A METHODOLOGY FOR OPTIMAL DESIGN OF COMPOSITE LAMINATES USING POLAR FORMALISM

M. Kazemi *

Environmental Sciences Research Center
Department of Mechanics
Eslamshahr Branch, Islamic Azad University
Eslamshahr, Iran

G. Verchery

Institut Supérieur des Matériaux et Mécaniques Avancés
Le Mans, France

ABSTRACT

An innovative optimization technique is presented for the design of composite laminated plates subjected to in-plane loads. A list of quasi-homogeneous laminates that can be used as angle-ply materials is proposed as a comprehensive solution for optimum lay-up. Two optimization procedures are performed: Dimensioning of the flexural stiffness and the elastic modulus, which provides the optimal orientations for the layers and offer highest in-plane resistance to composite laminated structures. The polar formalism for plane anisotropy is used to represent the flexural stiffness and elastic modulus tensors. Numerical examples are resolved for two materials with different elastic moduli.

Keywords: Composite, Optimization, Quasi-homogeneous, Angle-ply.

1. INTRODUCTION

The potential use of fiber-reinforced laminated composites in the development of lightweight structures is now well known. Composites have high strength-to-weight and stiffness-to-weight ratio. The present work is limited to layers or plies, and the fibers are assumed to be all parallel and continuous. The effective design of composite structures should be based on satisfying all requirements of a specific application.

The increased number of anisotropic properties and the complexity of composite behavior provide a couple of advantages and disadvantages for the designer. The possibility of attaining an optimal design that is safe against multiple failure mechanisms and accounts for the difficulty of selecting a large number of design variables makes mathematical dimensioning a natural tool for the design of laminated composite structures.

Structural dimensioning and optimization of laminated plates are not only determined by the size and thickness of surface sections, but factors such as material properties through the appropriate choice of fiber orientation, the number of layers and lay-up that make a composite laminate more efficient than conventional materials such as metals and alloys are also involved in this process Generally, an optimal design is associated with both the mechanical strength and the stiffness coefficients. There are also many applications in which the design of the laminate is principally governed by the flexural stiffness requirements. Furthermore, given the lack of understanding about the various failure modes of composite laminates, the optimization of flexural stiffness and elastic modulus provide the best preliminary steps of optimal design procedure.

The optimal design of the laminated composite materials has been the subject of broad analytical and numerical studies in the literature, *e.g.*, textbook of Haftka *et al.* [1] and Miki [2], but their works were limited to laminates of constant thickness without discontinuities under in-plane and bending loads using lamination theory. A compression strength model was proposed by Barbero [3] based on the damage mechanics and a Gaussian distribution of fiber misalignment, but the model used only three material parameters

More recently, Khani *et al.* [4] proposed an approach for maximizing the strength of variable stiffness panels using lamination parameters. It has been shown that strength, as an important design criterion, especially for structures with large stress gradients, should be taken into account. [4]. Liu *et al.* [5] and Setodeh *et al.* [6] used a two-step optimization procedure for a maximum stiffness design of laminated plates based on lamination parameters, but given the nature of numerical solutions, the results were not general.

Nik *et al.* [7] examined a set of meta-modeling techniques for the design optimization of composite lami-

* Corresponding author (masoud_kazemi@hotmail.com)

DOI : 10.1017/jmech.2015.98
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https://doi.org/10.1017/jmech.2015.98 Published online by Cambridge University Press

nates with variable stiffness. They considered three case studies with the first two related to the fiber path design of a plate under uniform compression and the third one concerned with the optimization of a composite cylinder under pure bending [7].

Zuo *et al.* [8] examined the effects of volume ratio, Young's modulus and Poisson ratio of the components on the stiffness of the composite as a classical optimization problem. Vannucci *et al.* [9] proposed several mathematical-analytical solutions based on the quasi-homogeneous concept for the bending design of the laminated plates. A general discussion of the optimal solutions to the problems of maximum stiffness, buckling load and modal frequencies for rectangular laminates was presented by Vannucci [10].

Verchery [11] proposed some rules and mathematical methods for stiffness design. He posited that the classical design rules were too restrictive and that more innovative designs were needed to find more efficient usages for composites.

In this study, a new design procedure, similar to all classical cases of in-plane and strength design, is proposed using invariant dimensionless material parameters. It is derived from the so-called "polar tensors invariants" proposed by Verchery [12], which is known as polar formalism or polar method. In this study, it is shown that the dependence of the final elastic properties of the optimal laminate upon such parameters is unexpected due to the material properties.

In this paper, a new optimization approach called dimensioning is presented based on the principle of the extreme ratio of mechanical coefficients. Principle of extreme ratio can be used for structural optimization such as anisotropic structures, as discussed by Christensen and Klarbring [13]. As an optimization tool, the dimensioning technique has already been used as a simple, accurate procedure for optimization of the homogeneous structures, but it has not been used for the optimal design of laminated plates.

The first part of this paper discusses the global solution of laminate lay-ups for identical layers. The quasi-homogeneous solution allows dealing with the complex problems of optimal design for composite laminates. Here, the quasi-homogeneous sequences, which can be used as angle-ply lay-up, is proposed as an optimal laminate configuration which provides the highest flexural strength with the same elastic behavior as classical homogeneous plates but without the elastic coupling.

An optimization procedure based on dimensioning technique is discussed in the two next sections of this paper: *i.e.*, the flexural stiffness dimensioning and elastic dimensioning or Young's modulus. This technique both provides the desirable range of the fiber orientation based on the design requirements and the appropriate constraints to the optimization problems.

In this study, the dimensioning technique is developed using the polar formalism. The polar method provides a simpler transformation rule of the elasticity tensor when a frame rotation is done. In this way, the influence of anisotropy is reduced to only two variables whose physical meaning is precisely determined. In addition, it is possible to distinguish anisotropy properties of layers and laminate independently of the reference frame the elastic moduli of which are known. We also have presented some numerical results based on analytical formulas derived in this paper. Numerical examples covering different features of laminated composite plates in terms of elastic modulus are proposed in the present study.

2. QUASI-HOMOGENEOUS LAMINATES THAT CAN BE USED AS ANGLE-PLY

A major challenge to the optimal design of the laminated plates is the anisotropy and coupling between in-plane and flexure behaviors caused by the heterogeneous thickness of the plate. The Love- Kirchhoff hypothesis, limited to the linear elastic behavior of laminated plates, simplified the analysis, but depending on the loading conditions and characteristics of materials, different kinds of elastic coupling may occur. For a laminated plate composed of various superposing laminas, the Classical Laminated Plate theory (CLPT) can be used to calculate stiffness tensors A, B and D [14]:

$$N = A\varepsilon + B\kappa$$
$$M = B\varepsilon + D\kappa \tag{1}$$

where N and M are tensors of in-plane resultant forces and bending moments respectively, while ε and κ are the mid-plane strains and curvature vectors respectively. The tensors A and D represent in-plane laminate moduli and flexural laminate stiffness respectively while the tensor B indicates in-plane/flexure coupling stiffness. According to the CLPT, for an N-layers laminate, the above stiffness tensors id dependent on the laminate characteristics (number of layers, material properties, thickness and orientation of each layer) [15]:

$$A, B, D = \frac{1}{n}\sum_{k=1}^{N} Q_k(\delta_k)(Z_k^n - Z_{k-1}^n) \tag{2}$$

where $Q_k(\delta_k)$ is the stiffness tensor of the k^{th} layer, whose material frame is oriented by angle δ_k with respect to the global axes of the plate, and the superscript m = 1, 2, 3 for A, B and D stiffness tensor respectively. Generally, elastic coupling is non zero ($B \neq 0$) for a laminate composed of anisotropic layers. Elastic coupling has some undesirable effects on composite laminates. It has been shown that symmetrical lay-ups can eliminate coupling behaviors, leading to the uncoupling of a laminate.

The use of symmetrical sequences does not guarantee the optimal lay-up as other design constraints may be neglected. In fact, it has been demonstrated that symmetric lay-ups represent only one trivial solution among numerous quai-trivial non-symmetric solutions that ensure elastic uncoupling [16,17]. The quasi-trivial solutions can help find the quasi-homogeneous (QH) sequences. The quasi-homogeneous laminates display a behavior similar to conventional homogeneous plates, but without the elastic coupling. Moreover, fiber orientations of QH laminate are variable, depending on the

https://doi.org/10.1017/jmech.2015.98 Published online by Cambridge University Press

choice of the designer in keeping with the design and/or optimization steps. To satisfy quasi-homogeneity conditions, the laminate must satisfy two conditions: First, the flexion-extension coupling must be eliminated, *i.e.* the coupling tensor (B) must be nulled ($B = 0$); Second, the laminate behavior must be to homogeneous plate in terms of extension and flexion, meaning that the homogenized stiffness tensors in extension and flexion must be equal [15]:

$$[A]^* = [D]^* \tag{3}$$

$$[A]^* = \frac{1}{h}[A], [D]^* = \frac{12}{h^3}[D] \tag{4}$$

where $[A]^*$ and $[D]^*$ represent homogenized flexion and extension stiffness tensors respectively, and h is the thickness of the laminated plate. The homogenized tensors are used to characterize mechanical properties independently of geometrical effects, as they represent inherent mechanical behavior of equivalent homogeneous materials.

The angle-ply composite laminates, widely used in many applications for their high strength and stiffness, provide optimal design in many cases. Thus, the optimal lay-up solution proposed in this paper is limited to the quasi-homogeneous sequences that can be used as an angle-ply with two opposite fiber orientations ($+\theta$ and $-\theta$). The result was a list of angle-ply sequences similar to homogeneous plate in terms of behavior, meaning they are elastically uncoupled and completely orthotropic in flexion and extension.

In this solution, the fibers orientations are not fixed and it is up to the designer to select them based on the other constraints. It is one of the important advantages of this solution.

This technique enables the designer to enlarge lay-up choices that are not limited to the symmetry sequences and provides an orthotropic composite laminate. Since this condition can be very difficult to achieve, most research preferred to examine orthotropic laminates that were not actually orthotropic. A list of quasi-homogeneous sequences that can be lay-up as angle-ply is shown in Table 1. The results can be used for laminates composed of identical layers, while the absolute value of the fiber orientations are equal but with opposite signs (Signs "+" or "–" respectively means $+\theta - \theta$).

Sometimes, for a laminate composed of multiple layers, there are several choices of angle-ply quasi-homogeneous sequences (symmetric, anti-symmetric or asymmetric). In these cases, it is desirable to take into account other design factors that may happen for some sequences. For example, it is shown that fatigue strength is reduced by repeating orientation in successive layers [18]. Repetition indices are shown in Table 1.

3. DIMENSIONING FLEXURAL STIFFNESS

This section seeks to introduce the procedure of dimensioning flexural stiffness as an effective mathematical tool for the optimal design of both angle-ply laminated plates and all other structures with anisotropic properties. Although the dimensioning procedure is useful for all types of loading conditions, this paper is limited to the thin laminated plates. It is demonstrated that in-plane loading caused buckling and it is a preponderant factor in the design of thin plates [19,20]. The principle of the extreme ratios of the mechanical coefficients is used for dimensioning flexural stiffness tensor using polar formalism [21,22].

3.1 Polar Formalism of the Flexural Stiffness Tensor

According to the theory of elasticity known as generalized Hooke's law, the strain (deformation) of an elastic object or material is proportional to the load-induced stress applied to it. However, since stresses and strains generally have multiple independent components for anisotropic materials, the proportionality factor may no longer be just a single real number, but rather a tensor, which can be calculated as follows [14]:

$$\sigma_{ij} = Q_{ijkl}\,\varepsilon_{kl} \quad (i, j, k, l = 1, 2) \tag{5}$$

where σ_{ij} and ε_{kl} are second order tensor known as stress tensor and strain tensor, respectively,

Q_{ijkl} is a fourth order tensor known as stiffness tensor. The components of this tensor are known under various titles as elastic constants, moduli and stiffness components. The above equation demonstrates the effect of the flexural stiffness components on the calculation of the stress components as a strength criterion in structure design. In this study, an orthotropic laminated plate investigated. Here, the material consists of three orthogonal planes as planes of material symmetry and four independent constants regardless of the coordinate axes orientations. Flexural stiffness tensor for orthotropic materials is written as follows:

$$Q = \begin{bmatrix} Q_{1111} & Q_{1122} & Q_{1112} \\ Q_{1122} & Q_{2222} & Q_{2212} \\ Q_{1112} & Q_{2212} & Q_{1212} \end{bmatrix} \tag{6}$$

Generally, layers are composed of isotropic matrix and anisotropic fibers known as anisotropic material. As a result, the laminates constituted by overlaying laminas with different fiber orientations displayed anisotropic behaviors. The analysis and design of laminated plates based on CLPT require transformation of elastic coefficients of layers, which are complex Cartesian calculations due to the fourth powers of trigonometric functions. Clearly, this issue complicates the anisotropy problems. The polar formalism can be used to resolve this complexity. The polar formalism gives a representation of any planar tensor by means of a complete set of independent invariants. These invariants are called polar parameters and a great advantage in the design of anisotropic structures is that they are directly linked to the different symmetries of the tensor. To resolve this complexity the polar representation of plane elasticity tensors can be used. The polar formalism was introduced by Verchery [12] and successively developed [9-11,15-17,23-27,30].

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Table 1 Examples of quasi-homogeneous sequences used as angle-ply.

Number of layers	Groupes		Angle-ply lay-up		Repetition index
8	4+/4–		-+ + - /+ - - +	Antisym[1].	2
12	6–/6+		- + - + + + /- - - + - +	Antisym.	3
16	1	8–/8+	- + + - + - - + /+ - - + - + + -	Sym[2]. $[8]_S$	2
	2	8–/8+	- + + - + - - + /- + + - + - - +	Antisym.[8]As	2
	3	8–/8+	- + + - - + + - / + - - + + - - +	Antisym	2
	4	8–/8+	- + - + + - + - / + - + - - + - +	Antisym.	2
	5	8–/8+	- - + + + + - - / + + - - - - + +	Antisym.2[8]	4
	6	8–/8+	- + - + + - + + / - - - - + + + -	Asym[3].	4
20	1	10–/10+	- + - + - + + - + + / - - + - - + - + - +	Antisym.	2
	2	10–/10+	- - + + - + + + - + / - + - - - + - - + +	Antisym.	3
	3	10–/10+	- + + - - + - + + + / - - - + - + + - - +	Antisym.	3
	4	10–/10+	- + + - + - + - - - / + + + - + - + - - +	Antisym.	3
	5	10–/10+	- + + + - - - + - - / + + - + + + - - - +	Antisym.	3
	6	10–/10+	- - + + + - + - + + / - - + - + - - - + +	Antisym.	3
	7	10–/10+	- + - + + - - + + + / - - - + + - - + - +	Antisym.	3
	8	10–/10+	- - + - + + + + + - / + - - - - - + - + +	Antisym.	5
	9	10–/10+	- + - + - + + + - - / + - - + + - - + - +	Asym.	3
	10	10–/10+	- + - + + + - - - + / - + - + + - + - - +	Asym.	3
	11	10–/10+	- + - + + + - - - + / - + + - - + - + + -	Asym.	3
	12	10–/10+	- + - + + + - - - - / + + + + - - - + - +	Asym.	4
	13	10–/10+	- - + + + + - + - - / + - + - + + - - +	Asym.	4
	14	10–/10+	- + - + - + + + - + / - - - - + + - + + -	Asym.	4
	15	10–/10+	- - + + + + + - - - / - - + - + + + - + -	Asym.	5
	16	10–/10+	- + - - + + + + + - / - - - + - + - + - +	Asym.	5
24	166 solutions	12+/12–	32 angle-ply antisymmetric		
			1 - + + - - + + - + - + -/- +- + - + + - - + + -	5 Angle-ply Symmetric	2
			$[12]_{sym}$ - + - + + + - - - + - +/+ - + - - - + + + - + -		3
			2 - + + + - - - - + - + + /+ + - + - - - - - + + + -		4
			4 - - + + + + - + - + - -/- - + - + - + + + + - -		4
			5 - + - + - + + + + - - - /- - - + + + + - + - + -		6

1. Antisymmetric stacking sequence
2. Symmetric stacking sequence
3. Asymmetric stacking sequence

Let $\tilde{Q}$ be the flexural stiffness tensor, the polar method allows representing its six distinct Cartesian components as functions of six independent parameters ($\tilde{T}_0$, $\tilde{T}_1$, $\tilde{R}_0$, $\tilde{R}_1$, $\tilde{\Phi}_0$, $\tilde{\Phi}_1$) called polar components of $\tilde{Q}$. If the material reference system is rotated by a desired angle φ , the flexural stiffness tensor can be expressed as [16]:

$$
\begin{aligned}
\tilde{Q}_{1111}(\varphi) &= \tilde{T}_0 + 2\tilde{T}_1 + \tilde{R}_0\cos 4(\tilde{\Phi}_0-\varphi) + 4\tilde{R}_1\cos 2(\tilde{\Phi}_1-\varphi)\\
\tilde{Q}_{1122}(\varphi) &= -\tilde{T}_0 + 2\tilde{T}_1 - \tilde{R}_0\cos 4(\tilde{\Phi}_0-\varphi)\\
\tilde{Q}_{1112}(\varphi) &= \tilde{R}_0\sin 4(\tilde{\Phi}_0-\varphi) + 2\tilde{R}_1\sin 2(\tilde{\Phi}_1-\varphi)\\
\tilde{Q}_{2222}(\varphi) &= \tilde{T}_0 + 2\tilde{T}_1 + \tilde{R}_0\cos 4(\tilde{\Phi}_0-\varphi) - 4R_1\cos 2(\tilde{\Phi}_1-\varphi)\\
\tilde{Q}_{2212}(\varphi) &= -\tilde{R}_0\sin 4(\tilde{\Phi}_0-\varphi) + 2\tilde{R}_1\sin 2(\tilde{\Phi}_1-\varphi)\\
\tilde{Q}_{1212}(\varphi) &= \tilde{T}_0 - \tilde{R}_0\cos 4(\tilde{\Phi}_0-\varphi)
\end{aligned}
\tag{7}
$$

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or in matrix form:

$$\begin{bmatrix} \tilde{Q}_{1111} \\ \tilde{Q}_{1122} \\ \tilde{Q}_{1112} \\ \tilde{Q}_{2222} \\ \tilde{Q}_{2212} \\ \tilde{Q}_{1212} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 & 4 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{T}_0 \\ \tilde{T}_1 \\ \tilde{R}_0 \cos 4(\tilde{\Phi}_0 - \varphi) \\ \tilde{R}_0 \sin 4(\tilde{\Phi}_0 - \varphi) \\ \tilde{R}_1 \cos 2(\tilde{\Phi}_1 - \varphi) \\ \tilde{R}_1 \sin 2(\tilde{\Phi}_1 - \varphi) \end{bmatrix} \tag{8}$$

In Eqs. (7) and (8) $\tilde{T}_0$, $\tilde{T}_1$ are scalars, $\tilde{R}_0$, $\tilde{R}_1$ are moduli and $\tilde{\Phi}_0$, $\tilde{\Phi}_1$ are polar angles. All of these parameters, as the polar components of flexural stiffness tensor, depend on the characteristics of the material. It has been demonstrated that $\tilde{T}_0$, $\tilde{T}_1$, $\tilde{R}_0$, $\tilde{R}_1$ and $\tilde{\Phi}_0$, $\tilde{\Phi}_1$ are invariant for a rotation of the reference axes. The polar angles $\tilde{\Phi}_0$ and $\tilde{\Phi}_1$ rely on the reference system. A rotation φ of the references axes transform the angles $\tilde{\Phi}_1$ and $\tilde{\Phi}_0$ into $\tilde{\Phi}_1 - \varphi$ and $\tilde{\Phi}_0 - \varphi$, respectively. Selecting one of the two polar angles set the orientation of the reference system and other polar angles as the difference $\tilde{\Phi}_0 - \tilde{\Phi}_1$ remains constant [15,16]. The polar representation of plane elasticity tensors has a number of advantages such as flexural stiffness tensor, flexural compliance tensor and elastic modulus tensor for the composite laminated plates. The polar formalism allows for easily eliminating some redundant mechanical variables from a design problem of a laminate composed of identical layers; so it simplifies the dimensioning procedure of mechanical coefficients of the composite laminates.

The inverse of Eq. (7) presents polar components (two scalars and two complexes) as a function of Cartesians using the following complex expression [16]:

$$\begin{aligned} 8\tilde{T}_0 &= \tilde{Q}_{1111} - 2\tilde{Q}_{1122} + 4\tilde{Q}_{1212} + \tilde{Q}_{2222} \\ 8\tilde{T}_1 &= \tilde{Q}_{1111} + 2\tilde{Q}_{1122} + \tilde{Q}_{2222} \\ 8\tilde{R}_0 e^{4i\Phi_0} &= \tilde{Q}_{1111} + 4i\tilde{Q}_{1112} - 2\tilde{Q}_{1122} - 4\tilde{Q}_{1212} - 4i\tilde{Q}_{1222} + \tilde{Q}_{2222} \\ 8\tilde{R}_1 e^{2i\Phi_1} &= \tilde{Q}_{1111} + 2i\tilde{Q}_{1112} + 2i\tilde{Q}_{1222} - \tilde{Q}_{2222} \end{aligned} \tag{9}$$

3.2 The Homogenized Polar Components of the Flexural Stiffness Tensor

The polar method can be used to represent both the stiffness tensor $Q_k(\theta_k)$ of the single layer and the stiffness tensor of the laminate. Equation (1) of the CLPT can be used to calculate the polar components of the flexural stiffness tensor ($\tilde{T}_0$, $\tilde{T}_1$, $\tilde{R}_0$, $\tilde{R}_1$, $\tilde{\Phi}_0$, $\tilde{\Phi}_1$) as the functions of layers (T_0, T_1, R_0, R_1, Φ_0, Φ_1):

$$\begin{aligned} &\tilde{T}_0 = \frac{1}{3}\sum_{k=1}^{N} T_0 (z_k^3 - z_{k-1}^3) \qquad \tilde{T}_1 = \frac{1}{3}\sum_{k=1}^{N} T_1 (z_k^3 - z_{k-1}^3) \\ &\tilde{R}_0 e^{4i\tilde{\Phi}_0} = \frac{1}{3}\sum_{k=1}^{N} R_0 e^{4i(\Phi_0 + \theta_k)} (z_k^3 - z_{k-1}^3) \\ &\tilde{R}_1 e^{4i\tilde{\Phi}_0} = \frac{1}{3}\sum_{k=1}^{N} R_1 e^{4i(\Phi_0 + \theta_k)} (z_k^3 - z_{k-1}^3) \end{aligned} \tag{10}$$

where angle θ_k is the fiber orientation of the k^{th} layer with respect to the laminate reference axes based on CLPT. The Cartesian components of the flexural stiffness tensor ($\tilde{Q}$) for laminates can be obtained using Eq. (1). For a laminated plate with h as the total thickness with N identical plies, Eq. (10) can be simplified as follows:

$$\begin{aligned} &\tilde{T}_0 = \frac{h^3}{12} T_0 , \ \tilde{T}_1 = \frac{h^3}{12} T \\ &\tilde{R}_0 = \frac{1}{3} R_0 \left\{ \left(\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \sin 4(\Phi_0 + \theta_k) \right)^2 + \left(\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \cos 4(\Phi_0 + \theta_k) \right)^2 \right\}^{\frac{1}{2}} \\ &\tilde{R}_1 = \frac{1}{3} R_1 \left\{ \left(\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \sin 2(\Phi_0 + \theta_k) \right)^2 + \left(\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \cos 2(\Phi_0 + \theta_k) \right)^2 \right\}^{\frac{1}{2}} \\ &\tilde{\Phi}_0 = \frac{1}{4} \tan^{-1} \frac{\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \sin 4(\Phi_0 + \theta_k)}{\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \cos 4(\Phi_0 + \theta_k)}, \ \tilde{\Phi}_1 = \frac{1}{2} \tan^{-1} \frac{\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \sin 2(\Phi_1 + \theta_k)}{\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \cos 2(\Phi_1 + \theta_k)} \end{aligned} \tag{11}$$

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The homogenized polar components $\tilde{T}_0^*, \tilde{T}_1^*, \tilde{R}_0^*, \tilde{R}_1^*$ within the laminated plate is defined as above polar components divided by $h^3/12$, but homogenization is unable to change the polar angles, as deduced from Eq. (11). For a laminated plate with an angle-ply lay-up, the homogenized polar components can be written as follows [26]:

$$\begin{matrix} \tilde{T}_0^* = T_0 \\ \tilde{T}_1^* = T_1 \end{matrix} \quad , \quad \begin{matrix} \tilde{R}_0^* \cos 4\tilde{\Phi}_0 = R_0 \cos 4\theta \\ \tilde{R}_1^* \cos 2\tilde{\Phi}_1 = R_1 \cos 2\theta \end{matrix} \tag{12}$$

The conditions on general elastic properties described above, *i.e.*, quasi-homogeneity and orthotropy, are mathematically expressed by [26]:

$$\tilde{\Phi}_0 - \tilde{\Phi}_1 = \tilde{K}\frac{\pi}{4}, \quad \tilde{K} = 0, 1 \tag{13}$$

The invariant $\tilde{K}$ determines the type of ordinary orthotropy (see [16]), and it can assume only the values 0 or 1, [26]. Vannucci [10] has shown the importance of this material invariant parameter in some problems of optimal design; namely, if a solution is optimal for $K = 0$, it is normally anti-optimal for $K = 1$ and inversely, and in some cases, the change of K can lead to a loss of uniqueness of the solution of an optimum problem [26].

3.3 Optimal Range of the Layer Orientation Angles

The design of the thin laminated plates depends on their strength against in-plane loads, which leads to global instability or delamination. The variation of in-plane components of flexural stiffness tensor with respect to the orthotropic axes is a major factor in calculation of the instability and consequently the optimal design of laminates [19-22,28,29].

In this section, a procedure to determine optimum range of fiber orientation of the angle-ply laminates is proposed. The optimal lay-up is selected from Table 1

The first component of flexural stiffness tensor ($\tilde{Q}_{1111}$) is a principal factor in the stability analysis and in-plane design of laminated plates. Kazemi [30] has been used this parameter to propose a

semi-analytical method for buckling analysis of composite laminates.

The dimensioning procedure uses the flexural stiffness tensor by imposing restrictions on extreme ratio of its first (in-plane) component (*i.e.* $\tilde{Q}_{1111}(\max)/\tilde{Q}_{1111}(\min)$). Here, a laminated plate composed of equal layers in terms of orientation $\pm\theta$ is examined. The calculation of polar components and $\tilde{Q}_{1111}(\varphi,\theta)$ is simplified thanks to the orthotropy condition and angle-ply sequences. The homogenized polar components of the laminate can be calculated using Eq. (12) as a function of the layer polar components and fiber orientation by imposing orthotropic criterion (Eq. (13)):

$$\begin{matrix} \tilde{T}_0^* = T_0 \\ \tilde{T}_1^* = T_1 \end{matrix} \quad , \quad \begin{matrix} \tilde{R}_0^* = R_0 |\cos 4\theta| \\ \tilde{R}_1^* = R_1 |\cos 2\theta| \end{matrix} \tag{14}$$

Henceforth, the homogenized variables in the above Equation are written without index '*'.

The in-plane component of flexural tensor in Eq. (7) is rewritten in the desired direction in terms of polar components:

$$\tilde{Q}_{1111} = \tilde{T}_0 + 2\tilde{T}_1 + \tilde{R}_0 \cos 4(\tilde{\Phi}_0 - \varphi) + 4\tilde{R}_1 \cos 2(\tilde{\Phi}_1 - \varphi) \tag{15a}$$

or

$$\tilde{Q}_{1111}(\varphi) = \tilde{T}_0 + 2\tilde{T}_1 + \tilde{R}_0^{\tilde{K}} \cos 4(\tilde{\Phi}_1 - \varphi) + 4\tilde{R}_1 \cos 2(\tilde{\Phi}_1 - \varphi) \tag{15b}$$

Orthotropic condition:

$$\tilde{R}_0^{\tilde{K}} = (-1)^{\tilde{K}} \tilde{R}_0, \quad (\tilde{K} = 0, 1) \tag{16}$$

Two different materials (HM carbon/epoxy and E-Glass/Epoxy) in terms of elastic modulus which were used for numerical analysis are shown in Table 2.

The polar curves in Fig. 1 show the influence of the fiber orientation θ of angle-ply (four angles 0°, 30°, 60°, 90°) on the flexural stiffness component $\tilde{Q}_{1111}(\varphi,\theta)$ for $\varphi = 0°$ to 90°.

Relative extremes of $\tilde{Q}_{1111}(\varphi,\theta)$ are examined to find the optimal orientation layers for a fixed fiber orientation (θ):

$$\frac{\partial \tilde{Q}_{1111}(\varphi)}{\partial \varphi} = 0 \Rightarrow 4\tilde{R}_0^{\tilde{K}} \sin 4(\tilde{\Phi}_1 - \varphi) + 8\tilde{R}_1 \sin 2(\tilde{\Phi}_1 - \varphi) = 0 \tag{17}$$

The ratio $\Re = \tilde{R}_1 / \tilde{R}_0^{\tilde{K}}$ is defined, assuming that there may be two cases as follows:

(a) If $|\Re| \geq 1$: There are two extrema in the direction of orthotropic axes as follows:

$$\varphi_1 = \tilde{\Phi}_1, \quad \varphi_2 = \tilde{\Phi}_1 + \frac{\pi}{2} \tag{18}$$

The sign of the second derivative $\tilde{Q}_{1111}(\varphi,\theta)$ determines the directions of extrema.

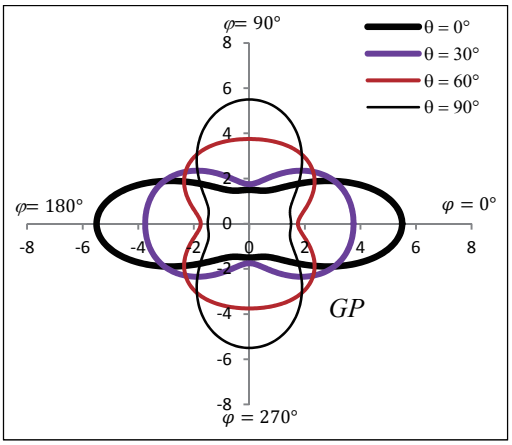
(b) If $|\Re| \leq 1$: There are 4 extrema depending on parameters $\tilde{K}$, including 2 extrema in angular direction φ_1 and φ_2, which are orthotropic axes, and 2 extrema in the direction of φ_3 as follows:

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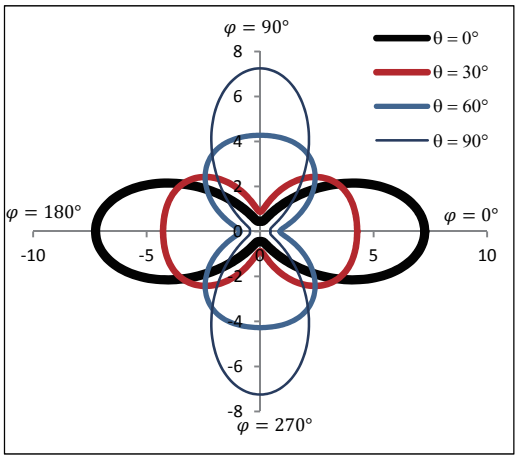
Table 2 Elastic properties and polar component of lamina.

Characteristics	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	v_{12}
E Glass / Epoxy	46.0	10.0	4.6	0.31
HM Carbon / Epoxy	230.0	14.4	4.9	0.32

Characteristics	T_0 (GPa)	T_1 (GPa)	$(-1)^K R_0$ (GPa)	R_1 (GPa)	Φ_0 (Rad)	Φ_1 (Rad)
E Glass / Epoxy	8.7	7.9	4.1	4.6	0	0
HM Carbon / Epoxy	32.0	31.9	27.0	27.1	0	0



(a) E-glass/epoxy



(b) HM Carbon/epoxy

Fig. 1 Polar variations of flexural stiffness component $\tilde{Q}_{1111}(\varphi,\theta)$ for an angle-ply.

$$\varphi_1 = \tilde{\Phi}_1\,,\quad \varphi_2 = \tilde{\Phi}_1 + \frac{\pi}{2}\,,\quad \varphi_3 = \tilde{\Phi}_1 \pm \frac{1}{2}\cos^{-1}\left(-\frac{R_1}{R_0^{\tilde{K}}}\right) \quad (19)$$

Note that relative extremes are assigned to the above angles depending on the parameter $\tilde{K}$. Table 3 summarizes the extreme $\tilde{Q}_{1111}$ in terms of $\Re$ ratio.

The flexural stiffness dimensioning determines the desired range of layer orientation based on the design restriction [20]. Thus, it is possible to find a fixed value for extrema ratio as $\tilde{Q}_{1111}(\max)/\tilde{Q}_{1111}(\min) \le l$.

Table 3 Extrema directions of flexural stiffness tensor component $\tilde{Q}_{1111}(\varphi,\theta)$.

$\Re = \frac{\tilde{R}_1}{\tilde{R}_0^{\tilde{K}}}$	−1		0	1
φ maximum	$\tilde{\Phi}_1$	$\tilde{\Phi}_1 \pm \frac{1}{2}\cos^{-1}(-\Re)$	$\tilde{\Phi}_1$	$\tilde{\Phi}_1 + \frac{\pi}{2}$
φ minimum	$\tilde{\Phi}_1 + \frac{\pi}{2}$	$\tilde{\Phi}_1 + \frac{\pi}{2}$	$\tilde{\Phi}_1 \pm \frac{1}{2}\cos^{-1}(-\Re)$	$\tilde{\Phi}_1$

The results of dimensioning $\tilde{Q}_{1111}$ with respect to φ as a function of layer orientation (θ) for an angle-ply laminate are shown in Figs. 2, 3 and 4. The curves are drawn for two cases of composite materials, *i.e.* glass fibre and HM Carbon fibre, using elastic properties and polar component of Table 2.

If the desired value of l is 2 and 4 for glass fiber and HM carbon fiber respectively, both cases will produce the same range of desired fiber orientation ($32° \le \theta \le 58°$) as shown in Figs. 2, 3, and 4. It is important to note that this procedure can be used to determine the constraints of optimization problems. The selection of the stiffness tensor component depends on the loading conditions in a special case. In this paper, the in-plane component of the flexural stiffness tensor is used as in-plane loads are principal criteria for the design of thin composite plates.

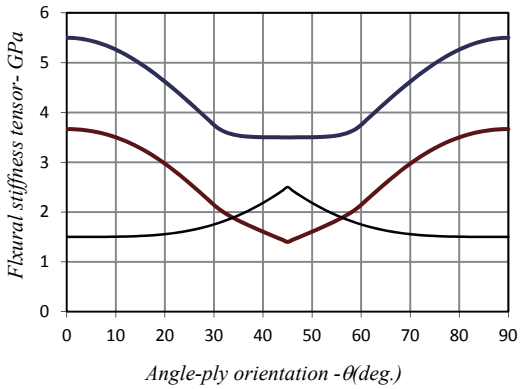
4. DIMENSIONING THE ELASTIC MODULUS

The elastic modulus is more practical and provides a better sense of the structural resistance. This section is aimed at developing the dimensioning procedure for elastic modulus using polar formalism to investigate the effect of the elastic modulus on the optimal range of fiber orientation. The numerical results are compared to those of the previous section. There is an inverse relationship between the in-plane elastic modulus ($\tilde{E}_1$) and the in-plane component of flexural compliance tensor ($\tilde{S}_{1111}$), *i.e.* $\tilde{E}_1 = \tilde{S}_{1111}^{-1}$.

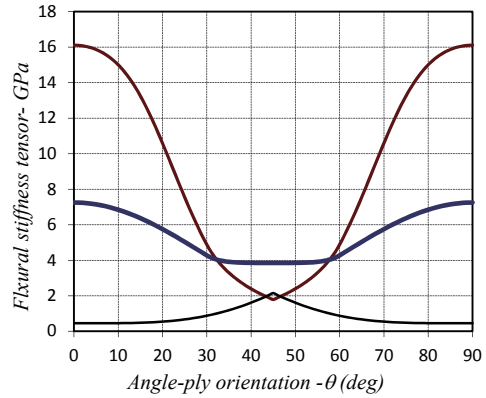
4.1 Polar Formalism of the Flexural Compliance Tensor

The fourth order Cartesian tensor of the flexural compliance ($\tilde{S}$) can be represented by the polar components similar to that of flexural stiffness tensor. The flexural compliance tensor $\tilde{S}_{ijkl}(i,j,k,l=1,2)$ is defined as opposite of the flexural stiffness tensor $\tilde{Q}_{ijkl}(i,j,k,l=1,2)$ [14]; The symbols $\tilde{t}_0, \tilde{t}_1, \tilde{r}_0, \tilde{r}_1, \tilde{\varphi}_0, \tilde{\varphi}_1$ are known as the polar components of the flexural compliance tensor [27]. The polar parameters $\tilde{t}_0, \tilde{t}_1$ are scalars, $\tilde{r}_0, \tilde{r}_1$ are moduli and $\tilde{\varphi}_0, \tilde{\varphi}_1$ are polar angles. The in-plane component of the flexural compliance tensor $\tilde{S}_{1111}(\varphi,\theta)$ in the desired direction of angle φ is represented as follows:

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(a) E-glass/epoxy



(b) HM Carbon/epoxy

Fig. 2 Dimensioning flexural stiffness tensor, $\tilde{Q}_{1111}$.

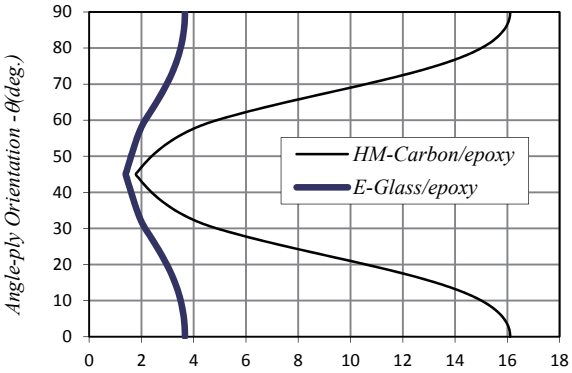
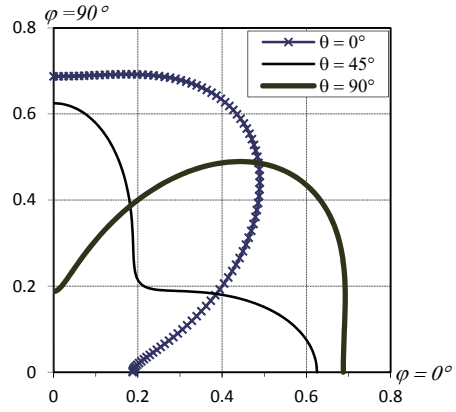


Fig. 3 Dimensioning flexural stiffness $\tilde{Q}_{1111}$: A comparison of two materials.

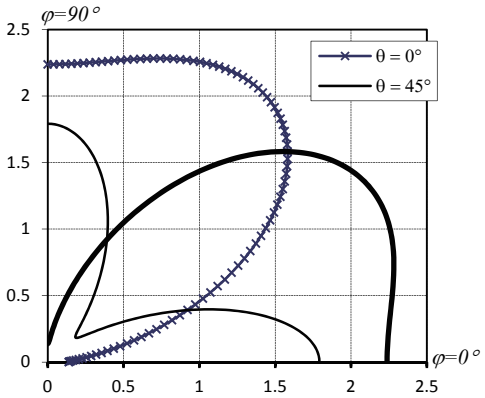
$$\tilde{S}_{1111}(\varphi,\theta) = \tilde{t}_0(\theta) + 2\tilde{t}_1(\theta) + \tilde{r}_0(\theta)\cos 4(\tilde{\varphi}_0 - \varphi) + 4\tilde{r}_1(\theta)\cos 2(\tilde{\varphi}_1 - \varphi) \quad (20)$$

The polar curves in Fig. 5 show the effect of layer orientation θ (three angles of 0°, 45°, and 90°) on $\tilde{S}_{1111}(\varphi,\theta)$. Since the polar curves in Fig. 5 are symmetric with respect to the x-y axes, they are drawn only in a quarter of the Cartesian plane.

The polar components of $\tilde{S}_{1111}$ are calculated as the functions of $\tilde{Q}_{1111}$ as are listed in Table 4. The orthotropic condition of the compliance tensor is applied as follows:

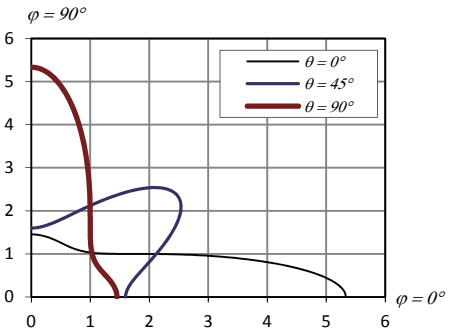


(a) E-glass/epoxy

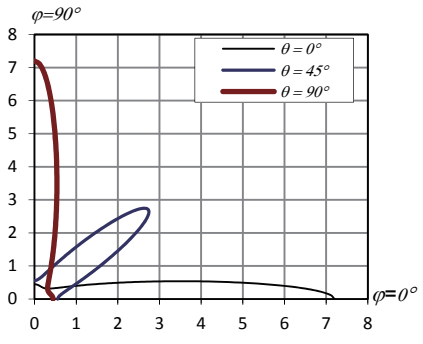


(b) HM Carbon/epoxy

Fig. 4 Polar curves of the in-plane compliance stiffness, $\tilde{S}_{1111}(\varphi,\theta)$ for three angle-ply orientation (θ).



(a) E-glass/epoxy



(b) HM Carbon/epoxy

Fig. 5 Polar curves of the in-plane elastic modulus, $\tilde{E}_1(\phi,\theta)$ for three angle-ply orientation (θ).

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Table 4 Polar components of the flexural compliance tensor ($\tilde{S}$).

Polar component of $\tilde{S}$	$\tilde{k}$ even	$\tilde{k}$ odd
$\tilde{t}_0(\theta)$	$4(\tilde{T}_0\,\tilde{T}_1-\tilde{R}_1^{\,2})/\Delta(\theta)$	
$\tilde{t}_1(\theta)$	$(\tilde{T}_0^{\,2}-\tilde{R}_0^2)/\Delta(\theta)$	
$\tilde{r}_0(\theta)$	$4(\tilde{R}_1^{\,2}-\tilde{T}_1\tilde{R}_0)/\Delta(\theta)$	$4(\tilde{R}_1^{\,2}-\tilde{T}_1\tilde{R}_0)/\Delta(\theta)$
$\tilde{r}_1(\theta)$	$2\tilde{R}_1(\tilde{T}_0-\tilde{R}_0)/\Delta(\theta)$	$2\tilde{R}_1(\tilde{T}_0-\tilde{R}_0)/\Delta(\theta)$
$\tilde{\varphi}_0$	$\frac{1}{4}\tan^{-1}\frac{\tilde{R}_1^2\sin 4\tilde{\Phi}_1-\tilde{T}_1\tilde{R}_0\sin 4\tilde{\Phi}_0}{\tilde{R}_1^2\cos 4\tilde{\Phi}_1-\tilde{T}_1\tilde{R}_0\cos 4\tilde{\Phi}_0}$	
$\tilde{\varphi}_1$	$\frac{1}{2}\tan^{-1}\frac{\tilde{R}_0^2\sin 4\tilde{\Phi}_1-\tilde{T}_1\tilde{R}_0\sin 4\tilde{\Phi}_0}{\tilde{R}_1^2\cos 4\tilde{\Phi}_1-\tilde{T}_1\tilde{R}_0\cos 4\tilde{\Phi}_0}$	
$\Delta(\theta)$	$16\tilde{T}_1(\tilde{T}_0^{\,2}-\tilde{R}_0^2)-32\tilde{R}_1^2(\tilde{T}_0-\tilde{R}_0)$	$16\tilde{T}_1(\tilde{T}_0^{\,2}-\tilde{R}_0^2)-32\tilde{R}_1^2(\tilde{T}_0+\tilde{R}_0)$

$$\tilde{\varphi}_0-\tilde{\varphi}_1=\tilde{k}\frac{\pi}{4} \tag{21}$$

The invariant $\tilde{k}$ determines the type of ordinary orthotropy and it can assume only the values 0 or 1.

4.2 Optimal Range of the Layer Orientation Angles

The modulus of elasticity of the composite laminates in angular direction of φ can be obtained by inversing similar compliance component expressed in Eq. (20):

$$\tilde{E}_1(\varphi,\theta)=S_{1111}{}^{-1}(\varphi,\theta)$$
$$=\left[\tilde{t}_0(\theta)+2\tilde{t}_1(\theta)+\tilde{r}_0(\theta)\cos 4(\tilde{\varphi}_0-\varphi)+4\tilde{r}_1(\theta)\cos 2(\tilde{\varphi}_1-\varphi)\right]^{-1} \tag{22}$$

Or by applying orthotropic condition (Eq. (21)):

$$\left[t_0(\theta)+2\tilde{t}_1(\theta)+\tilde{r}_0^{\tilde{k}}(\theta)\cos 4(\tilde{\varphi}_1-\varphi)+4\tilde{r}_1(\theta)\cos 2(\tilde{\varphi}_1-\varphi)\right]^{-1} \tag{23}$$

where $\tilde{r}_0^{\tilde{K}}(\theta)=(-1)^{\tilde{K}}\tilde{r}_0(\theta)\quad(\tilde{k}=0,1)$

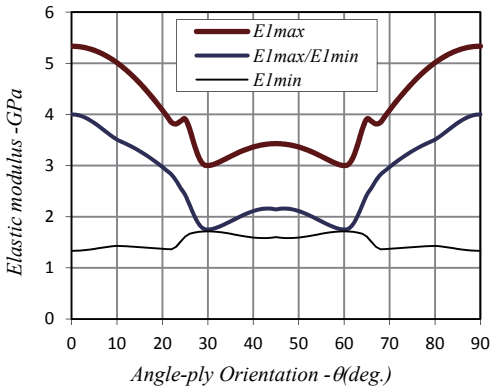
The polar curves in Fig. 6 show the influence of fiber orientation θ (three angles of 0°, 45°, and 90°) on $\tilde{E}_1(\varphi,\theta)$ as a function of the desired angle (φ). Since the polar curves in Fig. 6 are symmetric with respect to the x-y axes, they are drawn only in a quarter of the Cartesian plane.

The extrema of $\tilde{E}_1$ with respect to the desired angle φ for the fixed fibre orientation of angle-ply (θ) can be obtained as: $\partial\tilde{E}_1(\varphi,\theta)/\partial\varphi=0$ or

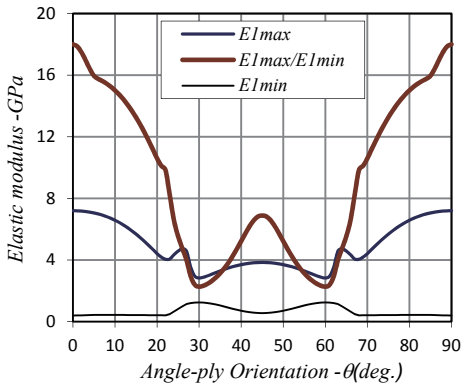
$$\sin 2(\tilde{\varphi}_1-\theta)\left[\tilde{r}_0^{\tilde{k}}\cos 2(\tilde{\varphi}-\theta)+\tilde{r}_1\right]=0 \tag{24}$$

(a) If $\tilde{r}_1/\tilde{r}_0>1\ \Rightarrow\tilde{E}_1(\varphi)$ has two extrema in directions of orthotropic axes that are $\tilde{\varphi}_1$, $\tilde{\varphi}_1+\pi/2$.

(b) If $\tilde{r}_1/\tilde{r}_0<1\ \Rightarrow\ \tilde{E}_1(\varphi)$ has four extrema in directions of $\tilde{\varphi}_1$, $\tilde{\varphi}_1+\pi/2$, $\tilde{\varphi}_1\pm 1/2\cos^{-1}(-\tilde{r}_1/\tilde{r}_0^{\tilde{k}})$



(a) E-glass/epoxy



(b) HM Carbon/epoxy

Fig. 6 Dimensioning of the in-plane elastic modulus tensor, $\tilde{E}_1(\varphi,\theta)$.

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Under both conditions $\tilde{E}_1(\varphi)$ has four extrema in directions of π/2. Note that the extrema directions of $\tilde{E}_1$ are inverse of those of $\tilde{S}_{1111}$. The results are given in Table 5.

Dimensioning procedure of the in-plane elastic modulus is similar to that of a flexural stiffness tensor. The results are shown in Fig. 7. This procedure leads to an optimal range of layer orientation based on the strength restriction, therefore allowing us to fix an upper limit for the extreme ratio: $\tilde{E}_1(\max)/\tilde{E}_1(\min) \le l$. The terms $\tilde{E}_1(\max)$ and $\tilde{E}_1(\min)$ correspond respectively to the maximum value and the minimum value of $\tilde{E}_1$ ($\varphi = 0°$ to $\varphi = 90°$) for a fixed angle-ply orientation angle θ (for example, in $\theta = 1°$ there are 2 extremes values of E_1 in the range of $0° \le \varphi \le 90°$).

The upper limit of extreme ratio should be selected on the basis of elastic strength, so the best minimum ratio is $l = 2$ for material with lower elastic modulus (E-glass/epoxy in this study). For materials with higher elastic strength (HM-carbon-epoxy in this study), it is selected on the basis of elastic modulus ratio between two materials to generalize the results. If we have $l = 2$ for E-glass/epoxy and $l = 4$ for HM-carbon/epoxy, both cases can generate the same range of optimal fiber orientation of an angle-ply, as deduced from Fig. 7:

$$27° \le \theta \le 37° \quad \text{and} \quad 53° \le \theta \le 63 \tag{25}$$

The results of the dimensioning for flexural stiffness tensor ($\tilde{Q}_{1111}$) as well as in-plane elastic modulus ($\tilde{E}_1$) can be merged to obtain best desired range of angle-ply orientation as are shown in Fig. 8. In numerical example that is examined in this work, for both materials the same range of optimal angle-ply orientations are obtained as follows:

$$32° \le \theta \le 37° \quad \text{and} \quad 53° \le \theta \le 58° \tag{26}$$

Table 5 Extrema directions of $\tilde{E}_1(\varphi, \theta)$.

$\frac{\tilde{r}_1}{\tilde{r}_0^{\tilde{k}}}$	$\frac{\tilde{r}_1}{\tilde{r}_0^{\tilde{k}}} < 1$		$\frac{\tilde{r}_1}{\tilde{r}_0^{\tilde{k}}} \ge 1$
	$\tilde{k} = 0$	$\tilde{k} = 1$	
φ maximum	$\tilde{\varphi}_1 \pm \frac{1}{2}\cos^{-1}\left(-\frac{\tilde{r}_1}{\tilde{r}_0}\right)$	$\tilde{\varphi}_1 + \frac{\pi}{2}$	$\tilde{\varphi}_1 + \frac{\pi}{2}$
φ minimum	$\tilde{\varphi}_1$	$\tilde{\varphi}_1 \pm \frac{1}{2}\cos^{-1}\left(-\frac{\tilde{r}_1}{\tilde{r}_0}\right)$	$\tilde{\varphi}_1$

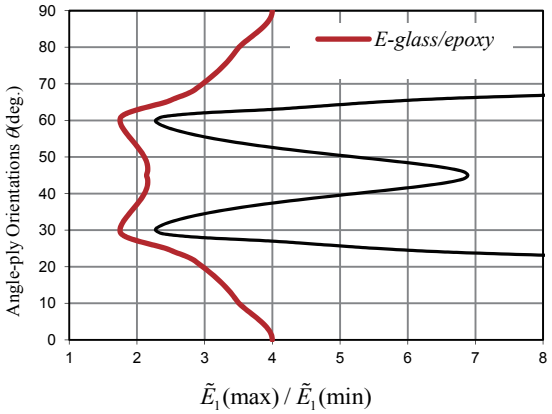
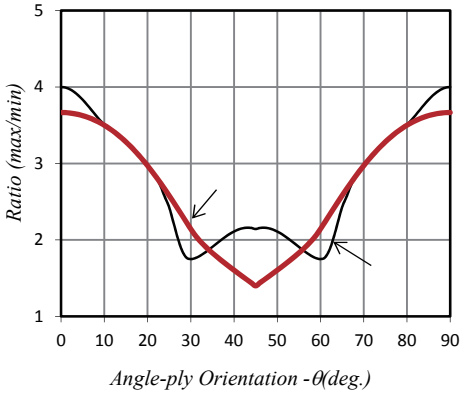
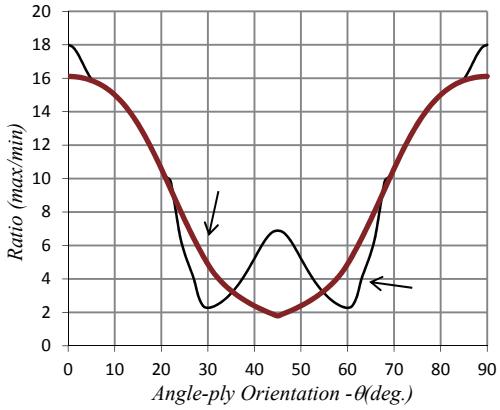


Fig. 7 Dimensioning of elastic modulus tensor: Comparison between two materials.



(a) E-glass/epoxy



(b) HM Carbon/epoxy

Fig. 8 Resultant of two optimization approach.

5. CONCLUSIONS

The optimal design of composite laminates is complex due to anisotropy and multiplicity of variables. In this study, an innovative method was proposed for determining two important anisotropy parameters of composite plates, *i.e.* laminate lay-up and fiber orientation.

The quasi-homogeneous laminates used as an angle-ply was proposed as a general solution of the optimal lay-up. The optimal lay-up consisted of a list of angle-ply sequences with where similar to the homoge-

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neous plate in terms of behaviour, which means they are elastically uncoupled and completely orthotropic in flexion and extension. An advantage of this solution is that the fiber orientations are unfixed and the designer can select them based on other constraints.

Polar representation method of plane elasticity tensors is used to distinguish isotropic and anisotropic parts of flexural stiffness tensor and elastic modulus, a task which was impossible in the conventional Cartesian method.

An innovative dimensioning procedure was presented: Dimensioning flexural stiffness tensor and elastic modulus. These two coefficients were selected for their major influence in the design of thin laminated plates.

Dimensioning procedure provides a range of optimal fiber orientations that can be used as constraints in optimization problems.

The numerical results suggested that for two materials with different elastic properties, the same range of optimal orientation in each dimensioning procedure could be obtained.

ACKNOWLEDGMENTS

ISAT of France and Professor Shahram Aivazzadeh supporting the first author through *Aides à la Formation de Recherche,* is graefully acknowledged.

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(Manuscript received February 25, 2015,
accepted for publication September 3, 2015.)

https://doi.org/10.1017/jmech.2015.98 Published online by Cambridge University Press