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Enumerating combinatorial classes of the complex polynomial vector fields in $\mathbb{C}$

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Abstract. In order to understand the parameter space Ξ_d of monic and centered complex polynomial vector fields in $\mathbb{C}$ of degree d, decomposed by the combinatorial classes of the vector fields, it is interesting to know the number of loci in parameter space consisting of vector fields with the same combinatorial data (corresponding to topological classification with fixed separatrices at infinity). This paper answers questions posed by Adam L. Epstein and Tan Lei about the total number of combinatorial classes and the number of combinatorial classes corresponding to loci of a specific (real) dimension q in parameter space, for fixed degree d; these numbers are denoted by c_d and $c_{d,q}$, respectively. These results are extensions of a result by Douady, Estrada, and Sentenac, which shows that the number of combinatorial classes of the structurally stable complex polynomial vector fields in $\mathbb{C}$ of degree d is the Catalan number C_{d-1}. We show that enumerating the combinatorial classes is equivalent to a so-called *bracketing problem*. Then we analyze the generating functions and find closed-form expressions for c_d and $c_{d,q}$, and we furthermore make an asymptotic analysis of these sequences for d tending to ∞. These results are also applicable to special classes of quadratic and Abelian differentials and singular holomorphic foliations of the plane.

1. *Introduction*

The space Ξ_d of monic and centered single-variable complex polynomial vector fields of degree d is parameterized by its $d-1$ complex coefficients. The space $\Xi_d \simeq \mathbb{C}^{d-1}$ is decomposed into combinatorial classes where the vector fields within each class have the same *combinatorial data set* (to be defined). Each of these combinatorial classes is a connected manifold with well-defined (real) dimension q, which is the dimension of the combinatorial class as a subspace in Ξ_d.

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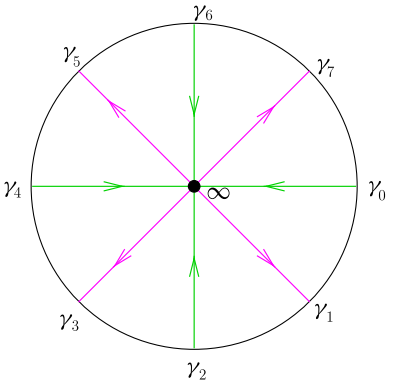


FIGURE 1. The point at ∞ is a pole of order $d-2$ for vector fields $\xi_P \in \Xi_d$. There are $2(d-1)$ trajectories γ_ℓ which meet at infinity with asymptotic angles $2\pi\ell/2(d-1)$, $\ell \in \{0, 1, \ldots, 2d-3\}$. When the labelling index ℓ is even, the trajectories are called *incoming* to ∞ and, when the index ℓ odd, they are called *outgoing* from ∞.

Counting the combinatorial classes will help us to better understand Ξ_d, and the techniques utilized may also prove valuable to those who are considering other enumerative problems in dynamical systems.

In this paper, we study the properties of the sequences c_d (the total number of combinatorial classes in Ξ_d) and $c_{d,q}$ (the number of combinatorial classes of dimension q in Ξ_d). These will be referred to as the *simplified problem* and the *complete problem*, respectively. We first show in §3 that these problems are equivalent to so-called *bracketing problems*, where one counts the number of pairings of parentheses satisfying desired conditions [**4**]. Then, for each problem, a recursion equation and implicit expressions for the algebraic generating functions are calculated (§3.4), closed-form expressions for c_d and $c_{d,q}$ are calculated (§4), and asymptotic questions are considered (§5). In particular, we compute the asymptotic growth of the sequence c_d and prove that the discrete probability distribution $\pi_{d,q} = c_{d,q}/c_d$ converges to a normal distribution for $d \to \infty$. Lastly, we consider in §6 some related enumeration problems one might be interested in solving.

2. *Background and definitions*

We present now a summary of some necessary concepts and definitions, and we include a new description of the combinatorial data set; it is different from the one presented in [**3**] (it is in fact more similar to the definition of the combinatorial invariant in [**6**]), but it is much more useful for our purposes here. For further details, please see [**3, 6**].

It can be shown that ∞ is a pole of order $d-2$ for vector fields $\xi_P \in \Xi_d$. There are $2(d-1)$ trajectories γ_ℓ which meet at infinity with asymptotic angles $2\pi\ell/2(d-1)$, $\ell \in \{0, 1, \ldots, 2d-3\}$. When the labelling index ℓ is even, the trajectories are called *incoming* to ∞ and, when the index ℓ odd, they are called *outgoing* from ∞ (see Figure 1).

There are $2d-2$ accesses to ∞ defined by the trajectories at infinity. An *end* e_ℓ is infinity with access between $\gamma_{\ell-1}$ and γ_ℓ (see Figure 2). An *odd end* is an end e_k labelled by an odd index k, and an *even end* is an end e_j labelled by an even index j.

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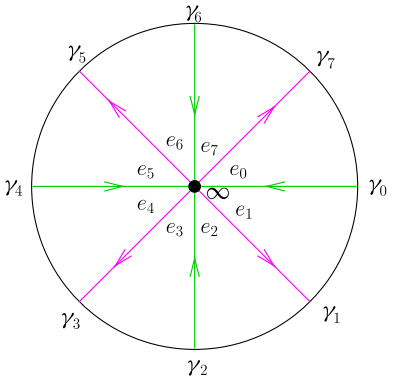


FIGURE 2. There are $2d-2$ accesses to ∞ defined by the trajectories at infinity. An *end* e_ℓ is infinity with access between $\gamma_{\ell-1}$ and γ_ℓ. An *odd end* is an end e_k labelled by an odd index k, and an *even end* is an end e_j labelled by an even index j.

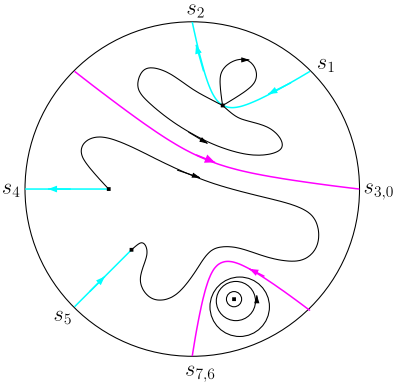


FIGURE 3. *Separatrices* s_ℓ are the maximal trajectories of ξ_P incoming to and outgoing from ∞. They are labelled by the $2(d-1)$ asymptotic angles $2\pi\ell/2(d-1)$, $\ell \in \{0, 1, \ldots, 2d-3\}$. The separatrix structure can be equivalently represented in a *separatrix disk model*, by labelling the points $\exp(2\pi i\ell/(2d-2))$, $\ell = 0, \ldots, 2d-3$ on $\mathbb{S}^1$ by s_ℓ and joining the points in the same equivalence class. In the figure, there are two homoclinic separatrices $s_{3,0}$ and $s_{7,6}$, and four landing separatrices s_1, s_2, s_4, and s_5. A few extra trajectories (in black) have been drawn in to better see the structure of the vector field. In this disk model, there are two sepal cells, one $\alpha\omega$ cell, and one center cell.

Separatrices s_ℓ are the maximal trajectories of ξ_P incoming to and outgoing from ∞. They are labelled by the $2(d-1)$ asymptotic angles $2\pi\ell/2(d-1)$, $\ell \in \{0, 1, \ldots, 2d-3\}$.

A separatrix s_ℓ is called *landing* if $\bar{s}_\ell \backslash s_\ell = \zeta$, where ζ is an equilibrium point for ξ_P. A separatrix $s_\ell = s_{k,j}$ is called *homoclinic* if $\bar{s}_{k,j} \backslash s_{k,j} = \emptyset$ (see Figure 3). A separatrix for a vector field $\xi_P \in \Xi_d$ can only be either homoclinic or landing. A homoclinic separatrix $s_{k,j}$ is labelled by the one odd index k and the one even index j corresponding to its two asymptotic directions at infinity. The *separatrix graph* $\Gamma_P = \bigcup_{\ell=0}^{2d-3} \hat{s}_\ell$ completely determines the topological structure of the trajectories of a vector field (see, for instance, **[1, 11]**).

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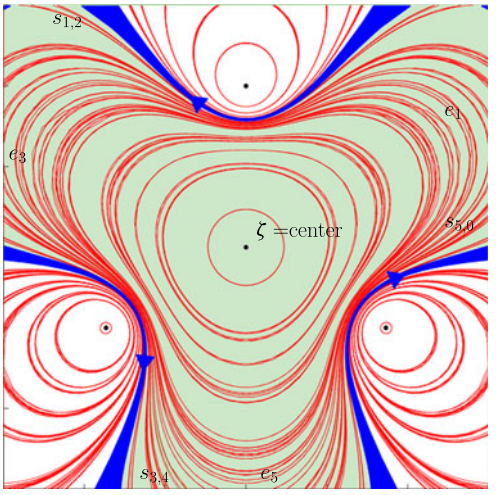


FIGURE 4. Pictured are the trajectories of a vector field with four center zones: one odd center zone (shaded) with homoclinic separatrices $s_{5,0}$, $s_{1,2}$, and $s_{3,4}$ and ends e_1, e_3, and e_5 on the boundary; and three even center zones, each with one homoclinic separatrix and one end on the boundary.

In [**3**], the separatrix structure is encoded via an equivalence relation and a marked subset H on $\mathbb{Z}/(2d-2)$, where, in short, H tells which separatrices are homoclinic and, for the remaining separatrices (which are landing), the equivalence relation essentially tells which s_ℓ land at the same equilibrium point (the formal definition will be given in §2.2).

This separatrix structure can be equivalently represented in a *separatrix disk model*, by labelling the points $\exp(2\pi \mathrm{i}\ell/(2d-2))$, $\ell = 0, \ldots, 2d-3$ on $\mathbb{S}^1$ by s_ℓ and joining the points in the same equivalence class (see Figure 3).

Conversely, if we define an abstract *combinatorial data set*, $(\sim, H)$, which consists of a non-crossing equivalence relation $\sim$ and a marked subset H on $\mathbb{Z}/(2d-2)$ which satisfy certain properties (to be discussed), then it is known that there exists a monic and centered complex polynomial vector field of degree d whose separatrix structure matches the data set [**3**].

2.1. *Zones.* The connected components Z of $\mathbb{C}\backslash\Gamma_P$ are called *zones*. There are three types of zones for vector fields in Ξ_d, and the types of zones are determined by the types of their boundaries.

- A *center zone* Z contains an equilibrium point, which is a center, in its interior. Its boundary consists of one or several homoclinic separatrices and the point at infinity. If a center zone is on the left of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on the boundary ∂Z, then the center zone has n odd ends $e_{k_1}, \ldots, e_{k_n}$ at infinity on ∂Z and the zone is called either a *counter-clockwise center zone* or an *odd center zone*. If a center zone is on the right of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on the boundary ∂Z, then the center zone has n even ends $e_{j_1}, \ldots, e_{j_n}$ at infinity on ∂Z and the zone is called either a *clockwise center zone* or an *even center zone* (see Figure 4).
- A *sepal zone* Z has exactly one equilibrium point on the boundary, which is both the α-limit point and the ω-limit point for all trajectories in Z (i.e. $\zeta_\alpha = \zeta_\omega$). This

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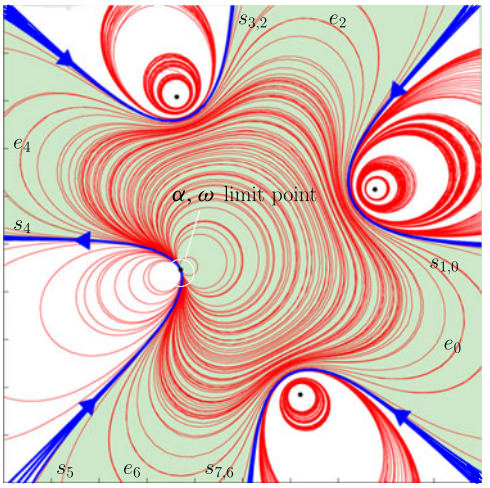


FIGURE 5. Pictured are the trajectories of a vector field with an even sepal zone (shaded). On the boundary of the sepal zone is the double equilibrium point which is both the α- and the ω-limit point of the trajectories; one incoming landing separatrix s_4 and one outgoing landing separatrix s_5; three homoclinic separatrices $s_{1,0}$, $s_{3,2}$, and $s_{7,6}$; and four ends at infinity e_0, e_2, e_4, and e_6. There is an odd sepal zone (not shaded) which shares the equilibrium point and the landing separatrices with the shaded sepal zone, but it has no homoclinic separatrices and only one odd end e_5 on the boundary.

equilibrium point is necessarily a multiple equilibrium point. The boundary ∂Z contains exactly one incoming and one outgoing landing separatrix, and possibly one or several homoclinic separatrices, and the point at infinity. If a sepal zone is to the left of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on its boundary, then it has $n+1$ odd ends on the boundary: $e_{k_1}, \ldots, e_{k_n}$ and e_{j_i+1} for some corresponding j_i, depending on how one orders the separatrices. In this case, it is called an *odd* sepal zone. Similarly, if a sepal zone is on the right of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on its boundary, then it has $n+1$ even ends on the boundary, $e_{j_1}, \ldots, e_{j_n}$ and e_{k_i+1} for some corresponding k_i, again depending on the ordering of the separatrices. In this case, it is called an *even* sepal zone (see Figure 5).

- An $\alpha\omega$-*zone* Z has two equilibrium points on the boundary, $\zeta_\alpha \neq \zeta_\omega$, the α-limit point and the ω-limit point for all trajectories in Z. The boundary ∂Z contains one or two incoming landing separatrices and one or two outgoing landing separatrices, possibly one or several homoclinic separatrices, and the point at infinity. If an $\alpha\omega$-zone is both on the left of n_1 homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_{n_1},j_{n_1}}$ and on the right of n_2 homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_{n_2},j_{n_2}}$ on the boundary, then the $\alpha\omega$-zone has n_1+1 odd ends ($e_{k_1}, \ldots, e_{k_{n_1}}$ and e_{j_i+1} for some corresponding j_i) and n_2+1 even ends ($e_{j_1}, \ldots, e_{j_{n_2}}$ and e_{k_i+1} for some corresponding k_i) on the boundary (see Figure 6).

Remark. It will be important to note that for a sepal zone, there is exactly one end whose index is not equal to an index of a homoclinic separatrix (in the notation above, e_{j_i+1} for an odd sepal zone and e_{k_i+1} for an even sepal zone). Similarly, for an $\alpha\omega$-zone, there are exactly one odd end and one even end, neither of whose indices coincide with any index of

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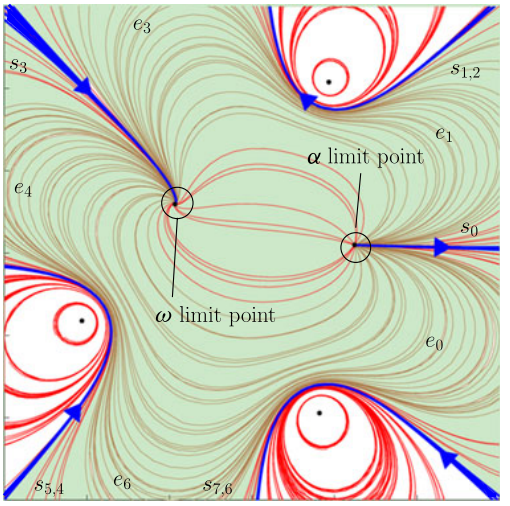


FIGURE 6. Pictured are the trajectories of a vector field with an $\alpha\omega$-zone (shaded). On the boundary of the zone are two equilibrium points: one is the α-limit point of the trajectories and the other is the ω-limit point of the trajectories. Also on the boundary are one incoming landing separatrix s_0 and one outgoing landing separatrix s_3. The zone is to the left of the homoclinic separatrix $s_{1,2}$ and on the right of the two homoclinic separatrices $s_{5,4}$, and $s_{7,6}$. Finally, the boundary contains two odd ends e_1 and e_3 and three even ends e_0, e_4, and e_6 at infinity.

a homoclinic separatrix (in the notation above, the odd and even ends are e_{j_i+1} and e_{k_i+1}, respectively).

In the separatrix disk model, the connected components of $\mathbb{D}\setminus\bigcup_\ell s_\ell$ are called *cells*. Cells are the abstract analog of zones, and are named accordingly (i.e. there are center cells, sepal cells, and $\alpha\omega$ cells). The types of cells are determined by the types of their boundaries (see Figure 3).

2.2. *Combinatorial data set.* Now that we have defined the types of zones and corresponding cells, we can state the definition of the combinatorial data set as presented in [**3**].

Definition. A *combinatorial data set* $(\sim, H)$ of degree $d \geq 2$ consists of an equivalence relation $\sim$ on $\mathbb{Z}/(2d-2)$ and a marked subset $H \subset \mathbb{Z}/(2d-2)$ satisfying:

(1) $\sim$ is non-crossing;
(2) if $\ell' \neq \ell''$, then $\ell' \sim \ell''$ and $\ell' \in H \Leftrightarrow \ell'' \in H$ and ℓ' and ℓ'' have different parity;
(3) every cell in the disk-model realization of $(\sim, H)$ is one of the five types: an $\alpha\omega$-cell, an odd or an even sepal cell, or an odd or an even center cell characterized as above.

2.3. *Transversals.* There is an equivalent way to encode the combinatorial structure of a vector field, which is much more useful for our purposes here. We define in this section the important structures needed to understand this equivalent definition of a combinatorial data set.

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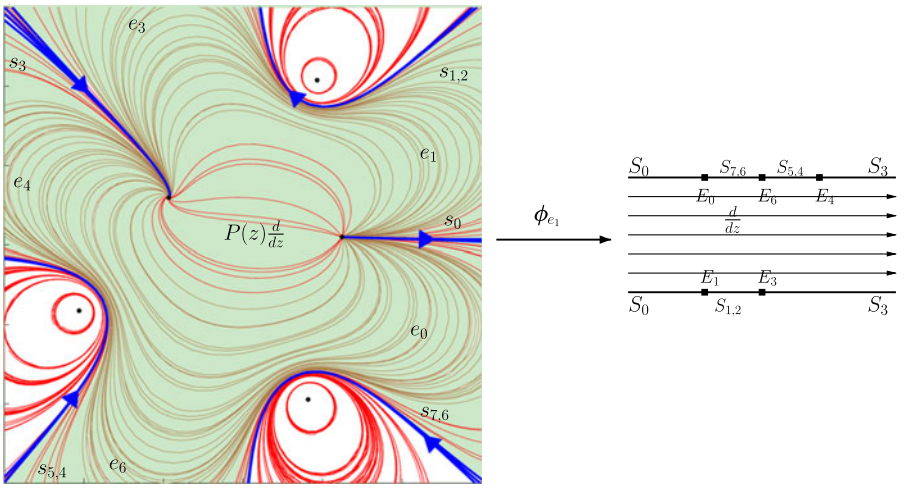


FIGURE 7. The image of an $\alpha\omega$-zone under ϕ is a horizontal strip.

In any simply connected domain avoiding zeros of P, the differential $dz/P(z)$ has an antiderivative, unique up to addition by a constant

$$\phi(z) = \int_{z_0}^{z} \frac{dw}{P(w)}.$$

Note that

$$\phi_*(\xi_P) = \phi'(z)P(z)\frac{d}{dz} = \frac{d}{dz}. \tag{1}$$

The coordinates $w = \phi(z)$ are, for this reason, called *rectifying coordinates*. We will call the images of zones under rectifying coordinates *rectified zones*. The rectified zones are of the following types.

- The image of an $\alpha\omega$-zone under ϕ is a horizontal strip (see Figure 7).
- The image of an odd sepal zone under ϕ is an upper half plane, and the image of an even sepal zone is a lower half plane (see Figure 8).
- The image of a center zone (minus a curve contained in the zone which joins the center ζ and ∞) under ϕ is a vertical half strip. It is an upper vertical half strip for an odd center zone and a lower vertical half strip for an even center zone (see Figure 9).

Via the rectifying coordinates, it is evident that there are a number of closed geodesics in $\hat{\mathbb{C}}\backslash\{\text{equilibrium pts}\}$ in the metric with length element $|dz|/|P(z)|$ through ∞. Among these are the h homoclinic separatrices, and there are s *distinguished transversals* (defined below).

Definition. The *distinguished transversal* $T_{k,j}$ is the geodesic in the metric $|dz|/|P(z)|$ joining the ends e_k and e_j, avoiding the separatrices and equilibrium points, where e_j is the left-most end on the upper boundary and e_k is the right-most end on the lower boundary of the strip which is the image of the $\alpha\omega$-zone the transversal is contained in (see Figures 10 and 11).

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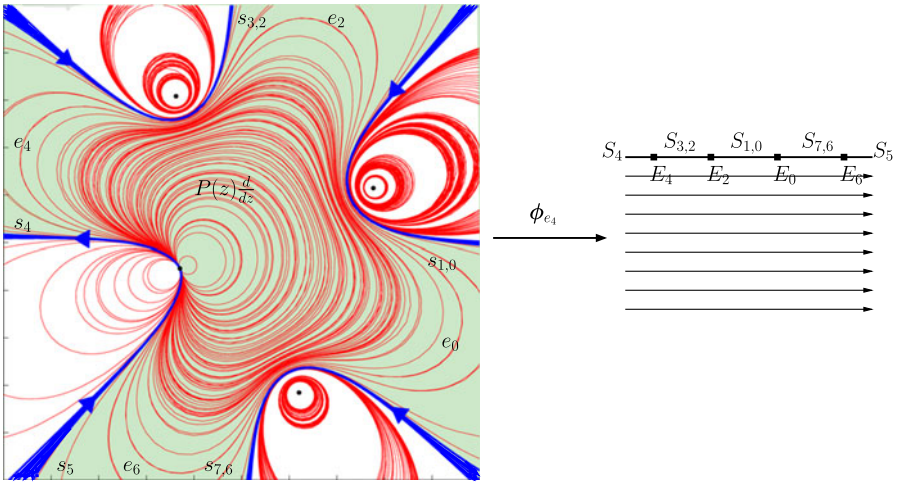


FIGURE 8. The image of an odd sepal zone under ϕ is an upper half plane, and the image of an even sepal zone is a lower half plane. In this figure, there is an even sepal zone mapped to a lower half plane.

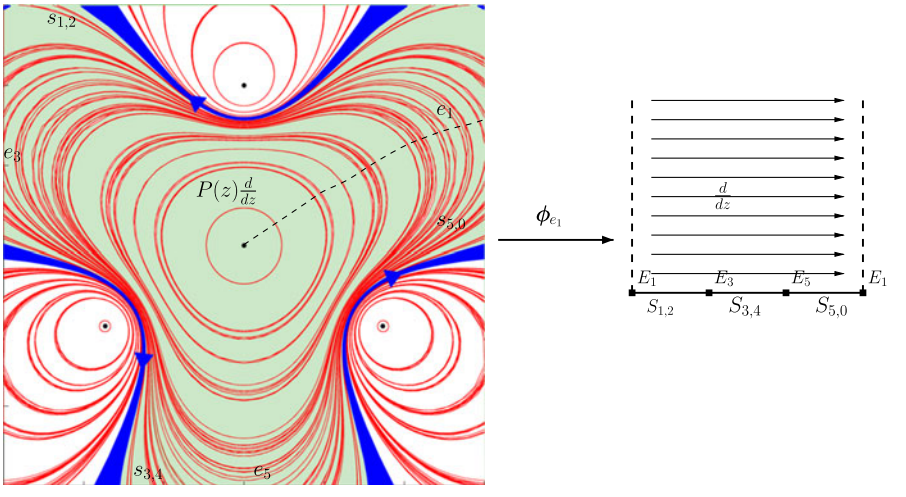


FIGURE 9. The image of a center zone (minus a curve contained in the zone which joins the center ζ and ∞) under ϕ is a vertical half strip. It is an upper vertical half strip for an odd center zone and a lower vertical half strip for an even center zone. In this case, there is an odd center zone mapped to an upper vertical half strip.

Note that from the way in which the distinguished transversal is chosen, the indices of the ends it joins are exactly those ends whose indices will never coincide with the indices of any homoclinic separatrices.

2.4. *Alternative combinatorial description.* We will prove that one can equivalently describe the combinatorics by the union of homoclinic separatrices $s_{k,j}$ and distinguished transversals $T_{k,j}$. We state the definition below, and prove that it is equivalent to the one presented before (from [**3**]).

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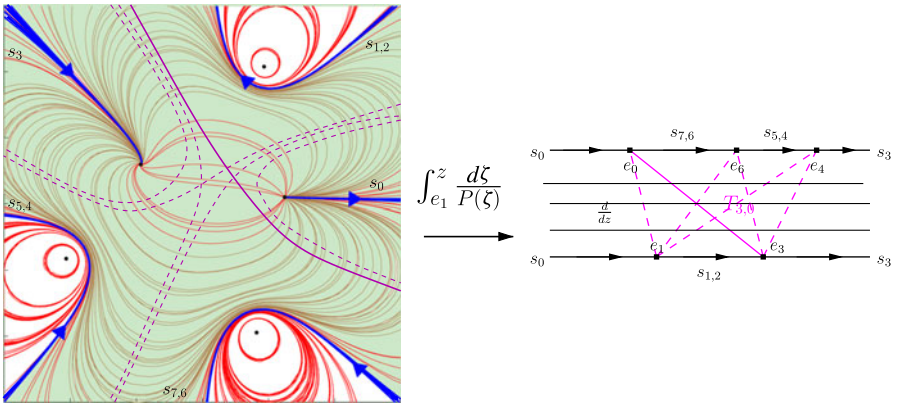


FIGURE 10. There may be several transversals which avoid the equilibrium points and separatrices (the dashed curves), but there is exactly one distinguished transversal for each $\alpha\omega$-zone (in this case, $T_{3,0}$).

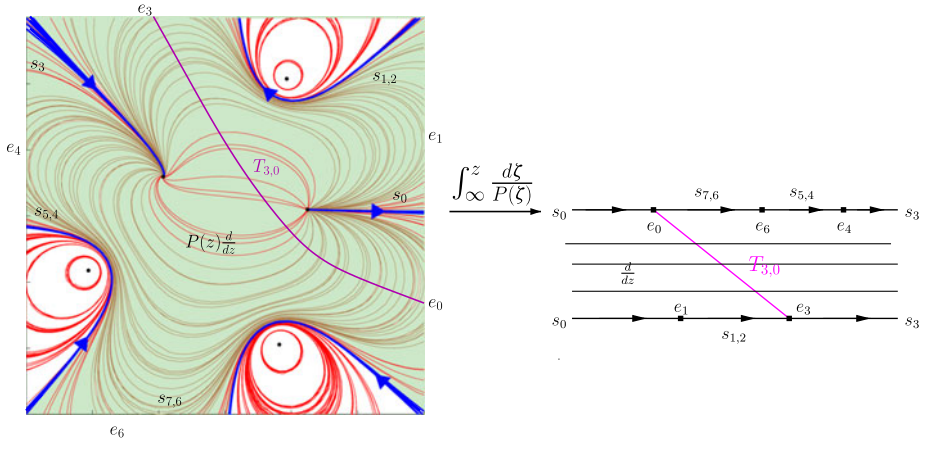


FIGURE 11. Each $\alpha\omega$-zone is isomorphic to a strip. We define the distinguished transversal to be the geodesic in the metric $|dz|/|P(z)|$ joining the ends e_k and e_j (in this figure, e_3 and e_0) where e_j is the left-most end on the upper boundary of the strip and e_k is the right-most end on the lower boundary of the strip. Since the indices of the distinguished transversal are the same as the indices for the two landing separatrices on the upper left and lower right boundaries of the strip, they can never coincide with the indices for a homoclinic separatrix.

Definition. A *combinatorial data set* (ι, H, T) of degree $d \geq 2$ consists of an involution $\iota : \mathbb{Z}/(2d-2) \to \mathbb{Z}/(2d-2)$ and marked subsets H and T satisfying:

- if $\iota(\ell) \neq \ell$, then ℓ and $\iota(\ell)$ have different parity (if one is even, then the other is odd);
- the involution ι is non-crossing;
- if $\iota(\ell) \neq \ell$, then either both ℓ and $\iota(\ell)$ are in H or both ℓ and $\iota(\ell)$ are in T.

This definition leads to a description of the combinatorics via non-crossing pairings of even and odd numbers. The pairing of indices in H represents pairings of indices k and j corresponding to homoclinic separatrices $s_{k,j}$, and the pairing of indices in T represents pairings of indices k and j corresponding to indices of ends e_k and e_j that the distinguished transversal $T_{k,j}$ joins. The indices which are not paired (i.e. ℓ such

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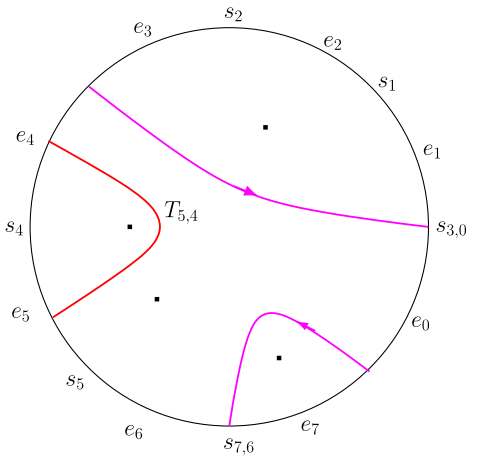


FIGURE 12. Pictured is the transversal disk model equivalent to the separatrix disk model pictured in Figure 3. There are two homoclinic separatrices $s_{3,0}$ and $s_{7,6}$, and there is one distinguished transversal $T_{5,4}$. Note that each connected component contains exactly one equilibrium point.

that $\iota(\ell) = \ell$) correspond to the indices of ends e_ℓ on the boundary of sepal zones which do not coincide with indices of any homoclinic separatrices. Essentially, we want to use the numbers $\mathbb{Z}/(2d-2)$ to stand for indices of separatrices for homoclinics, and indices of ends otherwise. Since these were chosen in the way described above, this never causes a conflict.

As one can draw a disk model with a separatrix graph given the first definition of combinatorial invariant, one can draw a transversal graph given the second definition. The transversal structure can be equivalently represented in a *transversal disk model*, by labelling the points $\exp(2\pi \mathrm{i}\ell/(2d-2))$, $\ell = 0, \ldots, 2d-3$ on $\mathbb{S}^1$ by s_ℓ and the points between them $\exp(2\pi \mathrm{i}(\ell - 1/2)/(2d-2))$ by e_ℓ, joining s_k and s_j corresponding to homoclinic separatrices $s_{k,j}$, and joining e_k and e_j corresponding to each distinguished transversal $T_{k,j}$ (see Figure 12). That the involution ι is non-crossing means that the transversal and homoclinic separatrices in the transversal disk model do not cross.

PROPOSITION 2.1. *The definition above is equivalent to the definition of combinatorial data set presented in* [**3**, Definition 2.2].

Proof. We basically need to prove that separatrix disk models are in one-to-one correspondence with transversal disk models.

If we start with a separatrix disk model, such that every cell is one of the five types, then we can construct a unique transversal disk model. Note first that the homoclinic separatrices stay the same. Then, for each $\alpha\omega$-zone, there are exactly one even end e_j and one odd end e_k whose indices do not coincide with any of the homoclinic indices on the boundary. Join these ends in the disk. Finally, remove all landing separatrices from the separatrix disk model, and what you are left with is a transversal disk model (see Figure 12).

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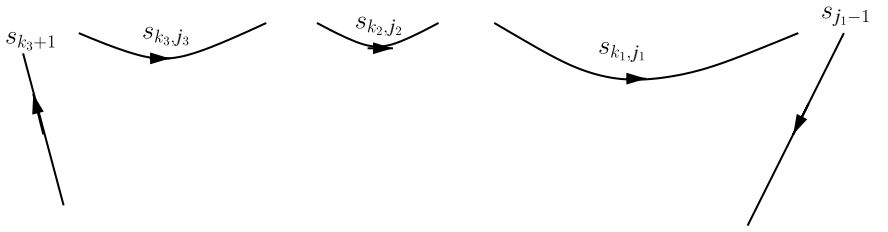


FIGURE 13. An open chain of three homoclinic separatrices s_{k_1,j_1}, s_{k_2,j_2}, and s_{k_3,j_3} such that $j_{i+1} = k_i + 1$ for $i = 1, 2$ and s_{j_1-1} and s_{k_3+1} are landing.

Now conversely given a valid transversal disk model (i.e. one satisfying the definition above), we again leave the homoclinic separatrices alone as before. The homoclinic separatrices and distinguished transversals decompose the transversal disk model into $h + s + 1$ connected components. Call each of these components a transversal cell (as not to confuse them with connected components with respect to a separatrix disk model). Let all s_ℓ on the boundary of each transversal cell be joined to the same point in the interior of the transversal cell. If there are no s_ℓ on the boundary of a transversal cell, then that component is a center cell (with respect to the separatrix disk model). Now remove the curves corresponding to the distinguished transversals. This procedure gives a separatrix disk model, and now we just need to show that this separatrix disk model has the allowable types of cells. As already mentioned, if there are only homoclinic separatrices on the boundary of a separatrix cell, then that component is a center cell. If there is an open chain of n (where we allow $n = 0$) homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ such that either $j_{i+1} = k_i + 1$ for $i = 1, \ldots, n-1$ and s_{j_1-1} and s_{k_n+1} are landing (see Figure 13) or if $k_{i+1} = j_i + 1$ for $i = 1, \ldots, n-1$ and s_{k_1-1} and s_{j_n+1} are landing (see Figure 14) on the boundary of the same separatrix cell, then either the two landing separatrices land at the same point, or they do not. If they do land at the same point, then s_{j_1-1} and s_{k_n+1} (or s_{k_1-1} and s_{j_n+1}) are the two landing separatrices on the boundary of a sepal cell. If they land at different points, then they must be separated by a distinguished transversal T_{k,k_n+1} for some k associated to another open homoclinic chain (or $T_{j_n+1,j}$ for some j associated to another open homoclinic chain) on the boundary of the same separatrix cell. Then s_{j_1-1} must land at the same point as s_k, since they are on the boundary of the same connected component in the transversal disk model (similarly for s_{k_1-1} and s_j) (see Figure 15). One can make a similar argument for the other open homoclinic chain involved, and then one must get an $\alpha\omega$-cell. □

3. *Setup of the bracketing problem*

We have seen that the combinatorial structure of a monic and centered complex polynomial vector field can be completely described by pairings of even integers with odd integers, and conversely that any such pairings as defined by the transversal combinatorial data set lead to the combinatorics for a monic and centered complex polynomial vector field. The goal is to utilize the alternative combinatorial description in order to convert the problem of

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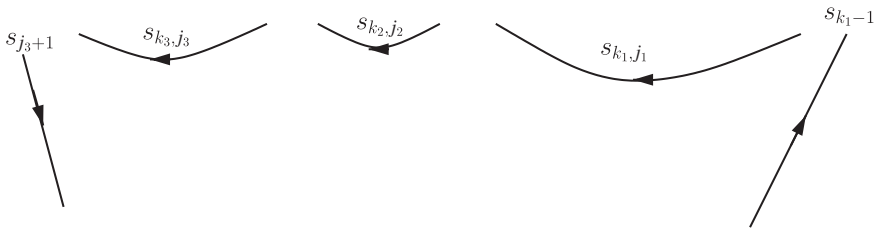


FIGURE 14. An open chain of three homoclinic separatrices s_{k_1,j_1}, s_{k_2,j_2}, and s_{k_3,j_3} such that $k_{i+1} = j_i + 1$ for $i = 1, 2$ and s_{k_1-1} and s_{j_n+1} are landing.

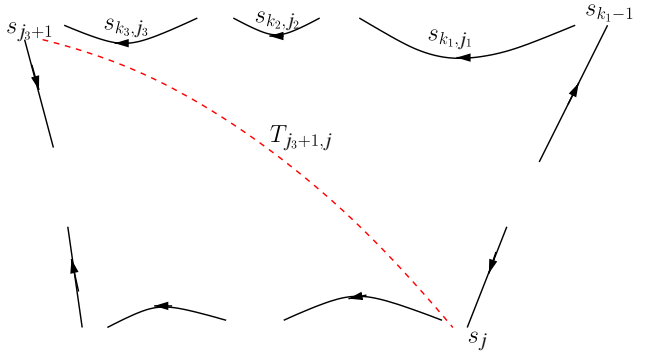


FIGURE 15. An open chain of three homoclinic separatrices s_{k_1,j_1}, s_{k_2,j_2}, and s_{k_3,j_3} such that $k_{i+1} = j_i + 1$ for $i = 1, 2$ and s_{k_1-1} and s_{j_n+1} are landing. If they land at different points, then they must be separated by a distinguished transversal $T_{j_3+1,j}$ for some j associated to another open homoclinic chain on the boundary of the same separatrix cell. Then s_{k_1-1} must land at the same point as s_j, since they are on the boundary of the same connected component in the transversal disk model.

counting the combinatorial classes into a so-called *bracketing problem*: a combinatorial problem involving pairings of parentheses placed in a string of elements (see [**4**]) in a *valid* way (to be defined).

In short, each homoclinic separatrix $s_{k,j}$ has exactly one even and one odd index associated with it, and each distinguished transversal $T_{k,j}$ has exactly one even and one odd index associated to it. Therefore, each combinatorial data set can be fully described by pairings of even and odd numbers, corresponding to these mutually disjoint pairs of indices, and hence they can be counted by a bracketing problem. In our case, we will have balanced parenthetical configurations placed in the string $0\ 1\ 2 \ldots 2d-3$, representing the indices in $\mathbb{Z}/(2d-2)$, and we will use round brackets $(\cdots)$ to pair an even and an odd number corresponding to homoclinics and square brackets $[\cdots]$ to pair an even and an odd number corresponding to distinguished transversals. We now elaborate on these ideas.

3.1. *Structurally stable vector fields.* For pedagogical reasons, we start by describing the case of counting structurally stable complex polynomial vector fields by a bracketing problem.

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THEOREM 3.1. (Douady, Estrada, and Sentenac) *The number of combinatorial classes for the structurally stable vector fields in Ξ_d is the Catalan number*

$$C_{d-1} = \frac{1}{d}\binom{2(d-1)}{d-1}. \tag{2}$$

Proof. For structurally stable vector fields, the combinatorics is completely describable by the $d-1$ transversals (since there are neither homoclinic separatrices nor sepal zones), and hence non-crossing pairings of each odd end e_k with an even end e_j (this is, in fact, how [**6**] defined the combinatorial invariant for the structurally stable vector fields). This pairing of ends can be represented by placing parentheses in the string $0\ 1\ 2 \ldots 2d-3$, where the elements paired by parentheses correspond to the labels of the ends we want to pair†. Note that this is equivalent to the number of ways to make *valid* pairings of $d-1$ pairs of parentheses, where the string $0\ 1\ 2 \ldots 2d-3$ is not actually necessary, since every end is paired with another. For example, if $d-1=3$, then the configurations in the table below are equivalent:

[01][2[34]5]	[ ][[ ]]
[0[12]3][45]	[[ ]][ ]
[0[1[23]4]5]	[[[ ]]]
[0[12][34]5]	[[ ][ ]]
[01][23][45]	[ ][ ][ ]

You might recognize then that the number of combinatorial classes (with the labelling of the separatrices) for the structurally stable monic and centered complex polynomial vector fields of degree d is just the Catalan number C_{d-1}. The proof is classical (see for instance [**5**]), but we include it here for completeness.

The goal is to write a recursion equation for C_n, the nth Catalan number, which counts the number of configurations matching n pairs of parentheses. The first character of any balanced configuration is an open parenthesis '['. Somewhere in the configuration is the matching ']' for the open one. In between that pair of parentheses is a balanced configuration, and to the right is another balanced configuration:

$$[A]B, \tag{3}$$

where A and B are balanced parenthetical configurations. Each A and B can have anywhere from 0 to $n-1$ pairs of parentheses, but together they must have exactly $n-1$ pairs of parentheses. So, if A has k pairs of parentheses, then B must have $n-k-1$ pairs of parentheses. Thus, we count all configurations where A has 0 pairs and B has $n-1$ pairs, A has 1 pair and B has $n-2$ pairs, etc. Add them up, and you get the total number of configurations with n balanced pairs of parentheses. Therefore,

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0, \quad C_0 = 1. \tag{4}$$

We will use this recursion equation and the generating function

$$f(z) = \sum_{n=0}^{\infty} C_n z^n \tag{5}$$

† A more familiar description might be evident in the disk model, where this pairing can be seen as the number of non-crossing handshakes of $2(d-1)$ people seated around a round table.

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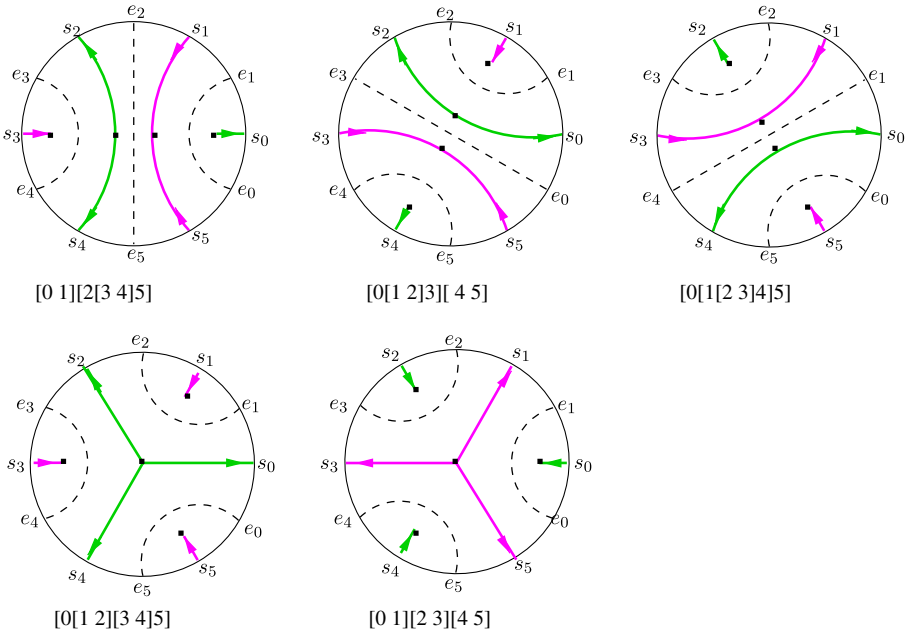


FIGURE 16. The five different disk models for the structurally stable vector fields of degree $d = 4$. The involution on the ends is marked by the dashed curves. The representation of the involution in brackets is displayed below each figure.

in order to derive a closed-form expression for C_n. Using (5), one can compute

$$[f(z)]^2 = C_0C_0 + (C_0C_1 + C_1C_0)z + (C_0C_2 + C_1C_1 + C_2C_0)z^2 + \cdots . \tag{6}$$

Combining this with (4), one arrives at

$$[f(z)]^2 = C_1 + C_2z + C_3z^2 + C_4z^3 + \cdots . \tag{7}$$

Therefore,

$$f(z) = C_0 + z[f(z)]^2. \tag{8}$$

Applying the quadratic formula,

$$f(z) = \frac{1 \pm \sqrt{1-4z}}{2z}, \tag{9}$$

and one must choose the branch

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z}, \tag{10}$$

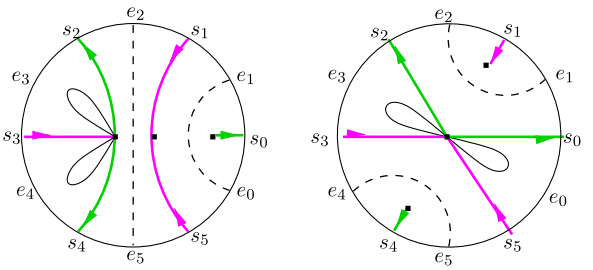
since $f(z) \to 1$ as $z \to 0$, which is desired since $f(z) = C_0 = 1$ (choosing the '+' branch gives $f(z) \to \infty$ as $z \to 0$). Expanding $\sqrt{1-4z} = (1-4z)^{1/2}$ using the binomial formula and making some arithmetic manipulations, one arrives at

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^n, \tag{11}$$

giving the desired result. □

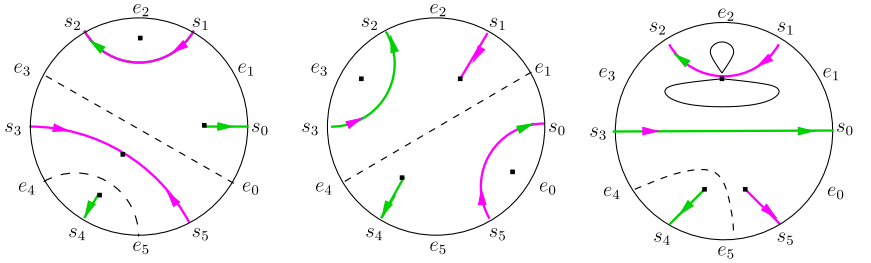
An example is given in Figure 16 to demonstrate this.

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[0 1][2 3 4 5] 0[1 2]3[4 5]

FIGURE 17. Disk models for two examples of vector fields of degree $d = 4$ having sepal zones and no homoclinic separatrices. The pairing of the ends is marked by the dashed curves. The representation of the combinatorics in brackets is displayed below each figure.



[0(1 2)3][4 5] (0[1(2 3)4]5) (0 1 2 3)[4 5]

FIGURE 18. Disk models for three examples of vector fields of degree $d = 4$ having sepal zones or/and homoclinic separatrices. The pairing of the ends is marked by the dashed curves. The representation of the combinatorics in brackets is displayed below each figure.

3.2. *Non-structurally stable vector fields.* The convention, as stated before, is that round parentheses $(\cdots)$ are used to mark pairings corresponding to homoclinic separatrices, square parentheses $[\cdots]$ are used to mark pairings corresponding to a distinguished transversal in each $\alpha\omega$-zone, and elements that are not paired correspond to the sepal zones.

We first look at the case when there are sepal zones but no homoclinic separatrices. Therefore, the zones can only be either $\alpha\omega$-zones, having exactly one even and one odd end, or sepal zones, having exactly one even or one odd end. The string $0\ 1\ 2 \ldots 2d-3$ represents the labels of the $2d-2$ ends in this case. The $\alpha\omega$-zones will then be denoted by square parentheses $[\cdots]$ in this string, and the labels corresponding to the ends of sepal zones are not paired by a set of parentheses. We see now that the string $0\ 1\ 2 \ldots 2d-3$ is necessary since the unpaired elements will be placeholders corresponding to the sepal zones. Two examples are given in Figure 17.

What then happens if we introduce homoclinic separatrices? Then it is no longer enough to consider pairings of ends. The string $0\ 1\ 2 \ldots 2d-3$ now represents the indices which may belong to either the ends e_ℓ or the separatrices s_ℓ (as was in the involution definition of the combinatorial data set). For every homoclinic separatrix $s_{k,j}$, we pair the numbers k and j by round parentheses $(\cdots)$. For example, $0\ 1\ \ldots (j \ldots k) \ldots$ or $0\ 1\ \ldots (k \ldots j) \ldots$. As usual, for every distinguished transversal $T_{k,j}$, we pair the numbers k and j by square parentheses $[\cdots]$ (see Figure 18).

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We can see from this that given any combinatorial data set, there is a unique bracketing representation defined in this way.

3.3. *Valid bracketings.* The bracketings must satisfy certain rules so that they are in accordance with what can happen for a vector field (and hence a combinatorial data set). Pairs of square and round parentheses placed in a string of elements $0\ 1\ 2 \ldots 2d-3$ are called a *valid bracketing* if:

(1) there are an equal number of right and left parentheses for each type;
(2) the number of left parentheses must be greater than or equal to the number of right, reading from left to right. *Example: ())((not valid*;
(3) there must be at least one element between successive left (respectively right) parentheses. *Example:* ((01)23) *not valid*;
(4) there must be an even number of elements in each pair of parentheses. *Example:* (012)3 *not valid*;
(5) and the square brackets must not cross the round brackets. *Example: ([)] not valid.*

It is clear that a valid bracketing as defined above leads to an admissible combinatorial data set.

3.4. *Recursion equations and generating functions.* We compute the recursion equation for p_n, the number of ways to place any number of round or square parentheses in a valid way in a string of n elements, by using the following algorithm to generate any possible valid parenthetical configuration in a string of n elements, which we will denote by s_n.

$$s_n = \begin{cases} \text{'}s\text{'}s_{n-1} & \text{if string does not start with parenthesis,} \\ \text{'}(s\text{'}s_{2a}\text{'}s)\text{'}s_b & \text{if string starts with round parenthesis,} \\ \text{'}[s\text{'}s_{2a}\text{'}s]\text{'}s_b & \text{if string starts with square parenthesis} \end{cases} \tag{12}$$

for non-negative integers a and b satisfying $2a+b+2=n$, and s representing numbers in $\{0, 1, \ldots, 2d-3\}$. Table 1 illustrates the construction of such strings from $n=1$ to $n=5$.

Then p_n satisfies the recursion equation

$$p_n = \begin{cases} p_{n-1} + 2\left(\displaystyle\sum_{2a+b+2=n} p_{2a}\, p_b\right), & n \geq 1 \text{ and } a, b \geq 0, \\ 1, & n = 0, \\ 0, & n < 0. \end{cases} \tag{13}$$

We are interested in the generating function

$$G(z) = \sum_{d=1}^{\infty} c_d z^d, \tag{14}$$

where $c_d = p_{2(d-1)}$, since for degree d the string we work with is $0 \ldots 2d-3$. Note that $G(0)=0$. For $n \geq 0$, let $q_n = p_{2n}$ and $r_n = p_{2n-1}$ ($r_0 = p_{-1} = 0$). Let also

$$f(z) = \sum_{n=0}^{\infty} q_n z^n \tag{15}$$

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TABLE 1. This table illustrates the construction of valid parenthetical configurations in strings of length n for $n = 1, \ldots, 5$ using the notation from the algorithm in equation (12). For example, for $n = 4$, the string contains four characters. The second column lists all of the possible configurations where the string does not start with a parenthesis, the fifth column lists all of the possible configurations where the string starts with a round parenthesis, and the sixth column lists all of the possible configurations where the string starts with a square parenthesis.

n	$'s's_{n-1}$	a	b	$'(s's_{2a}'s)'s_b$	$'[s's_{2a}'s]'s_b$
1	s				
2	ss	0	0	(ss)	$[ss]$
3	sss	0	1	$(ss)s$	$[ss]s$
	$s(ss)$				
	$s[ss]$				
4	$ssss$	0	2	$(ss)ss$	$[ss]ss$
	$ss(ss)$			$(ss)(ss)$	$[ss](ss)$
	$ss[ss]$			$(ss)[ss]$	$[ss][ss]$
	$s(ss)s$	1	0	$(ssss)$	$[ssss]$
	$s[ss]s$			$(s(ss)s)$	$[s(ss)s]$
				$(s[ss]s)$	$[s[ss]s]$
5	$sssss$	0	3	$(ss)sss$	$[ss]sss$
	$sss(ss)$			$(ss)s(ss)$	$[ss]s(ss)$
	$sss[ss]$			$(ss)s[ss]$	$[ss]s[ss]$
	$ss(ss)s$			$(ss)(ss)s$	$[ss](ss)s$
	$ss[ss]s$			$(ss)[ss]s$	$[ss][ss]s$
	$s(ss)(ss)$	1	1	$(ssss)s$	$[ssss]s$
	$s[ss](ss)$			$(s(ss)s)s$	$[s(ss)s]s$
	$s(ss)[ss]$			$(s[ss]s)s$	$[s[ss]s]s$
	$s[ss][ss]$				
	$s(ssss)$				
	$s[ssss]$				
	$s(s(ss)s)$				
	$s[s(ss)s]$				
	$s(s[ss]s)$				
	$s[s[ss]s]$				

and

$$g(z) = \sum_{n=1}^{\infty} r_n z^n. \tag{16}$$

Note that $zf = G$. One can deduce for $n \geq 0$

$$\begin{aligned} q_n = p_{2n} &= p_{2n-1} + 2\sum_{j=0}^{n-1} p_{2j}\, p_{2n-2j-2} \\ &= r_n + 2\sum_{j=0}^{n-1} q_j q_{n-j-1} \end{aligned} \tag{17}$$

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and

$$r_n = p_{2n-1} = p_{2n-2} + 2\sum_{j=0}^{n-1} p_{2j}\, p_{2n-2j-3}$$
$$= q_{n-1} + 2\sum_{j=0}^{n-1} q_j r_{n-j-1}. \tag{18}$$

From equations (17) and (18), one can deduce

$$f = 1 + g + 2zf^2 \quad \text{and} \quad g = zf + 2zfg, \tag{19}$$

respectively, giving

$$4z^2 f^3 - 4zf^2 + (z+1)f - 1 = 0. \tag{20}$$

Substituting $zf = G$, one gets that G satisfies the algebraic equation

$$G^3 - G^2 + \frac{(z+1)}{4}G - \frac{z}{4} = 0. \tag{21}$$

For the complete problem, namely giving the number of combinatorial classes of each dimension, we are interested in the generating function

$$G(z, t) = \sum_{d=1,q=0}^{\infty} c_{d,q} z^d t^q, \quad c_{d,q} = p_{2(d-1),q}, \tag{22}$$

where q is the real dimension of the combinatorial class in parameter space. Note that

$$G(z, 1) = \sum_{d=1,q=0}^{\infty} c_{d,q} z^d = G(z). \tag{23}$$

Following the same method as before,

$$s_{n,q} = \begin{cases} \text{'}s\text{'}s_{n-1,q} & \text{if string does not start with parenthesis,} \\ \text{'}(s\text{'}s_{2a,q_1}\text{'}s)\text{'}s_{b,q_2} & q_1 + q_2 + 1 = q, \\ \text{'}[s\text{'}s_{2a,q_1}\text{'}s]\text{'}s_{b,q_2} & q_1 + q_2 + 2 = q \end{cases} \tag{24}$$

for $2a + b + 2 = n$. The conditions on q_1 and q_2 come from the fact that each homoclinic separatrix (and hence each round pair of parentheses) contributes real dimension one, and each $\alpha\omega$-zone (and hence each square pair of parentheses) contributes complex dimension one (real dimension two). Manipulating the coefficients as in equations (17) and (18), we get the equations

$$f = 1 + g + z(tf^2 + t^2 f^2) \quad \text{and} \quad g = zf + tzfg + t^2 zfg. \tag{25}$$

We then have that $G = zf$ satisfies the algebraic equation

$$(t + t^2)^2 G^3 - 2(t + t^2)G^2 + (1 - z + z(t + t^2))G - z = 0. \tag{26}$$

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4. *Closed forms*

The best one can hope to achieve with respect to an enumerative problem is to find a closed-form representation for that what is being enumerated. The main tool we will use is the Lagrange–Bürmann inversion theorem. We will only use a simplified version and state it here for completeness.

THEOREM 4.1. (Lagrange–Bürmann inversion theorem) *Let $\phi(u)$ be a formal power series with $\phi_0 \neq 0$, and let $Y(z)$ be the unique formal power series solution of the equation $Y = z\phi(Y)$. Then the coefficient of $Y(z)$ of order n is given by*

$$[z^n]Y(z) = \frac{1}{n}[u^{n-1}]\phi(u)^n. \tag{27}$$

The notation $[z^n]\cdots$ means the coefficient of z^n in the power series expansion of the expression that follows.

4.1. *The simplified problem.* We first consider this problem for the single-index sequence c_d.

To apply the Lagrange–Bürmann inversion theorem, we need to write the algebraic equation (20) in the form $G = z\phi(G)$ for some formal power series ϕ. By simple arithmetic, one arrives at $G = z\phi(G)$ with

$$\phi(G) = \frac{1-G}{(1-2G)^2} \tag{28}$$

(note that $\phi(0) \neq 0$). Then, by Theorem 4.1,

$$c_d = [z^d]G(z) = \frac{1}{d}[u^{d-1}]\left(\frac{1-u}{(1-2u)^2}\right)^d. \tag{29}$$

Since

$$(1-u)^d = \sum_{n=0}^{d}\binom{d}{n}(-u)^n \tag{30}$$

and

$$\frac{1}{(1-2u)^{2d}} = \sum_{n=0}^{\infty}\binom{n+2d-1}{n}(2u)^n, \tag{31}$$

Cauchy multiplication gives

$$\begin{aligned}
[u^{d-1}]\left(\frac{1-u}{(1-2u)^2}\right)^d &= \sum_{n=0}^{d-1}\binom{n+2d-1}{n}2^n\binom{d}{d-1-n}(-1)^{d-1-n} \\
&= (-1)^{d-1}\sum_{n=0}^{d-1}\frac{(n+2d-1)!}{n!(2d-1)!}\frac{d!(-1)^n 2^n}{(d-1-n)!(n+1)!} \\
&= (-1)^{d-1}\sum_{n=0}^{d-1}\frac{(n+2d-1)!}{(2d-1)!}\frac{d!(-1)^n}{(d-1-n)!}\frac{2^n}{(2)_n n!}
\end{aligned} \tag{32}$$

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since $(2)_n = 2 \cdot 3 \cdots (2+n-1) = (n+1)!$, and $(-1)^n(1-d)_n = (d-1)!/(d-n-1)!$ gives

$$\begin{aligned} & d(-1)^{d-1} \sum_{n=0}^{d-1} \frac{(n+2d-1)!}{(2d-1)!} \frac{(1-d)_n 2^n}{(2)_n n!} \\ & = d(-1)^{d-1} \sum_{n=0}^{d-1} \frac{(2d)_n (1-d)_n}{(2)_n} \frac{2^n}{n!} \\ & = d(-1)^{d-1} {}_2F_1([1-d, 2d]; [2]; 2). \end{aligned} \tag{33}$$

We utilize (29) and Euler's formula:

$${}_2F_1([a, b]; [c]; z) = (1-z)^{-a} {}_2F_1\left([a, c-b]; [c]; \frac{z}{z-1}\right) \tag{34}$$

(found in Graham *et al* [**10**], for instance) for the Gaussian hypergeometric function

$${}_2F_1([a, b]; [c]; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{35}$$

where $(x)_n = x(x+1) \cdots (x+n-1)$ is the Pochhammer symbol. With $z = 2$, $c = 2$, $a = 1-d$, and $b = 2-2d$, equation (29) and (34) give the hypergeometric function stated in the following theorem.

THEOREM 4.2. *The number of combinatorial invariants for* Ξ_d *is*

$$c_d = {}_2F_1([2-2d, 1-d]; [2]; 2). \tag{36}$$

This series has a finite number of terms due to $1-d \in \mathbb{Z}^-$ being non-positive for $d \geq 1$.

4.2. *The complete problem.* We first find $[z^d]G(z, t)$. Following the same method as before, we note

$$\phi(G) = \frac{1 + (1 - (t+t^2))G}{(1-(t+t^2)G)^2} \tag{37}$$

and, by Theorem 4.1, we again have

$$c_d(t) := [z^d]G(z, t) = \frac{1}{d}[u^{d-1}]\left(\frac{1 + (1-(t+t^2))u}{(1-(t+t^2)u)^2}\right)^d. \tag{38}$$

Again utilizing Cauchy multiplication, we have

$$c_d(t) = \frac{1}{d} \sum_{n=0}^{d-1} \binom{n+2d-1}{n} (t+t^2)^n \binom{d}{d-1-n} (1-t-t^2)^{d-1-n}, \tag{39}$$

and rewriting gives

$$= \sum_{n=0}^{d-1} \frac{(-1)^n (2d)_n (1-d)_n}{(2)_n n!} (t+t^2)^n (1-t-t^2)^{d-1-n}. \tag{40}$$

Recognizing that this is the right-hand side of equation (34), with $a = 1-d$, $b = 2-2d$, $c = 2$, and $z = t+t^2$, this simplifies beautifully to

$$c_d(t) = {}_2F_1([1-d, 2(1-d)]; [2]; t+t^2). \tag{41}$$

By expanding $(t+t^2)^n$, we get the following theorem.

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THEOREM 4.3. *The number of combinatorial invariants of dimension q for Ξ_d is*

$$c_{d,q} = [t^q]{}_2F_1([1-d, 2(1-d)]; [2]; t+t^2) = \sum_{n=0}^{d-1} \frac{(2-2d)_n(1-d)_n}{(2)_n n!} \binom{n}{q-n}. \quad (42)$$

5. *Asymptotic behavior*

The next characteristic one might want to examine is the asymptotic behavior of the sequences c_d and $c_{d,q}$.

5.1. *The simplified problem.* For c_d, we wish to utilize the relation

$$\frac{1}{R} = \limsup_{d\to\infty} \sqrt[d]{|c_d|}, \quad (43)$$

where R is the radius of convergence of the associated generating function. We first check that $R > 0$ by use of the implicit function theorem. We have

$$P(G, z) = G^3 - G^2 + \left(\frac{z+1}{4}\right)G - \frac{z}{4} = 0, \quad (44)$$

$$\left.\frac{\partial P}{\partial G}\right|_{(0,0)} = \frac{1}{4} \neq 0, \quad (45)$$

so it follows that $R > 0$. We next determine R exactly in order to make a more precise asymptotic estimation for c_d. The polynomial $P(G)$ behaves nicely; it has three roots, counted with multiplicity. That is, G has three branches, of which we are interested in the one going through $(z, G) = (0, 0)$. The radius of convergence of the power series expansion of G about $z = 0$ will be determined by the first singularity on the branch of interest. In this light, we wish to find all of the places the branches intersect, i.e. where the discriminant is zero. Calculation gives

$$\Delta(P(G)) = \frac{-z}{16}\left(z - \left(\frac{11 \pm 5\sqrt{5}}{-2}\right)\right), \quad (46)$$

so the discriminant $\Delta(P(G)) = 0$ when $z = 0$ and $z = -(11 \pm 5\sqrt{5})/2$. When $z = 0$, $P(G)$ has a double root at $G = 1/2$ and a single root at $G = 0$. The branch we are interested in is the one where $G = 0$. So, G is analytic in a neighborhood of $z = 0$, and the radius of convergence will be determined by the z with smallest modulus where there is a singularity. There are two numbers to check; we start with the one with smaller modulus, $-(11 - 5\sqrt{5})/2$. Using Sturm's theorem, it can be shown that $P(G)$ has three distinct real roots in the interval $]0, -(11 - 5\sqrt{5})/2[$. At $z = -(11 - 5\sqrt{5})/2$, $P(G)$ has a single root at $G = (1/2)(-1 + \sqrt{5})$ and a double root at $G = (1/4)(3 - \sqrt{5})$. Since $(1/4)(3 - \sqrt{5}) < (1/2)(-1 + \sqrt{5})$, we can conclude that the branch we are interested in intersects the point $(z, G) = (-(11 - 5\sqrt{5})/2, (1/4)(3 - \sqrt{5}))$. Hence, $R = -(11 - 5\sqrt{5})/2$. Now, since c_d is positive and increasing, equation (43) becomes

$$\frac{1}{R} = \lim_{d\to\infty} \sqrt[d]{c_d} \quad (47)$$

and, by the definition of a limit, we can conclude

$$c_d \sim (1/R)^d = \left(\frac{-2}{11 - 5\sqrt{5}}\right)^d \approx (11.09)^d. \quad (48)$$

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5.2. *The complete problem.* We consider the discrete probability distribution

$$\pi_{d,q} = \frac{c_{d,q}}{c_d}. \tag{49}$$

The goal is to show that this probability distribution is asymptotically of some known form, for example normal, when $d \to \infty$. We wish to use the same method as in [**7**]; compare with their Theorem 5.

THEOREM 5.1. *The distribution $\pi_{d,q}$ has mean μ_d and variance σ_d^2 that satisfy*

$$\mu_d \sim \kappa d, \quad \sigma_d^2 \sim \lambda d, \tag{50}$$

where κ and λ are algebraic numbers. The laws in each case are asymptotically normal for $d \to \infty$.

Proof. The method of the proof is taken from [**7**] and also draws on [**2, 8, 9**].

Consider the characteristic function of $\pi_d(q)$, normalized by shifting the mean by μ_d and the variance by σ_d^2:

$$f_d(t) = e^{-i\mu_d t/\sigma_d} G_d(e^{it/\sigma_d}), \tag{51}$$

where $G_d(t) = c_d(t)/c_d$ is the probability generating function. The goal is to show that the normalized characteristic functions $f_d(t)$ converge pointwise to the characteristic function $e^{-t^2/2}$ of the standard normal. Then the limit law follows by Levy's continuity theorem.

We look at

$$P(z, t, G) = (t + t^2)^2 G^3 - 2(t + t^2)G^2 + (1 - z + z(t + t^2))G - z. \tag{52}$$

By previous discussions, $G_0 = (3 - \sqrt{5})/4$ is a double root for $P(\rho, 1, G)$, where $\rho = -(11 - 5\sqrt{5})/2$, so, by Weierstrass preparation, there is an analytic factorization

$$P(z, t, G) = ((G - G_0)^2 + m_1(z, t)(G - G_0) + m_2(z, t)) \cdot H(z, t, G), \tag{53}$$

where $H(z, t, G)$ is analytic near $(\rho, 1, G_0)$, $H(\rho, 1, G_0) \neq 0$, and m_1 and m_2 are analytic at $(\rho, 1)$. Then, applying the quadratic formula,

$$G(z, t) - G_0 = \tfrac{1}{2}(-m_1(z, t) \pm \sqrt{m_1(z, t)^2 - 4m_2(z, t)}). \tag{54}$$

We determine our branch of interest (as we did in §3.1) by the following. Consider first (z, t) restricted to $0 \leq z < \rho(t)$ and $0 \leq t < 1$. Since $G(z, t)$ is real there (G has real coefficients), the discriminant $m_1(z, t)^2 - 4m_2(z, t)$ must also be real and non-negative. Furthermore, since G is increasing in z for fixed t (coefficients of G are non-negative), and since the discriminant vanishes at $\rho(t)$ and is hence decreasing in z, we need to take the minus sign, i.e.

$$G(z, t) - G_0 = \tfrac{1}{2}(-m_1(z, t) - \sqrt{m_1(z, t)^2 - 4m_2(z, t)}). \tag{55}$$

Set $D(z, t) := m_1(z, t)^2 - 4m_2(z, t)$. Consider the resultant

$$R(z, t) = \mathrm{Result}_G\left(P(z, t, G), \frac{\partial}{\partial G} P(z, t, G)\right), \tag{56}$$

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which is a polynomial whose restriction $R(z, 1)$ has, by the discussion in §5.1, a simple root at $\rho = -(11 - 5\sqrt{5})/2$. By the implicit function theorem, this root lifts to an analytic branch $\rho(t)$ of an algebraic function, for t near 1:

$$R(\rho(t), t) = 0, \quad \rho(1) = \rho. \tag{57}$$

Therefore, the function $D(z, 1)$ has a simple real zero at $z = \rho$, and we can factorize

$$D(z, t) = (\rho(t) - z)K(z, t) \tag{58}$$

for some analytic K satisfying $K(\rho, 1) \neq 0$. So, the uniform family of singular expansions for $G(z, t)$ near $(\rho(t), t)$ takes the form

$$G(z, t) = c_0(t) + c_1(t)\sqrt{1 - z/\rho(t)} + \mathcal{O}(1 - z/\rho(t)), \tag{59}$$

uniformly with respect to t for t in a small neighborhood of 1, and with $\rho(t)$, $c_0(t)$, and $c_1(t)$ analytic at 1. If $G(z, t)$ satisfies equation (59), we let $\tilde{z} = z/\rho(t)$ and apply the results in [**8**] to obtain

$$\begin{aligned} c_d(t) := [z^d]G(z, t) &= (\rho(t))^{-d}[\tilde{z}^d]\{c_1(t)\sqrt{1 - \tilde{z}} + \mathcal{O}(1 - \tilde{z})\} \\ &= \frac{c_1(t)(\rho(t))^{-d}}{\Gamma(-1/2)d^{3/2}}(1 + \mathcal{O}(1/d)). \end{aligned} \tag{60}$$

The probability generating function is therefore a so-called 'quasi-power', i.e. $G_d(t)$ satisfies

$$G_d(t) = \frac{\gamma(t)}{\gamma(1)}\left(\frac{\rho(1)}{\rho(t)}\right)^d\left(1 + \mathcal{O}\left(\frac{1}{d}\right)\right). \tag{61}$$

If (61) holds, then for $d \to \infty$ and close enough to $w = 1$,

$$G_d(t) \sim \left(\frac{\rho(1)}{\rho(t)}\right)^d, \tag{62}$$

so, for fixed t, letting $d \to \infty$,

$$f_d(t) \sim e^{-i\mu_d t/\sigma_d}\left[\frac{r(0)}{r(it/\sigma_d)}\right]^d, \tag{63}$$

where $r(s) = \rho(e^s)$. Set $y(s) = \log(r(s)/r(0))$, expand $y(s)$ in a Taylor series, and use

$$\left[\frac{r(0)}{r(s)}\right]^d = \exp(d\,\log(r(0)/r(s)))$$

to get

$$f_d(t) \sim \exp\{-i\mu_d t/\sigma_d - d(ity'(0)/\sigma_d - t^2 y''(0)/2\sigma_d^2 + \mathcal{O}(t^3/\sigma_d^3))\}. \tag{64}$$

By calculation on (61),

$$\mu_d = G_d'(1) \sim -d\rho'(1)/\rho(1) = -dy'(0) \tag{65}$$

and

$$\sigma_d^2 = G_d''(1) + \mu_d(1 - \mu_d) \sim -d\left(\frac{\rho''(1) + \rho'(1)}{\rho(1)} - \left(\frac{\rho'(1)}{\rho(1)}\right)^2\right) = -dy''(0). \tag{66}$$

We obtain

$$f_d(t) \sim e^{-t^2/2} \tag{67}$$

for all t. □

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By differentiation of (61) and with help from MAPLE's *algeqtodiffeq* function in the GFUN package [**12**], we can calculate

$$\rho(1) = \frac{-11+5\sqrt{5}}{2}, \quad \rho'(1) = \frac{87-39\sqrt{5}}{2}, \quad \rho''(1) = \frac{702\rho(1)+716\rho'(1)}{-60}, \tag{68}$$

so

$$\kappa = \frac{\rho'(1)}{\rho(1)} = \frac{3(-3+\sqrt{5})}{-2} \approx 1.145\,898\,036 \tag{69}$$

and

$$\lambda = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2 = \frac{-60+29\sqrt{5}}{10} \approx 0.484\,597\,133. \tag{70}$$

6. *Alternative counting problems*

There are several variations on this enumeration problem that one might consider interesting. We discuss a few of these now.

6.1. *Distinguishing between real and complex dimension.* It has been suggested that one might want to count the complex dimensions (corresponding to the number of analytic invariants in $\mathbb{H}$ for a vector field ξ_P) and the real dimensions (corresponding to the number of analytic invariants in $\mathbb{R}_+$ for a vector field ξ_P) separately (see [**3, 6**] for definitions of the analytic invariants). This would correspond in the complete problem to a triple-indexed sequence $c_{d,s,h}$, where s is the number of complex analytic invariants and h is the number of real analytic invariants. We follow the same method as in §3.4. Let

$$s_{n,s,h} = \begin{cases} \text{'}s\text{'}s_{n-1,s,h} & \text{if string does not start with parenthesis,} \\ \text{'}(s\text{'}s_{2a,s_1,h_1}\text{'}s)\text{'}s_{b,s_2,h_2}, & s_1+s_2=s,\ h_1+h_2+1=h, \\ \text{'}[s\text{'}s_{2a,s_1,h_1}\text{'}s]\text{'}s_{b,s_2,h_2}, & s_1+s_2+1=s,\ h_1+h_2=h \end{cases} \tag{71}$$

be our string generating algorithm for $2a+b+2=n$. We define a recursion equation for $p_{n,s,h}$ in the usual way, and arrive at the generating function $G(z,t,w) = \sum c_{d,s,h} z^d w^s t^h$ satisfying (unsurprisingly)

$$(t+w)^2G^3 - 2(t+w)G^2 + (1-z+z(t+w))G - z = 0. \tag{72}$$

One can use methods as before to determine properties of $c_{d,s,h}$.

6.2. *Enumeration of moduli classes.* The combinatorial classes we have focused on distinguish between vector fields which are conformally conjugate by a rotation by a $(d-1)$st root of unity. Some may consider the problem of enumeration of classes in moduli space (i.e. without the labelling of the separatrices) more valuable. In order to study this, we utilize Pólya theory for the cyclic group isomorphic to the group generated by $e^{2\pi \mathrm{i}/(d-1)}$. We are interested in the sequence $\tilde{c}_d = [z^{d-1}]C(G(z))$, where $C(G(z))$ is the generating function for the number of combinatorial classes for degree d vector fields moduli rotation by a $(d-1)$st root of unity. Pólya theory gives

$$C(G(z)) = \sum_{k\geq 1} \frac{\phi(k)}{k} \log \frac{1}{1-G(z^k)}, \tag{73}$$

where $\phi(k)$ is the Euler totient function. This relies on prime factorization, so it does not seem likely that one can find a closed-form expression for $\tilde{c}_d$ as a function of d as of date.

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References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon and A. G. Maier. *Qualitative Theory of Second-Order Dynamic Systems*. Wiley, New York, 1973, Nauka, Moscow, 1967, English translation.

[2] E. A. Bender. Central and local limit theorems applied to asymptotic enumeration. *J. Combin. Theory Ser. A* **15** (1973), 91–111.

[3] B. Branner and K. Dias. Classification of polynomial vector fields in one complex variable. *J. Difference Equ. Appl.* **16**(5) (2010), 463–517.

[4] L. Comtet. *Advanced Combinatorics*. D. Reidel, Dordrecht, Holland, 1974.

[5] T. Davis. Catalan numbers. http://geometer.org/mathcircles/catalan.pdf.

[6] A. Douady, F. Estrada and P. Sentenac. Champs de vecteurs polynomiaux sur $\mathbb{C}$, unpublished manuscript.

[7] P. Flajolet and M. Noy. Analytic combinatorics of non-crossing configurations. *Discrete Math.* **204** (1999), 203–229.

[8] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.* **3**(2) (1990), 216–240.

[9] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[10] R. L. Graham, D. E. Knuth and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, Reading, MA, 1994.

[11] D. Neumann. Classification of continuous flows on 2-manifolds. *Proc. Amer. Math. Soc.* **48**(1) (1975), 73–81.

[12] B. Salvy and P. Zimmerman. GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Trans. Math. Software* **20**(2) (1994), 163–167.

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