, \lambda y), & \text{for } (x, y) \in D_{0,-}; \\ (\lambda y, \lambda(x - \exp(-\varepsilon))), & \text{for } (x, y) \in D_{1,+}. \end{cases}$$

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Actually, the definition of the map F on the other regions is irrelevant to our argument as long as F is a piecewise C^r expanding map. For completeness, we define

$$F(x,y)=\begin{cases}(-\lambda y,\lambda(x-\exp(-\varepsilon))), & \text{for } (x,y)\in D_{1,-};\\ (\lambda x,\lambda(y-\exp(-\varepsilon))), & \text{for } (x,y)\in D_{2,+};\\ (\lambda x,\lambda(y+\exp(-\varepsilon))), & \text{for } (x,y)\in D_{2,-};\\ (\lambda(x-\exp(-\varepsilon))),\lambda(y-\exp(-\varepsilon)), & \text{for } (x,y)\in D_{3,+};\\ (\lambda(x-\exp(-\varepsilon))),\lambda(y+\exp(-\varepsilon)), & \text{for } (x,y)\in D_{3,-}.\end{cases}$$

This construction of the map F depends on the functions f and η, which will be specified later. Note that the map F is a piecewise C^r expanding map if η is sufficiently small in the C^1 sense.

For $m\ge 1$ and $0\le k\le \gamma^m-1$, we will define open rectangles

$$R(k,m)=\left\{(x,y)\in\mathbb{R}^2\;\middle|\;\begin{array}{c}B(k,m)<y<B(k,m)+H(k,m),\\ L(k,m)<x<L(k,m)+W(k,m)\end{array}\right\}$$

by setting numbers $B(k,m)$, $L(k,m)$, $W(k,m)$ and $H(k,m)$ below. Then we will define the functions f and η so that:

- F maps $R(k,m)$ into $R(k+1,m)$ for $0\le k<\gamma^m-1$, and
- F maps $R(\gamma^m-1,m)$ into $R(0,m+1)$,

for sufficiently large m. First, we set

$$B(k,m)=\exp(\varepsilon k)\Bigg(\sum_{i=0}^{\infty}\exp\Bigg(-\varepsilon\sum_{j=0}^{2i+1}\gamma^{m+j}\Bigg)\Bigg),$$

$$L(k,m)=\exp(\varepsilon k)\Bigg(\exp(-\varepsilon\gamma^m)+\sum_{i=0}^{\infty}\exp\Bigg(-\varepsilon\sum_{j=0}^{2i+2}\gamma^{m+j}\Bigg)\Bigg)$$

for $0\le k\le\gamma^m-1$. We gave these definitions so that

$$B(k+1,m)=\lambda B(k,m),\quad L(k+1,m)=\lambda L(k,m)$$

for $0\le k\le\gamma^m-2$, and that

$$B(0,m+1)=\lambda(L(\gamma^m-1,m)-\exp(-\varepsilon)),$$
$$L(0,m+1)=\lambda B(\gamma^m-1,m).$$

Observe that, if m is sufficiently large, we have

$$B(k,m)\le B(\gamma^m-1,m)<\exp(-\varepsilon\gamma^{m+1}) \tag{2}$$

and

$$B(\gamma^{m+1}-k,m+1)<B(\gamma^m-k,m)/2, \tag{3}$$

$$0<L(\gamma^m-k,m)-\exp(-\varepsilon k)<\frac{\exp(-\varepsilon(k-1))-\exp(-\varepsilon k)}{6} \tag{4}$$

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for $1 \le k \le \gamma^m$. Next we define $H(k,m)$. When $0 \le k < \gamma^m - \gamma^{m-1}$, we set

$$H(k,m) = \exp\left((\varepsilon - \log 2)k + \sum_{i=0}^{m-1} \varepsilon\gamma^i - (1-\gamma^{-1})\log 2 \sum_{i=1}^{[m/2]} \gamma^{m-2i}\right).$$

On the other hand, when $\gamma^m - \gamma^{m-1} \le k \le \gamma^m - 1$, we set

$$H(k,m) = \exp\left(\varepsilon k + \sum_{i=0}^{m-1} \varepsilon\gamma^i - (1-\gamma^{-1})\log 2 \sum_{i=0}^{[m/2]} \gamma^{m-2i}\right).$$

We define $W(k,m)$ for $0 \le k \le \gamma^m - 1$ by

$$W(k,m) = \exp\left(\varepsilon k + \sum_{i=0}^{m-1} \varepsilon\gamma^i - (1-\gamma^{-1})\log 2 \sum_{i=0}^{[(m-1)/2]} \gamma^{m-2i-1}\right).$$

We gave these definitions so that

$$\begin{aligned} W(k+1,m) &= \lambda W(k,m), && \text{for } 0 \le k \le \gamma^m - 2,\\ H(k+1,m) &= (\lambda/2)H(k,m), && \text{for } 0 \le k < \gamma^m - \gamma^{m-1},\\ H(k+1,m) &= \lambda H(k,m), && \text{for } \gamma^m - \gamma^{m-1} \le k \le \gamma^m - 2 \end{aligned}$$

and that

$$\begin{aligned} W(0,m+1) &= \lambda H(\gamma^m - 1, m),\\ H(0,m+1) &= \lambda W(\gamma^m - 1, m). \end{aligned}$$

Observe that, from condition (1) on ε, we have

$$W(\gamma^m - k, m) < \exp(-\varepsilon\gamma^{m+2}) < (\exp(-\varepsilon(k-1)) - \exp(-\varepsilon k))/6, \tag{5}$$

$$H(\gamma^m - k, m) < B(\gamma^m - k, m) < \exp(-\varepsilon\gamma^{m+1}), \tag{6}$$

for $1 \le k \le \gamma^m$, provided that m is sufficiently large.

Put $I_k = [\exp(-\varepsilon k), \exp(-\varepsilon(k-1))]$ for $k \ge 1$ and let J_k be the left one-third of I_k, that is,

$$J_k = [\exp(-\varepsilon k), \exp(-\varepsilon k) + (\exp(-\varepsilon(k-1)) - \exp(-\varepsilon k))/3].$$

Let K_k be the interval containing J_k such that $K_k - J_k$ consists of two intervals with length

$$\Delta(k) = (\exp(-\varepsilon k) - \exp(-\varepsilon(k+1)))/6.$$

The intervals K_k are mutually disjoint.

Now we specify the functions f and η. Let M be a large integer that we will choose below. Then

(I) define $f(x) = 0$ and $\eta(x) = 0$ for $\exp(-\varepsilon\gamma^M) \le x \le \exp(-\varepsilon)$,

(II) define $f(x) = 0$ and $\eta(x) = 0$ for $x \notin \cup_{k=1}^{\infty} K_k$, and

(III) define, for $x \in J_k$ with $k > \gamma^M$,

$$\begin{aligned} f(x) &= B(\gamma^{m(k)} - k, m(k)) + (1/2)H(\gamma^{m(k)} - k, m(k));\\ \eta(x) &= (1/2)H(\gamma^{m(k)} - k, m(k)); \end{aligned}$$

where $m(k)$ is the integer satisfying $\gamma^{m(k)-1} < k \le \gamma^{m(k)}$.

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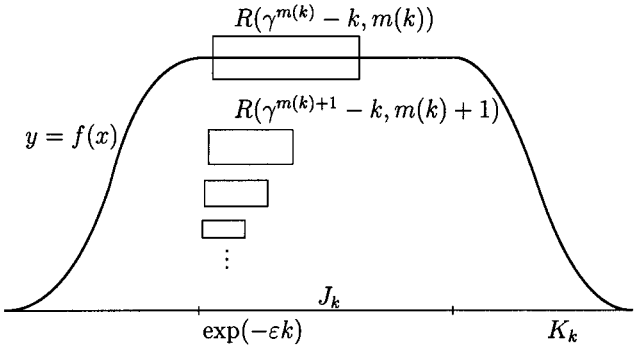


FIGURE 2. Rectangles in $J_k \times [0, 1]$.

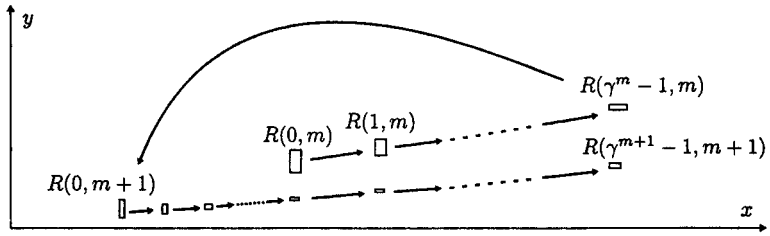


FIGURE 3. The orbit of the rectangle $R(0, m)$.

It remains to define the functions f and η on $K_k - J_k$ with $k \geq \gamma^M$. Let us consider a interval K_k with $k \geq \gamma^M$. From (2) and (6), each of two functions f and η takes a constant value smaller than $2\exp(-\varepsilon\gamma^{m(k)+1})$ on J_k. On the other hand, we have

$$\Delta(k) > \varepsilon\exp(-\varepsilon(k+1))/6 > \varepsilon\exp(-\varepsilon\gamma^{m(k)} - \varepsilon)/6.$$

Since we set $\gamma > r$, the ratio $2\exp(-\varepsilon\gamma^{m(k)+1})/\Delta(k)^r$ converges to zero as $k \to \infty$. Hence, we can define the functions f and η on the sets $K_k - J_k$, $k \geq \gamma^M$, so that:

- the functions f and η are of class C^r on K_k;
- the value and the derivatives up to order r of the functions f and η vanish at the end points of the interval K_k; and
- the C^r norm of the functions f and η on the interval K_k converge to 0 as $k \to \infty$.

Then the functions f and η are C^r on the interval $[0, \exp(-\varepsilon)]$. Moreover, if we take M sufficiently large, the function η can be taken so small in the C^1 sense that F is a piecewise C^r expanding map.

We have defined a piecewise C^r expanding map $F : D \to D$. Observe that the following hold if M is sufficiently large (see figures 2 and 3).

(i) For $k > \gamma^M$, the rectangles $R(\gamma^m - k, m)$, $m \geq m(k)$, are contained in $J_k \times [0, 1]$ from (4) and (5).

(ii) For $k > \gamma^M$, the graph of f restricted to the interval J_k is a horizontal segment that passes through the center of the rectangle $R(\gamma^{m(k)} - k, m(k))$. Other rectangles

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$R(\gamma^m - k, m)$, $m > m(k)$, in $J_k \times [0, 1]$ are below the graph and, hence, contained in the region $D_{0,-}$ from (3) and (6).

(iii) If $m > M$ and $0 \leq k < \gamma^m - \gamma^{m-1}$, then F maps each of two open rectangles $R(k, m) \cap D_{0,+}$ and $R(k, m) \cap D_{0,-}$ onto $R(k + 1, m)$.

(iv) If $m > M$ and $\gamma^m - \gamma^{m-1} \leq k < \gamma^m - \gamma^M$, the rectangle $R(k, m)$ is contained in $D_{0,-}$ and F maps $R(k, m)$ onto $R(k + 1, m)$.

(v) If $m > M$ and $\gamma^m - \gamma^M \leq k \leq \gamma^m - 2$, the rectangle $R(k, m)$ is contained in $D_{0,+}$ and F maps $R(k, m)$ onto $R(k + 1, m)$ because $\eta(x) = 0$ for $\exp(-\varepsilon\gamma^M) \leq x \leq \exp(-\varepsilon)$.

(vi) F maps $R(\gamma^m - 1, m)$ onto $R(0, m + 1)$ if $m > M$.

Therefore, we have, for $m > M$:

- $F(R(k, m)) \subset R(k + 1, m)$ for $0 \leq k \leq \gamma^m - 2$; and
- $F(R(\gamma^m - 1, m)) = R(0, m + 1)$.

Now the properties (A) and (B) hold for $B = R(0, M + 1)$.

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