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Holomorphic correspondences mating Chebyshev-like maps with Hecke groups

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Abstract. We prove structure theorems for holomorphic correspondences realizing matings between pinched polynomial-like maps with connected Julia sets and Hecke groups (Fuchsian representations of free products $C_2 * C_n$ of cyclic groups of orders 2 and n). We show that such matings are generated by two-to-two subcorrespondences if and only if the maps are Chebyshev-like. We describe the dynamical behaviour of these matings in detail in the case $n = 4$ and we present a conjectured combinatorial description of the connectedness locus in parameter space in this case.

1. *Introduction*

An m-to-n *holomorphic correspondence* on the Riemann sphere $\hat{\mathbb{C}}$ is a multivalued map $z \to w$ defined by a relation $p(z, w) = 0$, where $p(z, w)$ is a polynomial of degree m in z and degree n in w. Here we require that $p(z, w)$ has no square factors, so that a generic point z has n distinct images w and a generic point w has m distinct inverse images z. The notion of a holomorphic correspondence is a generalization of that of a rational map and also of a finitely generated group of fractional linear maps, and in appropriate circumstances the dynamical behaviour of a holomorphic correspondence under iteration can exhibit similarities to that of a rational map or a Kleinian group (a discrete group of fractional linear maps). Perhaps the most striking examples are those of matings between a polynomial P and a Kleinian group G, where the Riemann sphere is partitioned into regions restricted to which the action of the correspondence is conformally conjugate to the standard action of P on its filled Julia set or the standard action of G on a component of its ordinary set. The first such examples were exhibited in [**5**]. We describe below how families of matings occur between holomorphic maps of degree $n - 1$ and Hecke groups abstractly isomorphic to $C_2 * C_n$ (the free product of cyclic groups of orders 2 and n) with $n \geq 3$, we show that such matings are generated by two-to-two subcorrespondences if and only if the maps are Chebyshev-like, and we investigate in detail the structure of limit sets and parameter spaces in the case $n = 4$.

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2. *Hecke groups and the Böttcher coordinate*

2.1. *Hecke groups.* For each integer value of $n \geq 3$, the *Hecke group* H_n is the subgroup of $PSL(2, \mathbb{R})$ generated by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & 1 \\ -1 & -2\cos(\pi/n) \end{pmatrix}.$$

For each $n \geq 3$ this is a discrete and faithful representation of the free product of cyclic groups C_2 and C_n, generated by σ and ρ, respectively. The Hecke group H_n acts on the complex upper half-plane with limit set $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. Note that H_3 is the *modular group* $PSL(2, \mathbb{Z})$.

PROPOSITION 2.1. *For each $n \geq 3$ there is a homeomorphism $h : \hat{\mathbb{R}}_+ = [0, \infty] \to I = [0, 1]$ conjugating the actions of $\sigma\rho, \ldots, \sigma\rho^{n-1} \in H_n$ on $\hat{\mathbb{R}}_+$ to the actions of 'right shift of base $n-1$ expressions of points of I, with insertion of $n-2, n-3, \ldots, 0$'.*

Proof. For $1 \leq j \leq n-1$, the maps

$$\sigma\rho^j : [0, \infty] \to [\sigma\rho^j(0), \sigma\rho^j(\infty)]$$

are order-preserving homeomorphisms. However, each of these maps is a contraction in the spherical metric on $\hat{\mathbb{R}}_+ \subset \hat{\mathbb{C}}$, and their images partition $\hat{\mathbb{R}}_+$ since $\sigma\rho(\infty) = \infty$, $\sigma\rho^{j+1}(\infty) = \sigma\rho^j(0)$ for $0 < j < n-1$, and $\sigma\rho^{n-1}(0) = 0$. The result follows. □

2.2. *The Böttcher coordinate.* Another place where the action of the shift on $n-1$ symbols arises very naturally is on the boundary of the filled Julia set of a polynomial of degree $n-1$, in which case the complement of this filled Julia set is a disc, parameterized by the *Böttcher coordinate*:

THEOREM 2.1. (Douady and Hubbard [**8**]) *If $P(z)$ is a polynomial of degree $n-1$ such that all (non-infinite) critical points have bounded orbits, then the Riemann map*

$$\phi : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus K(P)$$

tangent to the identity at infinity, from the complement of the unit disc $\mathbb{D}$ to the complement of the filled Julia set $K(P)$ of P, conjugates the map $z \to z^{n-1}$ to the map $z \to P(z)$.

If the boundary $\partial K(P)$ of $K(P)$ is locally connected, ϕ extends to a continuous map from $\partial\mathbb{D}$ onto $\partial K(P)$ and so we can then express the multivalued function $z \to P^{-1}(z)$ on $\partial K(P)$ as a quotient of the right shift on base $n-1$ expressions of points of the circle.

If we now take the upper half-plane, with its H_n action, and we glue the origin to the point at infinity, then by Proposition 2.1 the forward dynamics of H_n on the part of the boundary corresponding to $\hat{\mathbb{R}}_+$ can be matched to the backwards dynamics of a degree $n-1$ polynomial P on $\partial K(P)$, and the backwards dynamics of H_n on the part of the boundary corresponding to $\hat{\mathbb{R}}_-$ can be matched to the forwards dynamics of P on $\partial K(P)$. This leads to the idea of *mating* H_n with a polynomial. Although at first sight this may seem artificial, we see that such matings arise in a very natural context, that of *holomorphic correspondences*.

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3. *Holomorphic correspondences and matings*

3.1. *Holomorphic correspondences.*

A *correspondence* $\mathcal{F}$ on $\hat{\mathbb{C}}$ is a multivalued map with multivalued inverse. It is *holomorphic* if it is defined by a polynomial relation, i.e. there exists a polynomial $p(z, w)$ such that

$$z \to w \Leftrightarrow p(z, w) = 0.$$

Remark 3.1. Equivalently, a holomorphic correspondence is a relation whose graph is a (singular) Riemann surface in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Also equivalently, a holomorphic correspondence is any multivalued function of the form $\pi_2 \circ \pi_1^{-1}$, where $\pi_1 : \Gamma \to \hat{\mathbb{C}}$ and $\pi_2 : \Gamma \to \hat{\mathbb{C}}$ are branched covering maps from some Riemann surface Γ onto $\hat{\mathbb{C}}$ (see [**7**]).

Example 3.1. If Q is any rational map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree n then its *covering correspondence* Cov^Q, defined by

$$z \to w \Leftrightarrow Q(z) = Q(w)$$

is an n-to-n holomorphic correspondence, and its *deleted covering correspondence* Cov_0^Q, defined by

$$z \to w \Leftrightarrow \frac{Q(z) - Q(w)}{z - w} = 0$$

is an $(n-1)$-to-$(n-1)$ holomorphic correspondence.

If M is a Möbius transformation then the composition $M \circ \mathrm{Cov}_0^Q$ is the $(n-1)$-to-$(n-1)$ correspondence defined by

$$z \to w \Leftrightarrow \frac{Q(z) - Q(M^{-1}(w))}{z - M^{-1}(w)} = 0.$$

3.2. *Matings.*

Definition 3.1. We say that an $(n-1)$-to-$(n-1)$ holomorphic correspondence $\mathcal{F}$ is a *mating* between the Hecke group H_n and a map if there exists a conformal involution J on $\hat{\mathbb{C}}$ carrying $\mathcal{F}$ to $\mathcal{F}^{-1}$, and $\hat{\mathbb{C}}$ is partitioned into Ω and $\Lambda = \Lambda_- \cup \Lambda_+$, where:

(i) Ω is open, simply-connected and completely invariant under both $\mathcal{F}$ and J; moreover, there is a conformal homeomorphism ϕ from the complex upper half-plane to Ω conjugating the action of σ to that of $J|_\Omega$, conjugating the action of $\sigma\rho, \ldots, \sigma\rho^{n-1}$ to that of $\mathcal{F}|_\Omega$, and extending to a continuous map at 0 and ∞, sending both to a single fixed point of J;

(ii) Λ_- is closed, backward invariant, and $\mathcal{F}$ restricted to Λ_- as domain and codomain is a single-valued holomorphic map;

(iii) $\Lambda_+ = J(\Lambda_-)$ and $\Lambda_- \cap \Lambda_+$ is a single point.

Figures 1 and 2 illustrate two examples of such matings. In the first, the Hecke group is H_4 mated with a cubic whose filled Julia set is the closure of the union of countably many copies of the Douady rabbit. In the second, H_5 is mated with a quartic whose filled Julia set is the closure of the union of countably many copies of the basilica ($z \to z^2 - 1$).

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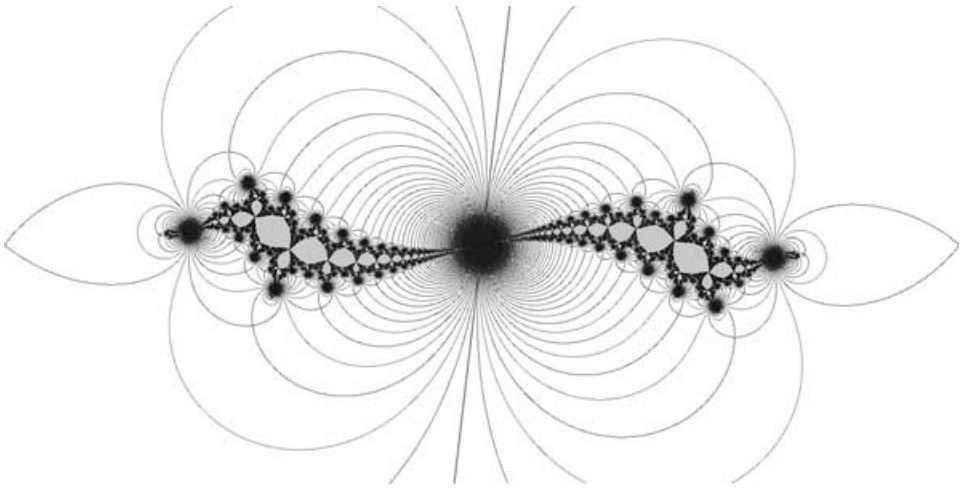


FIGURE 1. A mating between H_4 and a cubic (computer plot by X. Buff).

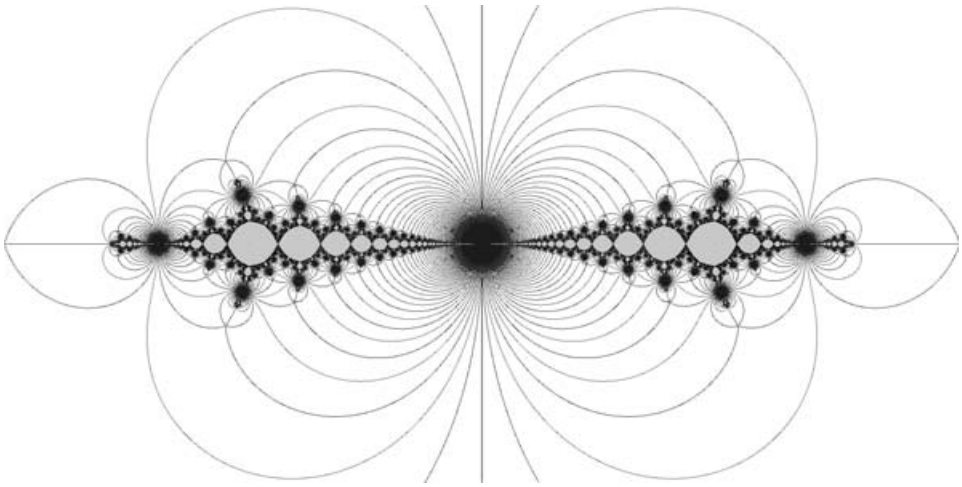


FIGURE 2. A mating between H_5 and a quartic (computer plot by X. Buff).

4. *Structure theorems for matings*

4.1. $C_2 * C_n$ *as an* $(n-1)$*-to-*$(n-1)$ *correspondence.* Let σ, ρ be fractional linear transformations, that is to say elements of $PSL(2,\mathbb{C})$, of orders 2 and n, respectively. Together σ and ρ generate a representation of the free product $C_2 * C_n$ as a subgroup $G \subset PSL(2,\mathbb{C})$. Consider the correspondence $\mathcal{H}_G$ defined by the relation

$$(w - \sigma\rho(z))(w - \sigma\rho^2(z)) \dots (w - \sigma\rho^{n-1}(z)) = 0,$$

which we also denote by

$$\mathcal{H}_G : z \to \{\sigma\rho z, \sigma\rho^2 z, \dots, \sigma\rho^{n-1} z\}.$$

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This is an $(n-1)$-to-$(n-1)$ holomorphic correspondence on the Riemann sphere. The grand orbits of the correspondence $\mathcal{H}_G$ are the orbits of the group G.

Definition 4.1. Let J be a (Möbius) involution on $\hat{\mathbb{C}}$. We say that an $(n-1)$-to-$(n-1)$ correspondence $\mathcal{F}$ is *J-reversible* if J conjugates $\mathcal{F}$ to its inverse $\mathcal{F}^{-1}$. For example $\mathcal{H}_G$, as defined above, is σ-reversible.

Now let $\mathcal{F}$ be any $(n-1)$-to-$(n-1)$ holomorphic correspondence on the Riemann sphere and let G be a faithful discrete representation of $C_2 * C_n$. Let U be an open set in $\hat{\mathbb{C}}$ completely invariant under the action of $\mathcal{F}$.

Definition 4.2. We say that $\mathcal{F}$ has a *$C_2 * C_n$-like action on U* if there exists an open G-invariant set $V \subset \hat{\mathbb{C}}$, and a conformal homeomorphism $\phi : V \to U$ conjugating the action of $\mathcal{H}_G$ on V to that of $\mathcal{F}$ on U. If $\mathcal{F}$ is J-reversible we say that the $C_2 * C_n$-like action is *compatible* with J if $J|_U$ is $\phi\sigma\phi^{-1}$.

THEOREM 4.1. *If $\mathcal{F}$ is a J-reversible $(n-1)$-to-$(n-1)$ holomorphic correspondence that is $C_2 * C_n$-like on an open set, compatibly with J, then $\mathcal{F}$ has the form $\mathcal{F} = J \circ \mathrm{Cov}_0^Q$ for some rational map Q of degree n.*

Proof. The correspondence $\mathcal{G} = (J \circ \mathcal{F}) \cup (\textit{identity})$ is an equivalence relation on U, since it is conjugate to $z \to \{z, \rho z, \ldots, \rho^{n-1} z\}$ on the upper half-plane. We will see that this implies that $\mathcal{G}$ is an equivalence relation on the whole of $\hat{\mathbb{C}}$. There are only finitely many singular points of $\mathcal{G}$ on $\hat{\mathbb{C}}$ (that is to say points z such that either $\mathcal{G}(z)$ or $\mathcal{G}^{-1}(z)$ contains less than n points). Given any non-singular point $z_1 \in \hat{\mathbb{C}}$ choose a path γ from z_1 to a non-singular point $z_0 \in U$, avoiding any singular points *en route*. By analytic continuation along γ from z_0 to z_1, it follows that $\mathcal{G}$ is an equivalence relation on the orbit of z_1. Since singular points are isolated $\mathcal{G}$ is also an equivalence relation on their orbits, by continuity.

Away from the orbits of singular points, the quotient $\hat{\mathbb{C}}/\mathcal{G}$ inherits a complex structure from $\hat{\mathbb{C}}$ and this extends to a Riemann surface structure on the whole of $\hat{\mathbb{C}}/\mathcal{G}$. Since the projection $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}/\mathcal{G}$ is conformal away from singular orbits it follows that Q is a branched covering, that $\hat{\mathbb{C}}/\mathcal{G}$ is a surface of genus zero (a sphere), and hence finally that Q is a rational map. □

Definition 4.3. A point z is called a *co-critical* point of Q if either z is a critical point of multiplicity greater than one, or $Q(z)$ is the image of a critical point of Q other than z. For Q of degree n the co-critical points of Q are those that have $(n-2)$ or fewer distinct images under Cov_0^Q.

By a *transversal* for Q we mean a maximal subset D_Q of $\hat{\mathbb{C}}$ on which Q is injective.

THEOREM 4.2. *Let $\mathcal{F}$ be a holomorphic correspondence which is a mating between a Hecke group and a map. Then:*

(i) *$\mathcal{F}$ is of the form $J \circ \mathrm{Cov}_0^Q$ for some polynomial Q and involution J;*

(ii) *there is a simply-connected transversal D_Q for Q, with boundary a Jordan curve, such that $\overline{D'_Q} \cap J(\overline{D'_Q})$ consists of a single point, which is both a simple critical point of Q and a fixed point of J (here D'_Q denotes $\hat{\mathbb{C}} \setminus D_Q$); and*

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(iii) *all (finite) co-critical points of Q are in $\Lambda_+ \cup \Lambda_-$, where $\Lambda_+ = \bigcap_{n=0}^{\infty} \mathcal{F}^n(D_Q)$ and $\Lambda_- = J(\Lambda_+)$.*

Proof. Part (i) is immediate from the previous theorem since by the definition of a mating $\mathcal{F}$ is H_n-like on Ω, compatibly with J. To find a transversal D_Q of the form described in part (ii) we choose as fundamental domain D for the action of ρ on the upper half-plane the set which has as its boundary the hyperbolic geodesic arcs from the fixed point of ρ to 0 and to ∞ and the right-hand half of the positive real axis between 0 and ∞. Now $D_Q = \phi(D) \cup \Lambda_+$ is a transversal for Q and has the properties stated; the condition (iii) on co-critical points follows from the fact that when restricted to Ω, the action of the correspondence $\mathcal{F}$ is conjugate to that of H_n on the upper half-plane. □

Remark 4.1. This theorem has a converse. If an $(n-1)$-to-$(n-1)$ holomorphic correspondence $\mathcal{F}$ is of the form $J \circ \mathrm{Cov}_0^Q$ for some involution J and satisfies (ii) and (iii), then $\mathcal{F}$ is necessarily a mating between H_n and a map. See Corollary 7.1 in §7.1 for a proof in the case when $n = 4$ and Q is the Chebyshev polynomial T_4. The method of proof presented in this example can be applied in general.

5. *Correspondences and matings generated by two-to-two subcorrespondences*

5.1. *Two-to-two correspondences.* By the *graph* of a correspondence f we mean the algebraic curve $\Gamma(f) = \{(z, w) : P(z, w) = 0\} \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ defining the (multivalued) assignment $f : z \to w$. Each two-to-two correspondence comes equipped with an action of a pair of involutions I_-, I_+ on its graph, namely the covering transformations for the projections $\tilde{Q}_-, \tilde{Q}_+$ from $\Gamma(f)$ onto the z-sphere and the w-sphere, respectively. Note that f can be written $\tilde{Q}_+ \tilde{Q}_-^{-1}$. For a generic two-to-two correspondence the group generated by I_+ and I_- will be the infinite dihedral group, but if the product $I_+ I_-$ is of order n they generate an action of the (finite) dihedral group D_{2n}, of order $2n$.

We recall that the possible topological types of the graph of a two-to-two correspondence are quite restricted.

LEMMA 5.1. *The graph Γ of any two-to-two holomorphic correspondence f on the Riemann sphere is of one of the following topological types:*

(i) *a torus (elliptic curve);*
(ii) *a sphere (with a self-intersection or non-smooth point);*
(iii) *a pair of spheres (intersecting or touching).*

Proof. For details see §2 of [**6**], but for the convenience of the reader we sketch the argument here. The correspondence f is defined by an equation of the form

$$w^2 A(z) + w B(z) + C(z) = 0$$

where $A(z)$, $B(z)$ and $C(z)$ are quadratic polynomials. (We remark that formally we should separately write z and w in homogeneous coordinates so that we can treat the point ∞ correctly: see [**6**, §2] or [**7**, §2.3].) It follows that f has four forward singular points (counted with multiplicity), where by a *forward singular point* we mean a point z_0 which has just one image w_0, rather than two. Thus the projection $\tilde{Q}_- : \Gamma \to \hat{\mathbb{C}}$ is branched over four points of $\hat{\mathbb{C}}$. When two of these points coincide, we have a double point of Γ,

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a point where two sheets intersect. A calculation of Euler characteristics now yields the results claimed, namely (i) if there are four simple branch points, (ii) if there are two simple branch points and a double point and (iii) if there are two double points. □

5.2. *Subcorrespondences of $C_2 * C_n$.*

Definition 5.1. We say that a correspondence f is a *subcorrespondence* of $\mathcal{F}$ if the graph of f is a union of components of the graph of $\mathcal{F}$. We say that $\mathcal{F}$ is *generated* by the subcorrespondence f if the grand orbits of $\mathcal{F}$ are the grand orbits of f. We also say that $\mathcal{F}$ is generated by f and an involution J if the grand orbits of $\mathcal{F}$ are the grand orbits obtained by mixed iteration of f and J.

As earlier, let σ, ρ be Möbius transformations of orders 2 and n, respectively, let $G \subset PSL(2, \mathbb{C})$ be the representation of $C_2 * C_n$ that they determine and consider the two-to-two subcorrespondence

$$h_1 : z \to \{\sigma\rho z, \sigma\rho^{-1} z\}$$

of the $(n-1)$-to-$(n-1)$ correspondence $\mathcal{H}_G$ defined earlier. This is a two-to-two holomorphic correspondence on the Riemann sphere, with inverse

$$h_1^{-1} : w \to \{\rho^{-1}\sigma w, \rho\sigma w\}.$$

Observe that $h_1^{-1}h_1(z) = \{z, \rho^2 z, \rho^{-2} z\}$. It follows that if n is odd the orbit of any point z under the action of $C_2 * C_n$ is identical to the grand orbit of z under mixed forward and backward iteration of h_1, and if n is even the orbit of z under the action of $C_2 * C_n$ consists of the union of the grand orbits of z and σz under mixed iteration of h_1.

The graph of h_1 is the union of two intersecting spheres, namely the graph Γ_1 of the Möbius transformation $\sigma\rho$ and the graph Γ_2 of $\sigma\rho^{-1}$.

Each of the involutions I_- and I_+ interchanges Γ_1 and Γ_2. Explicitly,

$$I_- : (z, \sigma\rho z) \in \Gamma_1 \to (z, \sigma\rho^{-1} z) \in \Gamma_2$$
$$I_- : (z, \sigma\rho^{-1} z) \in \Gamma_2 \to (z, \sigma\rho z) \in \Gamma_1$$
$$I_+ : (z, \sigma\rho z) \in \Gamma_1 \to (\rho^2 z, \sigma\rho z) \in \Gamma_2$$
$$I_+ : (z, \sigma\rho^{-1} z) \in \Gamma_2 \to (\rho^{-2} z, \sigma\rho^{-1} z) \in \Gamma_1.$$

Thus

$$I_+ I_- : (z, \sigma\rho z) \in \Gamma_1 \to (\rho^{-2} z, \sigma\rho^{-1} z) \in \Gamma_1$$
$$I_+ I_- : (z, \sigma\rho^{-1} z) \in \Gamma_2 \to (\rho^2 z, \sigma\rho z) \in \Gamma_2.$$

If n is odd the involutions I_- and I_+ generate a dihedral group of order $2n$. If n is even they generate a dihedral group of order n.

Now consider the involution $\tilde{\sigma}$ defined on $\Gamma(h_1)$ by sending each (z, w) to $(\sigma w, \sigma z)$.

PROPOSITION 5.1.

(i) *The involutions I_- and $\tilde{\sigma}$ generate a dihedral group D_{2n} of order $2n$ (whether n is odd or even).*

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(ii) *The correspondence* $\mathcal{H}_G : z \to \{\sigma\rho z, \ldots, \sigma\rho^{n-1}z\}$ *is* $\sigma \circ \mathrm{Cov}_0^Q$, *where* Q *is the projection* $\hat{\mathbb{C}} = \Gamma(h_1)/I_- \to \Gamma(h_1)/D_{2n}$ *and* D_{2n} *is the dihedral group defined in (i).*

Proof. Both (i) and (ii) follow at once from direct calculation:

$$\tilde{\sigma} I_- : (z, \sigma\rho z) \in \Gamma_1 \to (\rho z, \sigma\rho^2 z) \in \Gamma_1$$
$$\tilde{\sigma} I_- : (z, \sigma\rho^{-1} z) \in \Gamma_2 \to (\rho^{-1} z, \sigma\rho^{-2} z) \in \Gamma_2. \qquad \square$$

5.3. *Matings generated by two-to-two subcorrespondences.*

THEOREM 5.1. *Let* J *be an involution and let* $\mathcal{F} = J \circ \mathrm{Cov}_0^Q$ *be an* $(n-1)$*-to-*$(n-1)$ *holomorphic correspondence which is* $C_2 * C_n$*-like on an open set* Ω*, compatibly with* J. *Suppose the two-to-two subcorrespondence* $\phi h_1 \phi^{-1}$ *of* $\mathcal{F}|_\Omega$ *extends to a* J*-reversible two-to-two subcorrespondence* f_1 *of* $\mathcal{F}$ *on the whole of* $\hat{\mathbb{C}}$. *Then* Q *is the projection* $\hat{\mathbb{C}} = \Gamma(f_1)/I_- \to \Gamma(f_1)/D_{2n}$, *where* D_{2n} *is the dihedral group generated by* I_- *and* $\tilde{J}$, *the involution of* $\Gamma(f_1)$ *defined by* $(z, w) \to (Jw, Jz)$.

Proof. We first note that on $\Gamma(f_1)|_\Omega$ the involutions I_- and $\tilde{J}$ generate a dihedral group of order $2n$, and restricted to Ω the function Q is the projection $(\Gamma(f_1)|_\Omega)/I_- \to (\Gamma(f_1)|_\Omega)/D_{2n}$ (by Proposition 5.1). By analytic continuation $(I_-\tilde{J})^n$ is the identity on the whole of $\Gamma(f_1)$, yielding an action of D_{2n} extending that on $\Gamma(f_1)|_\Omega$. It follows by the uniqueness of analytic continuation that Q is the projection $\Gamma(f_1)/I_- \to \Gamma(f_1)/D_{2n}$ on the whole of the Riemann sphere $\hat{\mathbb{C}} = \Gamma(f_1)/I_-$. $\square$

Theorem 5.1 has some immediate consequences. Let $g_1 = \sigma \circ f_1$, where f_1 is the two-to-two subcorrespondence of $J \circ \mathrm{Cov}_0^Q$ the existence of which is assumed in the hypothesis of Theorem 5.1. Then $g_1 = \sigma \circ f_1$, is a *symmetric* two-to-two subcorrespondence of Cov_0^Q, in other words $g_1 : z \to w$ if and only if $g_1 : w \to z$. This is because g_1 is symmetric on Ω, where it is $\{\rho, \rho^{-1}\}$, and the property of being symmetric extends by analyticity from Ω to the whole sphere.

Definition 5.2. If g is a symmetric two-to-two correspondence we denote by $g^{[j]}$ the jth iterate of g without backtracking, that is to say if $z \to w$ we do not allow $w \to z$ at the next stage unless z is the *sole* image of w. Thus each $g^{[j]}$ is also a symmetric two-to-two (or possibly one-to-one) correspondence.

Let f_j denote the two-to-two (or possibly one-to-one) correspondence $\sigma \circ g_1^{[j]}$, where $g_1 = \sigma \circ f_1$ as above, and let g_j denote $g_1^{[j]} (= \sigma \circ f_j)$.

COROLLARY 5.1. *In the situation of Theorem 5.1:*

(i) *the graph of the correspondence* $\mathcal{F} = J \circ \mathrm{Cov}_0^Q$ *is a union of graphs of the* J*-reversible two-to-two correspondences* $f_1, \ldots, f_{\lfloor (n-1)/2 \rfloor}$, *together with that of a* J*-reversible one-to-one correspondence* $f_{n/2}$ *when* n *is even;*

(ii) *when* n *is odd,* $J \circ \mathrm{Cov}^Q$ *is generated by any* f_j *with* j *coprime to* n; *when* n *is even,* $J \circ \mathrm{Cov}^Q$ *is generated by any such* f_j *together with* J.

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Proof. (i) It will suffice to show that Cov_0^Q splits into the symmetric two-to-two correspondences $g_1, \ldots, g_{\lfloor (n-1)/2 \rfloor}$, together with a symmetric one-to-one correspondence $g_{n/2}$ when n is even. However, (i) is true when restricted to Ω since here g_j is $\{\rho^j, \rho^{-j}\}$, and Cov_0^Q is $\{\rho, \rho^2, \ldots, \rho^{n-1}\}$, and hence (i) is true on the whole sphere by analyticity.

(ii) It is evident from the proof of (i) that for any n the correspondence Cov^Q is generated by any g_j with j coprime to n, so J together with $J \circ g_j$ generates $J \circ \mathrm{Cov}^Q$. However, $f_j^{-1} \circ f_j$ contains $g_j^{[2]}$, which when restricted to Ω is $\{\rho^{2j}, \rho^{-2j}\}$, so when n is odd and j is coprime to n there is a mixed iterate of f_j and f_j^{-1} which includes g_1 as one of its branches. It follows at once that the grand orbits of f_j are those of $J \circ \mathrm{Cov}^Q$. □

COROLLARY 5.2. *In the situation of Theorem 5.1, if $\Gamma(f_1)$ has genus zero then, up to pre- and post-composition by Möbius transformations, $Q = T_n$, the nth Chebyshev polynomial T_n (the polynomial which expresses $\cos n\theta$ in terms of $\cos\theta$).*

Proof. Choosing a coordinate ζ on the sphere $\Gamma = \Gamma(f_1)$ we may take I_- to be the involution $\zeta \to \zeta^{-1}$, and $\tilde{J}$ (an involution, which together with I_-, generates the dihedral group of order $2n$) to be $\zeta \to (e^{i\theta}\zeta)^{-1}$ for $\theta = 2\pi/n$. Note that up to conjugacy I_- and $\tilde{J}$ must have this form, since there is only one action of D_{2n} on the sphere, and we know that near its fixed point the composite $\tilde{J}I_-$ becomes ρ, rotation through $2\pi/n$, when projected to $\Gamma(f_1)/I_- = \hat{\mathbb{C}}$. The quotient map $\Gamma \to \Gamma/D_{2n}$ is now

$$\zeta \to (\zeta^n + \zeta^{-n})/2.$$

Taking $z = (\zeta + \zeta^{-1})/2$ as our coordinate on $\hat{\mathbb{C}} = \Gamma/I_-$, the function Q_n becomes the polynomial which expresses $(\zeta^n + \zeta^{-n})/2$ in terms of $z = (\zeta + \zeta^{-1})/2$. □

The following proposition establishes that in the situations that interest us, $\Gamma(f_1)$ does indeed have genus zero.

PROPOSITION 5.2. *If $\mathcal{F}$ is a mating between a Hecke group H_n and a map of degree $n-1$, and $\mathcal{F}$ has a two-to-two subcorrespondence f, then $\Gamma(f)$ has genus zero.*

Proof. The region Ω contains the ϕ-image, z_0 say, of the fixed point of ρ (where ϕ is the homeomorphism from the upper half-plane to Ω conjugating the Hecke group action there to that of $\mathcal{F}$ on Ω). It follows that $f(z_0)$ is a single point w_0, and that $f^{-1}(w_0)$ is the single point z_0. Thus (z_0, w_0) is a double point of $\Gamma(f)$, that is to say a point of self-intersection of the graph. However, $\Gamma(f)$ can have no other double point, since it has no other double point in Ω and by the definition of a mating it can have no double point in Λ_- or Λ_+ (being a two-to-one or one-to-two correspondence on these regions, respectively). We deduce from Lemma 5.1 that $\Gamma(f)$ has genus zero. □

5.4. *Examples of splittings of $J \circ \mathrm{Cov}_0^{T_n}$.* The first few Chebyshev polynomials are

$$T_2(z) = 2z^2 - 1$$
$$T_3(z) = 4z^3 - 3z$$
$$T_4(z) = 8z^4 - 8z^2 + 1$$
$$T_5(z) = 16z^5 - 20z^3 + 5z.$$

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The two-to-two subcorrespondence $g_1 = J \circ f_1$ of $\mathrm{Cov}_0^{T_n}$ (where f_1 is as in Theorem 5.1) is defined by pushing down the involution $\tilde{J}$ on $\Gamma(f_1)$ to a two-to-two correspondence on $\Gamma(f_1)/I_- = \hat{\mathbb{C}}$. Thus, in the notation of the proof of Corollary 5.2, the correspondence g_1 is

$$z = (\zeta + \zeta^{-1})/2 \to \{w_1 = (\zeta e^{-i\theta} + \zeta^{-1} e^{i\theta})/2,\ w_2 = (\zeta e^{i\theta} + \zeta^{-1} e^{-i\theta})/2\}.$$

Now

$$w_1 + w_2 = z(e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad w_1 w_2 = z^2 + (e^{2i\theta} + e^{-2i\theta} - 2)/4$$

so g_1 is the correspondence

$$z \to w \Leftrightarrow w^2 - 2zw\cos(2\pi/n) + z^2 - \sin^2(2\pi/n) = 0.$$

However, Cov^{T_n} also contains $g_1^{[j]}$, a correspondence of this form with $2j\pi/n$ in place of $2\pi/n$, for each j with $1 \le j \le n/2$. A check of degrees and a computation of scalar factors gives us that for n odd

$$T_n(w) - T_n(z) = 2^{n-1}(w - z) \prod_{j=1}^{(n-1)/2} (w^2 - 2zw\cos(2j\pi/n) + z^2 - \sin^2(2j\pi/n))$$

and for n even

$$T_n(w) - T_n(z) = 2^{n-1}(w - z)(w + z) \prod_{j=1}^{n/2-1} (w^2 - 2zw\cos(2j\pi/n) + z^2 - \sin^2(2j\pi/n)).$$

The correspondence $J \circ \mathrm{Cov}^{T_n}$ is defined by the curve $T_n(Jw) - T_n(z) = 0$. For example,

$$J \circ \mathrm{Cov}^{T_3} : z \to w \Leftrightarrow (Jw - z)((Jw)^2 + z(Jw) + z^2 - 3/4) = 0$$
$$J \circ \mathrm{Cov}^{T_4} : z \to w \Leftrightarrow (Jw - z)((Jw) + z)((Jw)^2 + z^2 - 1) = 0.$$

The correspondence $J \circ \mathrm{Cov}_0^{T_n}$ is given by the same curve with $(Jw - z) = 0$ deleted.

6. *Chebyshev-like maps*

When $J \circ \mathrm{Cov}_0^{T_n}$ is a mating between the Hecke group H_n and a map, we see that the map in question has a particular type of dynamics, associated with the Chebyshev polynomial T_{n-1} of one degree less than that of T_n. We first introduce some terminology.

6.1. *Pinched polynomial-like maps.*

Definition 6.1. (Douady and Hubbard [**9**]) Let U and V be subsets of the complex plane, homeomorphic to open discs and such that the closure of U is properly contained in V. Suppose $f : U \to V$ is a proper holomorphic map of degree d. Then f is called a polynomial-like map of degree d. The filled Julia set of f is defined to be

$$K_f = \bigcap_{n=0}^{\infty} f^{-n}(V).$$

Two polynomial-like maps f and g are said to be hybrid-equivalent if there exists a quasi-conformal homeomorphism ϕ conjugating f to g, with dilatation zero on K_f.

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A key tool in this subject is the following.

THEOREM 6.1. (The straightening theorem [**9**]) *Every polynomial-like map f is hybrid equivalent to a polynomial g of the same degree. If K_f is connected then g is unique.*

Unfortunately the maps we have to deal with have *pinch points*.

Definition 6.2. Let $U_1, \dots, U_k$ and V be open subsets of $\mathbb{C}$, each homeomorphic to a disc and such that:

(i) $U_i \subset V$ for $i = 1, \dots, k$;

(ii) ∂U_1 meets ∂V at a single point v_0, and $\partial U_i \cap \partial V = \emptyset$ for $i = 2, \dots, k$;

(iii) for each pair $i \neq j$ the intersection $\overline{U}_i \cap \overline{U}_j$ is either empty or contains a single point;

(iv) $\bigcup_1^k \overline{U}_i$ is connected and simply connected.

Then a holomorphic map f that sends each U_i onto V is called simply-pinched polynomial-like if f extends to a continuous map on the boundaries of the U_i, fixing v_0 and sending each non-empty $\overline{U}_i \cap \overline{U}_j$ to v_0.

As before, the set $K_f = \bigcap_{n=1}^{\infty} f^{-n}(V)$ is called the filled Julia set of f.

CONJECTURE 6.1. (The straightening conjecture for simply-pinched polynomial-like maps) *If f is a simply-pinched polynomial-like map with connected filled Julia set K_f, then on the interior of K_f the map f is conformally conjugate to a polynomial g of the same degree acting on its filled Julia set K_g, and this conjugacy extends to a topological conjugacy on the boundary of K_f.*

The maps which arise in our matings are simply-pinched polynomial-like maps of a particular type.

Definition 6.3. We say that a simply-pinched polynomial-like map $f : \bigcup_{i=1}^{\lfloor (n+1)/2 \rfloor} U_i \to V$ of degree $n-1$ is Chebyshev-like if:

(i) each $f_i = f|_{U_i} : U_i \to V$ with $1 \leq i \leq (n-1)/2$ is of degree 2, and if n is even $f_{n/2}$ is of degree 1;

(ii) each $\overline{U}_i \cap \overline{U}_j$ is empty unless $|i-j| = 1$ in which case $\overline{U}_i \cap \overline{U}_j$ is a single point, mapped by f to the fixed point $v_0 = \partial U_1 \cap \partial V$;

(iii) all critical points that lie in the interior of $\bigcup U_i$ are mapped to the *same* critical value $v_1 \in V$.

The following lemma is self-evident.

LEMMA 6.1. *The Chebyshev polynomial T_n has $n-1$ non-infinite critical points, all in the real interval $[-1, +1]$, and precisely two non-infinite critical values, -1 and $+1$. The critical points are non-degenerate and T_n alternately takes the values $+1$ and -1 at these. When n is odd, each critical value has exactly one inverse image that is not critical. When n is even, one critical value has $n/2$ inverse images, all critical points, and the other has $(n/2-1)$ inverse images that are critical and two other inverse images that are not critical points.*

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PROPOSITION 6.1. *If the* $(n-1)$*-to-*$(n-1)$ *correspondence* $\mathcal{F} = J \circ \mathrm{Cov}_0^{T_n}$ *is a mating between the Hecke group* H_n *and a holomorphic map of degree* $n-1$*, then the map in question is Chebyshev-like.*

Proof. Let $\mathcal{F} = J \circ \mathrm{Cov}_0^{T_n}$ be a mating between H_n and a map and let ϕ be a conformal homeomorphism from the upper half-plane to Ω conjugating $\{\sigma\rho, \ldots, \sigma\rho^{n-1}\}$ to $\mathcal{F}|_\Omega$. Let $\mathcal{L}$ denote the half of the imaginary axis in the upper half-plane and let $\mathcal{P}$ denote the polygon (with cusps at vertices) bounded by $\{\rho^j\mathcal{L} : 0 \le j < n\}$. Since $\phi^{-1}\sigma\phi$ extends to the involution J of $\hat{\mathbb{C}}$, the map ϕ extends to the endpoints 0 with ∞ of $\mathcal{L}$ and identifies these to the single point $v_0 = \Lambda_- \cap \Lambda_+$. Moreover, since ϕ conjugates $\{\rho^j, \rho^{-j}\}$ to a two-to-two correspondence g_j on Ω for each j, we deduce that ϕ extends to the vertices of $\mathcal{P}$, where it identifies each pair $\{\rho^j(0), \rho^{-j}(\infty)\}$ to a point. The quotient $\phi\mathcal{P}$ is a disc with $\lfloor n/2 \rfloor$ holes U_j, and if we let V denote the set obtained by 'filling' $\phi\mathcal{P}$, then the sets U_j and V are arranged as in the definition of Chebyshev-like map. Considering the behaviour of the maps $\sigma\rho^j$ on the boundary of $\mathcal{P}$ we see that each point $\overline{U}_j \cap \overline{U}_{j+1}$ maps under $\mathcal{F}$ to the fixed point v_0, and that each ∂U_j maps two-to-one to ∂V (except in the case of $j = n/2$ with n even, when the map is one-to-one). Hence, $\mathcal{F}$ maps each U_j two-to-one onto V (again with the possible exception of $U_{n/2}$). Now each such U_j contains a critical point c_j of $\mathcal{F}$. This means that $\mathcal{F}(c_j) = v_j$ has less than the prescribed number of images under $\mathcal{F}^{-1}$. Hence, $J(v_j)$ is a co-critical point of T_n. If n is odd there is only one such point by Lemma 6.1, and hence all the v_j are equal. If it is even there are two such co-critical points, but only one of them can lie in D_{T_n} (since the candidates share the same critical value under T_n). In our case, by construction, $J(V)$ is contained in a transversal for T_n, so again all the v_j are equal. □

If we arrange the fixed point v_0 of Definition 6.3 to be the origin, then up to multiplication by scalars, there is only one genuine polynomial of degree $m = n-1$ which can satisfy (i), (ii) and (iii) of that definition, namely the polynomial R_m, conjugate to $T_m(z)$, defined by

$$R_m(z) = \tfrac{1}{2}(1 - T_m(1-2z)).$$

This maps the real interval $[0, 1]$ to itself, has all its non-infinite critical points in this interval and has critical values 0 and 1. As is easily verified by comparing roots and leading coefficients, $R_m(z)$ may be written

$$R_m(z) = 2^{2m-2}z \prod_{j=1}^{m-1} \left(\sin^2\left(\frac{j\pi}{m}\right) - z \right).$$

Not all scalar multiples μR_m are Chebyshev-like in the sense of the definition above. For example, this cannot be so if the fixed point 0 of μR_m is attracting. A necessary condition for μR_m to be Chebyshev-like is that the fixed point 0 is the landing point of exactly one external ray. Let $\mathcal{R}_m$ denote the connectivity locus for the family μR_m, that is to say the set of values of the parameter μ such that filled Julia set of μR_m is connected. Let $\mathcal{R}_{m,0}$ denote the limb of $\mathcal{R}_m$ corresponding to values of μ where the fixed point 0 of μR_m is the landing point of exactly one external ray. The straightening conjecture for simply-pinched polynomial-like maps implies that for each mating of H_n

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with a holomorphic map of degree $m = n - 1$, the holomorphic map is $\mu R_m(z)$ for some scalar $\mu \in \mathcal{R}_{m,0}$. The parameter space version of this conjecture says that there is a bijection between the Teichmüller space of matings and the set $\mathcal{R}_{m,0}$.

The family $\mu R_2(z)$ contains all quadratic maps $z \to z^2 + c$ (up to conjugacy). Matings of such maps with H_3, the modular group, were discussed in [**5**]. Our main concern in the remainder of this paper is the description of the dynamics of maps in the family $z \to \mu R_m(z)$ in the next case, $m = 3$. However, before leaving the general case we note that when m is odd there is a direct connection between the dynamics of $z \to \lambda T_m(z)$ and $z \to \lambda^2 R_m(z)$, so that anything we can say about the latter will have a direct interpretation in terms of the former.

PROPOSITION 6.2. *When m is odd, for any value of $\lambda \in \mathbb{C}$,*

$$\lambda^2 R_m(z^2) = (\lambda T_m(z))^2.$$

Thus, λT_m is a lift of $\lambda^2 R_m$ via the double covering $z \to z^2$.

Proof. Writing $z = \cos\theta$ and using the defining property of the Chebyshev polynomials T_m, and also the fact that T_m is an *odd* function when m is odd, we have

$$\begin{aligned}\lambda^2 R_m(\cos^2\theta) &= \frac{\lambda^2}{2}(1 - T_m(1 - 2\cos^2\theta)) = \frac{\lambda^2}{2}(1 - T_m(-\cos 2\theta)) \\ &= \frac{\lambda^2}{2}(1 + T_m(\cos 2\theta)) = \frac{\lambda^2}{2}(1 + \cos 2m\theta) = \lambda^2 \cos^2 m\theta = (\lambda T_m(\cos\theta))^2.\end{aligned}$$

□

Remark 6.1. For each odd value of m the map T_m is the unique polynomial (up to conjugacy) of degree m with non-degenerate critical points which has the property that $z^2 \to (T_m(z))^2$ is conjugate to $z \to T_m(z)$.

It follows at once from the proposition above that every periodic orbit of $z \to \lambda^2 R_m(z)$ lifts to a periodic orbit of $z \to \lambda T_m(z)$, with the period either the same or doubled, the Julia set of $z \to \lambda^2 R_m(z)$ lifts to that of $z \to \lambda T_m(z)$, and the connectivity locus $\mathcal{R}_m$ of the family $\{\mu R_m\}$ is double covered by the connectivity locus $\mathcal{T}_m$ of the family $\{\lambda T_m\}$, via $\mu = \lambda^2$. The limb $\mathcal{T}_{m,0}$ of $\mathcal{T}_m$, consisting of those values of λ for which the external rays of angles $\frac{1}{4}$ and $\frac{3}{4}$ both land at $z = 0$, projects bijectively onto the limb $\mathcal{R}_{m,0} \subset \mathcal{R}_m$.

In the remainder of the paper we look at correspondences of the form $J \circ T_4$. We prove that some of these do indeed realize matings between H_4 and a Chebyshev-like map, and we investigate their dynamics, giving a conjectured description of the moduli space. Suitably modified versions of the results presented here hold for all n, and the techniques required to prove them are no more complicated than those used below.

7. *Dynamics of correspondences $\mathcal{F} = J \circ \mathrm{Cov}^{T_4}$*

We consider the four-to-four correspondences $\mathcal{F} = J \circ \mathrm{Cov}^Q$ where J is an involution and $Q = T_4$ is the Chebyshev polynomial $T_4(z) = 8z^4 - 8z^2 + 1$.

Our aim is to find correspondences of this form that realize matings between H_4 and a degree 3 Chebyshev-like map. By Theorem 4.2, one of the fixed points of J, call it p,

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is a simple critical point of Q. This reduces the family of possible J, and thus the family of $\mathcal{F}$ realizing matings, to a one-parameter family (the parameter being the other fixed point q of J). Moreover, also by Theorem 4.2, we are only considering those parameters q for which there exists a transversal D_Q with the property that $\overline{D'_Q} \cap J(\overline{D'_Q}) = p$. The following lemma is easy to prove.

LEMMA 7.1. *If $\overline{D'_Q} \cap J(\overline{D'_Q}) = p$, then there exists a fundamental domain D_J of J with $D_J \cup D_Q = \hat{\mathbb{C}} - p$. Moreover, $\mathcal{F}(D_Q) \subset \overline{D_Q}$ with $\partial D_Q \cap \partial \mathcal{F}(D_Q) = p$.*

Now Q has critical points ∞, 0 and $\pm 1/\sqrt{2}$. The points $\pm 1/\sqrt{2}$ map to the same critical value under Q, while associated with 0 we have the co-critical points ± 1. Thus, the singular fibres of Q (those containing less than four points) are $\{\infty\}$, $\{1/\sqrt{2}, -1/\sqrt{2}\}$ and $\{-1, 0, 1\}$. As a transversal for Q we may choose the disc

$$D_Q = \left\{ (x, y) \;\middle|\; x \geq \frac{1}{\sqrt{2}}, |y| < \sqrt{\frac{2x^2 - 1}{2}} \right\}.$$

The correspondence Cov^Q is given by

$$z \to w \Leftrightarrow (z - w)(z^2 + w^2 - 1)(z + w) = 0,$$

the union of the identity, a two-to-two symmetric correspondence g and a one-to-one symmetric correspondence $g^{[2]}$ (the involution $z \leftrightarrow -z$). We note that the singular points of g (the points z having a single image w) are ∞, -1 and $+1$. Each of the singular points $\pm 1/\sqrt{2}$ for Cov^Q remains a singular point for $\mathrm{Cov}_0^Q = g \cup g^{[2]}$, since each has the property that its image under $g^{[2]}$ coincides with one of its images under g. We choose J so that it has $1/\sqrt{2}$ as one of its fixed points. Thus, it is of the form

$$J(z) = \frac{(q + 1/\sqrt{2})z - \sqrt{2}q}{2z - q - 1/\sqrt{2}}.$$

The correspondence $\mathcal{F} = J \circ \mathrm{Cov}_0^Q$ is given by

$$z \to w \Leftrightarrow (z^2 + (Jw)^2 - 1)(z + Jw) = 0,$$

the union of a two-to-two correspondence $f = Jg$ and a one-to-one correspondence $A = Jg^{[2]}$. The singular points of $\mathcal{F}$ for forward iteration are just those of Cov_0^Q, namely $\infty, \pm 1$ and $\pm 1/\sqrt{2}$, and the singular points for backward iteration of $\mathcal{F}$ are the images under J of these points.

We define $\Lambda_+ = \bigcap_{n=0}^{\infty} \mathcal{F}^n(D_Q)$, $\Lambda_- = J(\Lambda_+)$ (note that $\Lambda_+ \cap \Lambda_- = 1/\sqrt{2}$), $\Lambda = \Lambda_+ \cup \Lambda_-$ and $\Omega = \hat{\mathbb{C}} - \Lambda$.

Let $\mathcal{M}$ denote the set of q in parameter space for which $\mathcal{F}_q = \mathcal{F}$ satisfies $\partial D_Q \cap \partial \mathcal{F}(D_Q) = 1/\sqrt{2}$ and for which Ω is simply connected.

7.1. *Group-like behaviour on $\Omega = \hat{\mathbb{C}} - \Lambda$.*

THEOREM 7.1. *Suppose $q \in \mathcal{M}$. Then the action of $\mathcal{F}(= \mathcal{F}_q)$ on the completely invariant set Ω is conformally conjugate to that of the three-to-three correspondence*

$$\mathcal{H} : z \to w \quad \text{if } (z - \sigma\rho(w))(z - \sigma\rho^2(w))(z - \sigma\rho^3(w)) = 0$$

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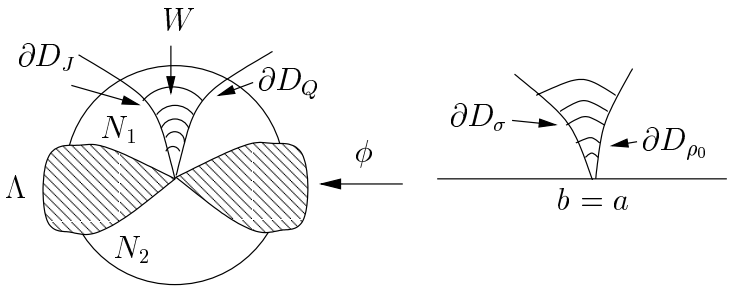


FIGURE 3. Images of curves under the Riemann map ϕ.

on the open complex upper half-plane, where σ and ρ are the generators of the Hecke group H_4.

Proof. Since Ω is simply connected there exists a Riemann map ϕ sending the upper half-plane $\mathbb{H}$ to Ω, with the point i mapping to the fixed point q of the involution J. Thus, ϕ^{-1} carries the involution J to the unique involution of the upper half-plane that has fixed point i, the involution $\sigma : z \to -1/z$. The only critical point of Q in Ω is at ∞, in a neighbourhood of which the four branches of Cov^Q are conjugate to rotations through $j\pi/2$, $j = 1, 2, 3, 4$, and thus on the whole of Ω the Riemann map ϕ^{-1} carries these four branches to the powers of a rotation ρ_0 of order 4 of the upper half-plane, about the point $\phi^{-1}(\infty)$. Hence, σ and ρ_0 generate a Fuchsian group H.

By Lemma 7.1, the sets $D_J \cap \Omega$ and $D_Q \cap \Omega$ map to fundamental domains D_σ and D_{ρ_0} for the actions of σ and ρ_0, respectively, on the (open) upper half-plane. Since $(D_Q \cap \Omega) \cup (D_J \cap \Omega) = \Omega$, these fundamental domains together cover all of the upper half-plane. Thus, by the usual Klein combination argument [**12**], the group H generated by σ and ρ_0 is a faithful representation of the free product $C_2 * C_4$.

It remains to prove that $H = H_4$ (up to conjugacy by an automorphism of the half-plane). To prove this we have to look more carefully at ∂D_Q and ∂D_J. We may choose D_Q and D_J such that in some neighbourhood of the contact point their boundaries are smooth curves, tangent to one another at the contact point (since locally both J and the branch of Cov_0^Q fixing the contact point are smooth involutions). We first observe that the boundaries of the fundamental domains D_σ and D_{ρ_0} have well-defined endpoints in $\mathbb{R}$, by applying Proposition 2.14 of [**15**]. This states that if ϕ is a conformal map from some domain U onto the unit disc, and γ is a curve in U with endpoints on the boundary of U, then $\phi(\gamma)$ is a curve in the unit disc with endpoints on the unit circle. In order to show that $H = H_4$, we need to prove that each of the endpoints of ∂D_σ coincides with one of the endpoints of ∂D_{ρ_0}. Let N be a sufficiently small neighbourhood of the contact point $1/\sqrt{2}$ in the correspondence plane. Then $N \cap \Omega$ is the disjoint union of two sets N_1 and N_2, as Λ cuts through N. See Figure 3. Let z_n and z_n' be sequences in $\partial D_J \cap N_1$ and $\partial D_Q \cap N_1$, respectively, and let w_n and w_n' be sequences in $\partial D_J \cap N_2$ and $\partial D_Q \cap N_2$, respectively, such that all four sequences have limit $1/\sqrt{2}$. Then, by the proposition just cited, $\lim_{n\to\infty} \phi^{-1}(\zeta_n)$ exists for $\{\zeta_n\}$ any of the four sequences above, and is equal to an endpoint of one of ∂D_σ or ∂D_{ρ_0}. Let $a = \lim_{n\to\infty} \phi^{-1}(z_n)$, $b = \lim_{n\to\infty} \phi^{-1}(z_n')$,

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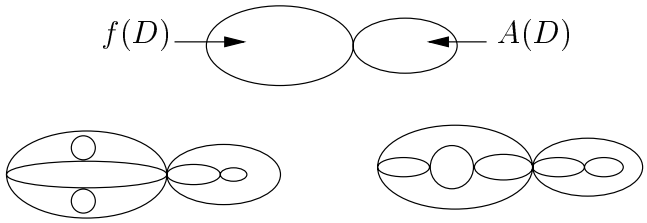


FIGURE 4. The picture on the top shows $\mathcal{F}(D)$. If $1 \in f(D)$, then $\mathcal{F}^2(D)$ looks like the bottom left picture. If $1 \in A(D)$, it looks like the bottom right picture.

$c = \lim_{n\to\infty} \phi^{-1}(w_n)$ and $d = \lim_{n\to\infty} \phi^{-1}(w'_n)$. We need to show that $a = b$ and $c = d$. This follows from Proposition 2.2 of [**15**], which states that if ϕ maps the open set $U \subset \mathbb{C}$ conformally into the unit disc, and we let $C(r)$ denote $U \cap \{|z - c| = r\}$ for a chosen point $c \in \mathbb{C}$, then there exists a decreasing sequence $r_n \to 0$ such that the length of the image curves $\phi(C(r_n))$ tends to zero as n tends to infinity. In our situation we let $\mathcal{C}_n$ be a sequence of nested circles about the contact point $1/\sqrt{2}$, with the r_n and the sets $C(r_n)$ as above. Let W denote the wedge in the neighbourhood N consisting of the closed region in N_1 bounded by ∂D_J and ∂D_Q. Since these boundary curves are both smooth and have a common tangent at the contact point, the intersection of each circle $\mathcal{C}_n$ and W is a closed, connected segment of the circle, which is a subset of $C(r_n)$. The images under ϕ^{-1} of these segments are curve segments s_n in the upper half-plane connecting ∂D_σ and ∂D_{ρ_0}, and getting closer and closer to the real line as $n \to \infty$. By the proposition just quoted, the length of the s_n tend to zero as $n \to \infty$, and therefore $a = b$. See Figure 3. The same argument shows that $c = d$. □

An application of Proposition 6.1 gives the following.

COROLLARY 7.1. *If $q \in \mathcal{M}$ then $\mathcal{F}_q$ is a mating between H_4 and a Chebyshev-like map.*

7.2. *Polynomial-like behaviour on* Λ. From now on we consider $\mathcal{F}(= \mathcal{F}_q)$ with $q \in \mathcal{M}$. The limit sets Λ_+ and Λ_- are the filled Julia sets of the Chebyshev-like maps $\mathcal{F}^{-1} : \mathcal{F}(D) \to D$ and $\mathcal{F} : J(\mathcal{F}(D)) \to J(D)$, respectively, where $D = \hat{\mathbb{C}} - D_J$. These maps have 'free' critical values 1 and $J(1)$, respectively, and clearly Λ is connected if and only if $1 \in \Lambda_+$. Recall that $\mathcal{F}^{-1} : \mathcal{F}(D) \to D$ consists of a pinched quadratic-like restriction (the inverse of the one-to-two correspondence f) and a homeomorphism (the inverse of A). Thus, $\mathcal{F}(D)$ consists of two topological discs whose boundaries touch at the point $A(1/\sqrt{2})$ (Figure 4 shows $\mathcal{F}(D)$ and the two possibilities for $\mathcal{F}^2(D)$). An induction gives the following.

LEMMA 7.2. *$\mathcal{F}^n(D)$ consists of a union of discs with disjoint interiors: any common boundary point of two discs is in the orbit of $1/\sqrt{2}$.*

The inverse $\mathcal{F}^{-1}$ restricted to each disc is either a one-to-one or (pinched) quadratic-like map. Each such quadratic-like restriction has critical value 1, the location of which determines whether or not it has connected filled Julia set. If it has, then by the

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straightening theorem (or Conjecture 6.1), Λ_+ contains a copy of a quadratic filled Julia set. In fact, we show that in this case Λ_+ is the closure of countably many copies of this Julia set.

7.3. *Sequences and addresses.* By Lemma 7.2, every point z in $\Lambda_+ = \bigcap_{n=0}^{\infty} \mathcal{F}^n(D)$ lies in the intersection of a sequence of nested discs D_n^z, where $D_{n+1}^z \subset D_n^z$. Each D_n^z is an image $w_1 \dots w_n(D)$ of the disc D where either $w_i = A$ or

$$w_i = \begin{cases} f & \text{if } 1 \in w_{i+1} \dots w_n(D), \\ \text{a branch of } f & \text{if } 1 \notin w_{i+1} \dots w_n(D). \end{cases}$$

Our aim is to describe the location of each point in Λ_+ in terms of this nested sequence. First we need a coherent way to describe the two distinct branches of the one-to-two correspondence f.

Suppose that there exists an integer n such that $1 \in f^i(D)$ for all $1 \le i \le n-1$ (so $f^{i+1}(D)$ consists of one connected component for all such i), but that $1 \notin f^n(D)$. Then $f^{n+1}(D)$ consists of two connected components. One of these, denoted by C, will contain $1/\sqrt{2}$ on its boundary as this is a fixed point of f. We denote the branch of f sending $f^n(D)$ to C by f_1 and the other by f_2. To extend the definition of f_1 and f_2 we note that the boundary of $f^n(D)$ touches the boundaries of other components of $\mathcal{F}^i(D)$ for $1 \le i \le n$ and on each of these, provided they do not contain the point 1 (when the branches cannot be distinguished), we define f_1 and f_2 so that we have continuity at the contact points. If no such n exists, let $\mathcal{C}$ denote the boundary of D minus the point $1/\sqrt{2}$. Then on $\mathcal{C}$ the map f defines two homeomorphisms, each mapping $\mathcal{C}$ to one of the components of the boundary of $f(D)$ minus the points $1/\sqrt{2}$ and $J(-1/\sqrt{2})$ (the two images of $1/\sqrt{2}$ under f). Denote one homeomorphism by f_1 and the other by f_2. Now define f_1 and f_2 on $A(D)$ so that we have continuity at the contact point $J(-1/\sqrt{2})$. Inductively, define f_1 and f_2 on components of $\mathcal{F}^n(D)$ for any n so as to have continuity at the contact points. Now:

- if $D_n^z = w_1 \dots w_n(D)$, then $D_{n+1}^z = v_1 \dots v_{n+1}(D)$, where v_{n+1} is either A or f and

$$v_i = \begin{cases} w_i & \text{if } w_i = A \text{ or if } w_i \text{ is a particular branch of } f, \\ f \text{ or a particular branch of } f & \text{if } w_i = f, \end{cases}$$

  for $1 \le i < n$;
- if the boundaries of the discs $D_n^z = w_1 \dots w_n(D)$ and $D_{n+1}^z = v_1 \dots v_{n+1}(D)$ touch, then $v_{n+1} = f$; the contact point is in the orbit of $1/\sqrt{2}$.

Definition 7.1. Let $z \in \Lambda_+$. Then an address of z is an infinite sequence $\{a_n\}_{n\in\mathbb{N}}$ in f_1, f_2, f and A given by:

- $a_i = A$ if in $D_i^z = w_1 \dots w_i(D)$ we have $w_i = A$;
- $a_i = f_j$, $j = 1, 2$, if there exists an integer $n > i$ such that in $D_n^z = v_1 \dots v_n(D)$ we have $v_i = f_j$;
- $a_i = f$ otherwise.

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We note that the address of the point $1/\sqrt{2}$ is either $\bar{f}$ (if $1 \in f^i(D)$ for all i) or $\bar{f}_1$. Moreover, the point $J(-1/\sqrt{2})$ is an image of $1/\sqrt{2}$ under *both* f and A and therefore has two addresses (either $\bar{f}$ and $A\bar{f}$, or $\bar{f}_1$ and $A\bar{f}_1$). The address of every point in Λ_+ that does not lie in the orbit of $1/\sqrt{2}$ is unique.

Definition 7.2.

- For a point $z \in \Lambda_+$ with address $a = \{a_n\}$, we say that the entry a_i is undecidable if $a_i = f$.
- The critical address is defined to be the address of the point 1, and we denote it by $c = \{c_n\}$.
- A domicile is a set of points in Λ_+ that share the same address. Thus, a domicile is the intersection of an infinite sequence of nested discs and, therefore, it is simply connected.
- The domicile containing the singular point 1 is called the critical domicile and denoted by K_c.

LEMMA 7.3. *The critical address contains an undecidable entry $c_m = f$ if and only if it is periodic of the form $c = \overline{c_1 \dots c_m}$, with $c_m = f$.*

Proof. Suppose that m is the smallest integer such that c_m is an undecidable entry. Then, by definition, $1 \in c_{m+1} \dots c_n(U_n)$, for all n, where the U_n are suitable images of D. So $c_m c_{m+1} \dots = c_1 c_2 \dots$ and therefore $c = \overline{c_1 \dots c_m}$. The proof in the other direction is obvious. □

Remark 7.1. Suppose that the critical address is periodic of minimal period m. We cannot write $c_1 \dots c_m(D)$ for the disc D_m^1 in $\mathcal{F}^m(D)$ containing the point 1. For example, the critical address might be $c = \overline{Af_1 f_1 Af}$. Then $D_5^1 = Af_1 f Af(D)$, so the third map here is f rather than f_1, which it is in the word $c_1 \dots c_5$. This is because the critical address tells us that the point 1 lies in $Af(D)$ and hence the distinct branches of f are not defined on $Af(D)$. We write $D_m^1 = s_1 \dots s_m(D)$.

The following proposition explains the appearance of copies of the Douady rabbit in Figure 1.

PROPOSITION 7.1. *If the critical address c is periodic of minimal period m, then the map*

$$g_c = (c_1 \dots c_m)^{-1} : c_1 \dots c_m s_1 \dots s_m(D) \to s_1 \dots s_m(D)$$

is polynomial-like of degree 1 if $c_m = A$, quadratic-like if $c_m = f$ and pinched quadratic-like in the special case $c = \bar{f}$. In each case the filled Julia set of g_c is the domicile K_c containing the point 1. If g_c is polynomial-like of degree one, then K_c consists of a single point. If g_c is quadratic-like then K_c is homeomorphic to the filled Julia set of a unique quadratic polynomial q in the Mandelbrot set, and the action of g_c on K_c is conformally conjugate to that of q on its filled Julia set. If g_c is pinched quadratic-like, then the straightening conjecture for pinched polynomial-like maps implies a similar result.

See Figure 5. The reason why it is not possible to consider the map $(s_1 \dots s_m)^{-1} : s_1 \dots s_m(D) \to D$ to get a quadratic-like map as in Proposition 7.1 is the complication described in Remark 7.1.

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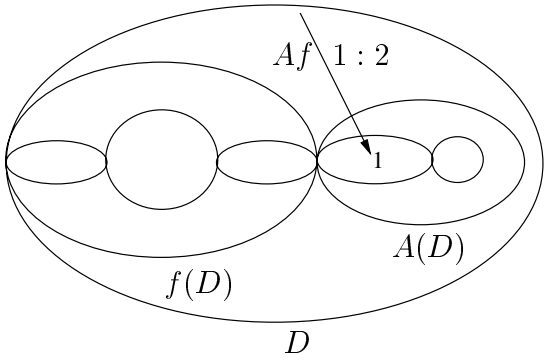


FIGURE 5. The map $(Af)^{-1} : Af(D) \to D$ is quadratic-like.

Proof. First suppose that $c_m = f$. Then, since $1 \in s_1 \dots s_m(D)$, the map $f : s_1 \dots s_m(D) \to f s_1 \dots s_m(D)$ is one-to-two, its image being homeomorphic to a disc. The fact that m is the minimal period of the address c implies that for any $1 \leq i < m$, the map $c_i \dots c_m : s_1 \dots s_m(D) \to c_i \dots c_m s_1 \dots s_m(D)$ is a one-to-one map. Unless the address is equal to $\bar{f}$, a case that needs some extra attention, the image $c_1 \dots c_m s_1 \dots s_m(D)$ is properly contained in the disc $s_1 \dots s_m(D)$, and the map $g_c = (c_1 \dots c_m)^{-1} : c_1 \dots c_m s_1 \dots s_m(D) \to s_1 \dots s_m(D)$ is quadratic-like. The critical value of this map is the point 1. However, 1 has critical address c and thus lies in the filled Julia set of the quadratic-like map g_c. The result now follows from the straightening theorem.

The same argument applies if $c_m = A$, except that in this case the map g_c will be one-to-one, and it is well known that the Julia set of such a map consists of a single point. If $c = \bar{f}$, then the map $f^{-1} : f(D) \to D$ is a *pinched* quadratic-like map, as the boundary of D touches that of $f(D)$. □

Definition 7.3. We say that the critical address c is quadratic-like if $c = \overline{c_1 \dots c_m}$ is periodic with $c_m = f$. (This includes the special case $c = \bar{f}$.)

PROPOSITION 7.2. *Suppose that the critical address is periodic. Then any domicile K that is not equal to K_c or one of its images consists of a single point.*

Proof. The proof is rather technical, and we only give an outline. Details can be found in [**10**]. Suppose that K is given by the address $t = \{t_n\}$.

Firstly, assume that there exists an integer N such that for all $n \geq N$, $t_n = f_1$. Then K is just an image of the domicile K_0 given by the address $\bar{f}_1$, and it suffices to prove the result for K_0. Moreover, the critical address cannot be $\bar{f}$ and, therefore, there exists an integer n such that $1 \in f^i(D)$ for all $1 \leq i \leq n-1$, but $1 \notin f^n(D)$. So f_1 is defined on $U = f^n(D)$ and sends U into itself, fixing $1/\sqrt{2}$. The first and second derivatives of f_1 at $1/\sqrt{2}$ are equal to 1 and $\alpha \neq 0$, respectively, so f_1 can be conjugated to a map with Taylor expansion $z - z^2 + o(z^3)$ about zero, where zero corresponds to $1/\sqrt{2}$. One can prove the assertion using standard results about such maps (see [**1**]). If the critical address does not satisfy the condition above, then one can apply the *method of divergence* ([**13**, Theorem 2.16]; or [**4**, Proposition 5.5]) as follows: the domicile K_0 is the intersection

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of an infinite sequence of nested discs D_n. It is possible to choose a subsequence $\{D_{n_j}\}$ such that for each j,

$$A_{n_j} = D_{n_j} - \bar{D}_{n_{j+1}}$$

is an open bounded annulus. Moreover, if one chooses the D_{n_j} carefully, one can ensure that each A_n is a d-fold cover of one of a finite number of annuli, where d is at most 2. This means that the moduli of the annuli A_n are bounded below and, therefore, that the sum of their moduli diverges. The annuli A_n are contained in the annulus $A = D - K_0$ and wind around the bounded component of the complement of A, which is equal to K_0. Thus, K_0 consists of a single point. □

The following corollary is immediate.

COROLLARY 7.2. *If the critical address is periodic of period m with $t_m = A$, then every domicile in Λ_+ consists of a single point.*

It follows from Proposition 7.2 that if $\mathcal{F}$ has a quadratic-like critical address then $\Lambda_+(\mathcal{F})$ is the closure of the union of countably many copies of the filled Julia set of some quadratic polynomial with connected Julia set. We expect that for each such quadratic and quadratic-like sequence c there exists $q \in \mathcal{M}$ such that $\mathcal{F}_q$ has critical address c and such that $\Lambda_+(\mathcal{F}_q)$ is made up of copies of the filled Julia set of the quadratic polynomial. In the following sections we establish results that go some way towards proving this conjecture.

8. *Real parameter values*

For reasons of space we omit all proofs in this section: the interested reader is invited to construct his or her own proofs or to refer to [**10**] where proofs of analogous results are given in detail.

LEMMA 8.1. *We have $\mathcal{M} \cap \mathbb{R} = [\sqrt{2}, 3\sqrt{2}/2]$.*

LEMMA 8.2. *Let $z \in D$ and $q \in \mathbb{R}$. Then:*

(i) *the images of z under $f = f_q$ are real if and only if $z \in I = [1/\sqrt{2}, 1]$; if $z \notin I$, then its images under f lie on either side of the real axis;*

(ii) *$A_q(z)$ is real if and only if z is real.*

Thus, the interval $I = [1/\sqrt{2}, 1]$ maps under f to the intervals $I_1 = [1/\sqrt{2}, f(1)]$ and $I_2 = [f(1), J(-1/\sqrt{2})]$. As before, we denote by f_1 the branch of f sending I to I_1 and by f_2 the branch of f sending I to I_2. Note that f_1 is orientation preserving and that f_2 is orientation reversing on I.

The ideas below are motivated by those of Milnor and Thurston in [**14**]. In their analysis of unimodal maps of the interval, they use the *itinerary* of the critical point to characterize the map. The analogue in our situation is the critical address. As is the case with the itineraries of unimodal maps, only certain critical addresses can occur when the correspondence is real.

Definition 8.1. We define an order on finite and infinite words in A, f_1, f_2 and f as follows. Firstly,

$$f_1 < f < f_2 < A.$$

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Now let $v = \{v_n\}$ and $w = \{w_n\}$ be two distinct words. Suppose that i is the smallest integer such that $v_i \neq w_i$. Then $v < w$ if either:

- $v_i < w_i$ and the number of occurrences of f_2 in $v_1 \dots v_{i-1}$ is even; or
- $v_i > w_i$ and the number of occurrences of f_2 in $v_1 \dots v_{i-1}$ is odd.

For any word $w = \{w_n\}$ in A, f_1, f_2 and f define the function $occ_w(i, k)$ for $i < k$ as follows:

$$occ_w(i,k) = \begin{cases} 0 & \text{if the number of occurrences of } f_2 \text{ in } w_i \dots w_k \text{ is even,} \\ 1 & \text{if the number of occurrences of } f_2 \text{ in } w_i \dots w_k \text{ is odd.} \end{cases}$$

Definition 8.2. We say that an infinite sequence $t = \{t_n\}$ in f_1, f_2, f and A is admissible if for all subsequences $t' = t_i t_{i+1} \dots$ for $i > 1$ we have

$$t' \leq t.$$

We note that the ordering of addresses reflects the order of points on the real line. Suppose that $x \in f_1(\mathbb{R}) \cap \mathbb{R}$, $y \in f_2(\mathbb{R}) \cap \mathbb{R}$ and $z \in A(\mathbb{R})$. then clearly $x < y < z$. This, together with the fact that f_1 and f_2 are orientation preserving and reversing, respectively, gives that if $v_1 \dots v_n$ and $w_1 \dots w_n$ are words in f_1, f_2, f and A, then

$$v_1 \dots v_n(x) < w_1 \dots w_n(y)$$

if and only if either:

- $v_1 \dots v_n < w_1 \dots w_n$; or
- $v_1 \dots v_n = w_1 \dots w_n$, $x < y$ and $occ(1, n) = 0$; or
- $v_1 \dots v_n = w_1 \dots w_n$, $x > y$ and $occ(1, n) = 1$.

LEMMA 8.3. *Every periodic admissible sequence* $c = \overline{c_1 \dots c_m}$ *is of the form* $c = \overline{A \dots A c_i \dots c_m}$ *for some* $i \geq 1$, *where each* c_i *is* f *or one of its branches.*

LEMMA 8.4. *If the parameter* q *is real then only admissible sequences can occur as the critical address.*

In order to emphasize dependence on parameters, we write $\mathcal{F}_q$ for the correspondence $\mathcal{F}$ and c_q for the critical address of $\mathcal{F}_q$.

THEOREM 8.1. *Given any admissible sequence* s, *there exists a parameter* $q \in \mathcal{M} \cap \mathbb{R}$ *such that the correspondence* $\mathcal{F}_q$ *has critical address* $c_q = s$. *If* $s = c_q = \overline{c_1 \dots c_m}$ *is quadratic-like and if* $m > 1$, *then the quadratic-like map* $g_q = (c_1 \dots c_m)^{-1}$ *is hybrid equivalent to a unique real quadratic polynomial* p_q *with* q *in the Mandelbrot set. Indeed if* c_q *is quadratic-like, of period* $m > 1$, *then there exists a whole interval* $I_s \subset \mathcal{M} \cap \mathbb{R}$ *such that for all* $q \in I_s$ *the correspondence* $\mathcal{F}_q$ *has critical address* $c_q = s$. *At the endpoints* $a < b$ *of* I_s *the quadratic-like maps* g_a *and* g_b *are hybrid equivalent to* $q_{1/4} : z \to z^2 + 1/4$ *and* $q_{-2} : z \to z^2 - 2$, *respectively. As* q *passes along* I_s, *the quadratic maps* p_q *pass through* $z \to z^2 + c$ *for all* c *in the intersection* $[-2, \frac{1}{4}]$ *of the Mandelbrot set with the real line.* □

Remark 8.1. The restriction $m > 1$ ensures that we have a genuine quadratic-like map: $m = 1$ corresponds to the special case $s = \bar{f}$, when we have a *pinched* quadratic-like map.

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9. *Models for* Λ

So far, we have attempted to analyse the dynamics of a correspondence $\mathcal{F} \in \mathcal{M}$ on its limit set Λ in terms of the branches of $\mathcal{F}$. However, to obtain a combinatorial description of Λ, it is also helpful to approach $\partial\Lambda$ from the *outside*, that is to say from Ω. By Theorem 7.1, we know that the action of $\mathcal{F}$ on Ω is conjugate to that of the group H_4 on the upper half-plane $\mathbb{H}$. The boundary $\partial\Lambda$ is locally connected if and only if this conjugacy extends to a continuous surjection $\mathbb{R} \cup \infty \to \partial\Lambda$ (by a well-known result of Carathéodory), and it is then possible to describe $\partial\Lambda$ as a quotient of the limit set $\mathbb{R} \cup \infty$ of the group.

THEOREM 9.1. *Suppose that $\mathcal{F} \in \mathcal{M}$ has quadratic-like critical address $c = \overline{c_1 \dots c_m}$ with $m > 1$ and that the quadratic-like map $g = (c_1 \dots c_m)^{-1}$ is hybrid-equivalent to $z \to z^2$. Then the map $\phi : \mathbb{H} \to \Omega$ extends to a continuous map from the boundary $\hat{\mathbb{R}}$ of $\mathbb{H}$ to the boundary of Ω, the quotient $\hat{\mathbb{R}}/\phi$ is homeomorphic to the boundary of Λ, and the action of the quotient of the correspondence $\mathcal{H}$ on $\hat{\mathbb{R}}/\phi$ is conjugate to that of $\mathcal{F}$ on $\partial\Lambda$.*

COROLLARY 9.1. *If $\mathcal{F} \in \mathcal{M}$ has quadratic-like critical address, such that the corresponding quadratic-like map is hybrid-equivalent to $z \to z^2$, then Λ is locally connected.*

Remark 9.1. The statements of Theorem 9.1 and Corollary 9.1 remain true if $z \to z^2$ is replaced by any other hyperbolic quadratic polynomial; we have restricted ourselves to the special case to simplify our exposition. We conjecture that the statements also remain true if $m = 1$ so that $c = \bar{f}$ and $g = f^{-1}$ is 'pinched-equivalent' to $z \to z^2$, or indeed to any hyperbolic quadratic.

Proof of Theorem 9.1. We start by extending ϕ to a map from $\mathbb{R}^+$ to $\partial\Lambda_+$.

By Theorem 7.1, there exists a Riemann map $\phi : \mathbb{H} \to \Omega$ conjugating the correspondence $\mathcal{H}$ to $\mathcal{F}$. Since it carries σ to J, and since $(JA)^2$ is the identity, we conclude that the map μ_1 is carried to the branch A of $\mathcal{F}$, and that the two maps μ_0 and μ_2 are carried to the two branches of f.

Now let D be the complement in $\hat{\mathbb{C}}$ of the transversal D_J of J. By the proof of Theorem 7.1, $\phi^{-1}(D - \Lambda_+)$ is a fundamental domain D_σ for σ. Moreover, $\phi^{-1}(\partial(D - \Lambda_+)) = \partial D_\sigma$ has two well-defined endpoints in $\mathbb{R}^+$ and we may take them to be 0 and ∞.

Suppose that $z \in \partial\Lambda_+$. Then z belongs to a particular domicile K_z given by a sequence of nested discs $K_z = \bigcap_{n=0}^{\infty} D_n^z$. The boundary of any disc D_n^z contains 2^k points in the orbit of $1/\sqrt{2}$, for some k . Moreover, these points lie in Λ_+ (since $1/\sqrt{2}$ does), and ∂D_n^z touches Λ_+ only at these points. Thus, since Λ_+ is connected, $D_n^z \cap \Omega$ consists of 2^k connected components. Since any connected component of $D_{n+1}^z \cap \Omega$ is contained in one connected component of $D_n^z \cap \Omega$, we get nested sequences of such components. The boundary of each such component contains an arc in ∂D_n^z that connects two points in the orbit of $1/\sqrt{2}$. In other words, it contains an image of $\mathcal{C} = \partial D - \{1/\sqrt{2}\}$. See Figure 6.

Firstly, suppose that $K_z = \{z\}$. Then any nested sequence S_n of components of $D_n^z \cap \Omega$ accumulates at $\partial K_z = \{z\}$. For each n, the boundary piece of S_n that is an image of $\mathcal{C}$ gets carried by the conjugacy ϕ^{-1} to an image of ∂D_σ under a word of length n in μ_0, μ_1 and μ_2. The interior of a component S_n must therefore map to an image of D_σ.

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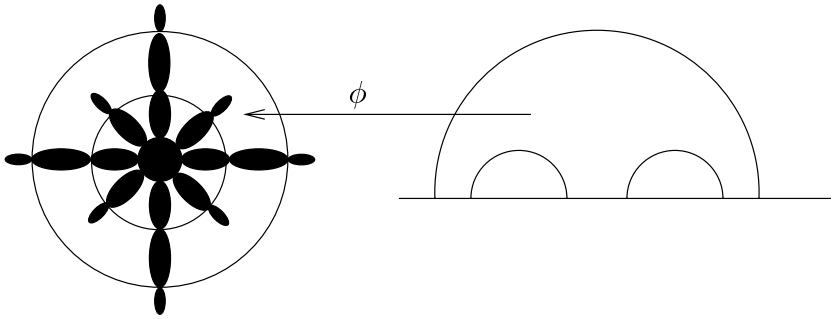


FIGURE 6. The black disc in the centre of figure on the left represents K_z. Surrounding it are the boundary circles of two discs from the nested sequence D_n^z. The other black parts show parts of Λ_+ that cut through the D_n^z. They meet these circles in points in the orbit of $1/\sqrt{2}$. We see the beginning of the nested sequences of the connected components of $D_n^1 \cap \Lambda_+$. Each such component maps under ϕ^{-1} to an image of D_σ.

Thus, the nested sequence S_n maps under ϕ^{-1} to a nested sequence of images Σ_n of D_σ. Such a sequence accumulates at exactly one point x on $\mathbb{R}^+$, by the standard properties of the group. For any point x arising in such a way, we define $\phi(x) = z$.

Now suppose that the domicile of z is the critical domicile K_c. Thus, we get a quadratic-like map $g = (c_1 \dots c_m)^{-1} : D_{2m}^1 \to D_m^1$, and

$$K_c = \bigcap_{n=0}^{\infty} (c_1 \dots c_m)^n (D_m^1) = \bigcap_{n=0}^{\infty} D_{nm}^1.$$

As before, D_m^1 contains 2^k points in the orbit of $1/\sqrt{2}$ on its boundary. Thus, since $D_{2m}^1 = c_1 \dots c_m(D_m^1)$, and since $c_1 \dots c_m$ is a one-to-two map, any disc D_{nm}^1 contains 2^{n+k-1} points in the orbit of $1/\sqrt{2}$ on its boundary, and therefore $D_{nm}^1 \cap \Omega$ consists of precisely 2^{n+k-1} connected components. Again, we get infinitely many nested sequences of such components, and each sequence accumulates on ∂K_c. See Figure 6. Moreover, each such sequence gives rise to a unique point x on $\mathbb{R}^+$, by the same reasoning as above.

Now $g : D_{2m}^1 \to D_m^1$ is hybrid-equivalent to $z \to z^2$, which is expanding on and outside its Julia set. Therefore, g is expanding on and outside ∂K_c, and we can conclude that any nested sequence of connected components of $D_{nm}^1 \cap \Omega$ accumulates at precisely one point in ∂K_c. (Note that it would have been sufficient to require that g is hybrid-equivalent to a *hyperbolic* quadratic in order to get this result.)

We also note that every point on ∂K_c is the accumulation point of at least one and at most two such sequences. Thus, our point $z \in \partial K_c$ gives rise to at most two points x_1 and x_2 in $\mathbb{R}^+$ and we define $\phi(x_1) = \phi(x_2) = z$.

We repeat this procedure for points in the boundary of any image of K_c: thus we cover all of $\partial \Lambda_+$.

One can check that this extension of ϕ is defined everywhere on $\mathbb{R}^+$ and by construction it is surjective and a continuous extension of ϕ. It follows that $\mathbb{R}^+/\phi$ is homeomorphic to $\partial \Lambda_+$ and since ϕ is the extension of a conjugacy, the quotient of $\mathcal{H}$ on $\mathbb{R}^+$ is conjugate to the forward action of $\mathcal{F}$ on $\partial \Lambda_+$.

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To extend ϕ to $\mathbb{R}^-$, we recall that the involution J conjugates the action of $\mathcal{F}$ on Λ_+ to that of $\mathcal{F}^{-1}$ on Λ_-. Similarly the involution $j : x \to -1/x$ conjugates the actions of μ_0, μ_1 and μ_2 on $\mathbb{R}^+$ to those of μ_2^{-1}, μ_1^{-1} and μ_0^{-1} (respectively) on $\mathbb{R}^- = \{x : -\infty \le x \le 0\}$. In other words it conjugates the action of $\mathcal{H}$ on $\mathbb{R}^+$ to that of $\mathcal{H}^{-1}$ on $\mathbb{R}^-$.

We define $\phi(x) = J\phi j(x)$ for all $x \in \mathbb{R}^-$. This ensures that the action of $\mathcal{H}^{-1}$ on $\mathbb{R}^-/\phi$ is conjugate to the action of $\mathcal{F}^{-1}$ on Λ_-.

One can check that the remaining branches of $\mathcal{H}$ are conjugate to the remaining branches of $\mathcal{F}$. □

Definition 9.1. Let $\sim$ be the equivalence relation on $\hat{\mathbb{R}}$ defined by $x \sim y$ if and only if $\phi(x) = \phi(y)$.

Thus $\hat{\mathbb{R}}/\sim$ is homeomorphic to $\partial\Lambda$ and we can construct a topological model of Λ, simply by glueing together points on $\hat{\mathbb{R}}$ which are identified by $\sim$. Our next task is to make the definition of $\sim$ as explicit as possible. This definition depends on and is completely determined by the critical address.

It is helpful to use an appropriate notation for the points of $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. Recall that the group H_4 acts on $\mathbb{H} \cup \hat{\mathbb{R}}$ via the grand orbits of the three-to-three correspondence $\mathcal{H}$ defined by

$$(w - \mu_0(z))(w - \mu_1(z))(w - \mu_2(z)) = 0,$$

where

$$\mu_0 = \sigma\rho^3 : z \to \frac{z}{\sqrt{2}z + 1}, \quad \mu_1 = \sigma\rho^2 : z \to \frac{\sqrt{2}z + 1}{z + \sqrt{2}} \quad \text{and} \quad \mu_2 = \sigma\rho : z \to z + \sqrt{2},$$

so $\{\mu_0, \mu_1, \mu_2\}$ is a generating set of H_4.

By Proposition 2.1 the action of $\mathcal{H}$ on $\hat{\mathbb{R}}_+$ is conjugate to that of the inverse of the shift map on the unit interval with points in their base three expansion. From now on we represent each point in $\hat{\mathbb{R}}_+$ by its infinite word in 0, 1 and 2, omitting the ternary point to indicate that we are working in $\hat{\mathbb{R}}_+$ rather than I.

Note that the points 0 and ∞ have expansions $\bar{0}$ and $\bar{2}$, respectively, and that a point lies in the orbit of 0 or ∞ under the group if and only if its expansion ends in $\bar{0}$ or $\bar{2}$.

LEMMA 9.1. *For any quadratic-like critical address $c = \overline{c_1 \dots c_m}$ with $m > 1$ we have:*

(i) $0 = \bar{0} \sim \bar{2} = \infty$ *and* $\phi(\bar{0}) = \phi(\bar{2}) = 1/\sqrt{2}$;

(ii) $\mu_0(\infty) = 0\bar{2} \sim 2\bar{0} = \mu_2(0)$ *and* $\phi(0\bar{2}) = \phi(2\bar{0}) = J(-1/\sqrt{2})$;

(iii) *any point in $\mathbb{R}^+$ in the orbit of $\bar{0}$ (i.e. with base three expansion ending in $\bar{0}$) is identified to a point in the orbit of $\bar{2}$ (with base three expansion ending in $\bar{2}$), and if x and y are two such points then $\mu_1(x) \sim \mu_1(y)$, and for $i = 0, 2$ we have either $\mu_i(x) = \mu_i(y)$ or $\mu_i(x) = \mu_{2-i}(y)$.*

Proof. Recall that the point $z = 1/\sqrt{2}$ in the correspondence plane has address $\bar{f}_1$. For any n, there are two connected components of $D_n^z \cap \Omega$ which have $1/\sqrt{2}$ on its boundary, thus we get two sequences of connected components accumulating at $1/\sqrt{2}$. These map via ϕ^{-1} to two distinct sequences $\{\Sigma_n^1\}$ and $\{\Sigma_n^2\}$ of nested images of D_σ.

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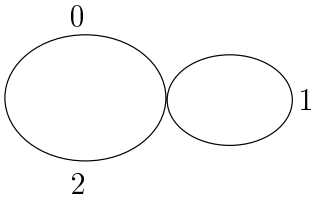


FIGURE 7. For any critical address $\mathcal{F}^1(D)$ looks like this. We can obtain this shape from $\mathbb{R}^+$ by glueing 0 to ∞ and $1/\sqrt{2}$ to $\sqrt{2}$.

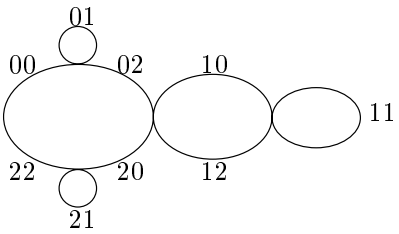


FIGURE 8. $\mathcal{F}^2(D)$ for the critical address $\bar{f}$.

Since $1/\sqrt{2}$ is also contained in ∂D, the boundary of each Σ_n^i must contain one of the endpoints, $\bar{0}$ or $\bar{2}$, of D_σ. It follows that $\Sigma_n^1 = \mu_0^n(D_\sigma)$ and $\Sigma_n^2 = \mu_2^n(D_\sigma)$. Thus, the first sequence accumulates at $\bar{0}$ and the second at $\bar{2}$. The rest follows from the construction of ϕ, and the fact that the two branches of f get carried by ϕ^{-1} to the maps μ_0 and μ_2, while A gets carried to μ_1. □

We see that on points in the orbit of $\bar{0}$ and $\bar{2}$ under μ_0, μ_1 and μ_2, we can define $\sim$ inductively, starting with $\bar{0} \sim \bar{2}$. The three examples below show how this definition varies as the critical address varies. For each critical address we give pictures of $\mathcal{F}^n(D)$ for $n = 1, 2, 3$. Pieces of the boundary of $\mathcal{F}^n(D)$ are labelled by words $i_1 \dots i_n$ in 0, 1 and 2 if they map under ϕ^{-1} to the arcs $\mu_{i_1} \dots \mu_{i_n}(\partial D_\sigma)$.

Example 1. Critical address $\bar{f}$ (we proceed on the assumption that the conjecture in Remark 9.1 is true). Figures 7, 8 and 9 give pictures of $\mathcal{F}^n(D)$ for $n = 1, 2, 3$. It quickly becomes apparent that $\sim$ has to be the closure of the relation defined by:

- $\bar{0} \sim \bar{2}$;
- $0\bar{2} \sim 2\bar{0}$;
- if $x \sim y$ and $x, y \neq \bar{0}, \bar{2}$, then $wx \sim wy$, for $w = 0, 1, 2$.

Moreover, we see that the two inverse branches of the pinched quadratic-like map f^{-1} on its Julia set correspond to the two maps μ_0 and μ_2. The set corresponding to the Julia set of the quadratic-like map is the set T, homeomorphic to a circle, that is obtained by cutting out from $\mathbb{R}^+$ open intervals of the form $(w_1 \dots w_n 2\bar{0}, w_1 \dots w_n 0\bar{2})$ which have $w_i \in \{0, 2\}$ and identifying their endpoints. It is apparent that T consists precisely of projections of points in $\mathbb{R}^+$ that have sequences in 0 and 2 only. The point $\bar{0} \sim \bar{2}$ corresponds to the fixed point $1/\sqrt{2}$ of f, while the point $2\bar{0} \sim 0\bar{2}$ corresponds to the image $f(1/\sqrt{2}) = J(-1/\sqrt{2})$ of $1/\sqrt{2}$.

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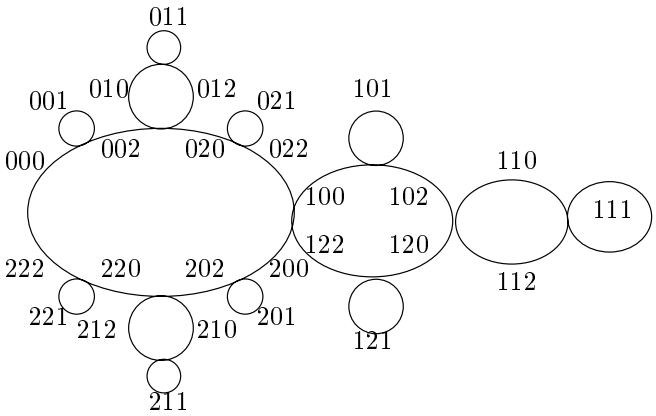


FIGURE 9. $\mathcal{F}^3(D)$ for the critical address $\bar{f}$.

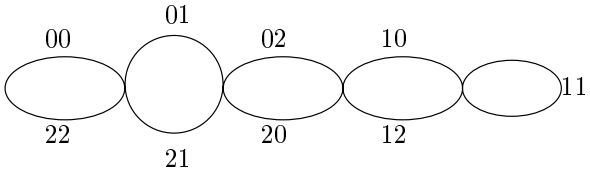


FIGURE 10. $\mathcal{F}^2(D)$ for the critical address $\overline{Af}$.

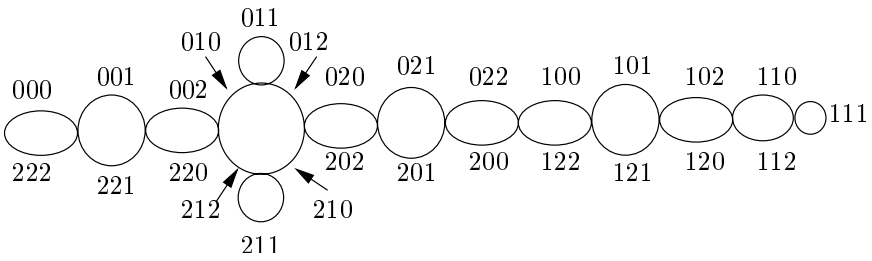


FIGURE 11. $\mathcal{F}^3(D)$ for the critical address $\overline{Af}$.

Example 2. Critical address $\overline{Af}$. Figures 10 and 11 show $\mathcal{F}^n(D)$ for $n = 2, 3$, together with the labelling coming from the group (the picture for $n = 1$ is the same as that in Figure 7). Here the rule for identifying points in $\mathbb{R}^+$ cannot be spotted as easily.

We note that the two inverse branches of the quadratic-like map $(Af)^{-1}$ correspond to the two maps $\mu_1\mu_0$ and $\mu_1\mu_2$.

Once again, $\mathbb{R}^+/\sim$ contains a set T homeomorphic to a circle, this time consisting of projections of points in $\mathbb{R}^+$ that have infinite words in the pairs 10 and 12 only. These are identified according to the same pattern as in Example 1. The set T corresponds to the Julia set of the quadratic-like map $(Af)^{-1}$ in Λ_+.

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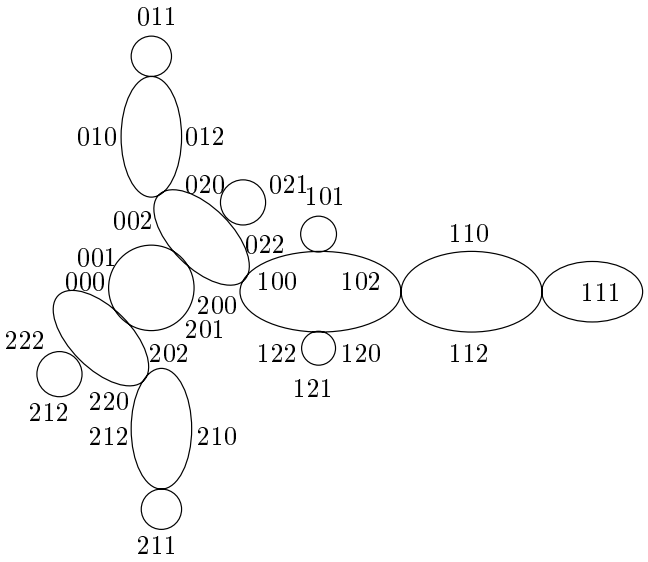


FIGURE 12. $\mathcal{F}^3(D)$ for the critical address $\overline{f_1Af}$.

We note that the point in T given by $\overline{10} \sim \overline{12}$ is a fixed point of $\mu_1\mu_0$ and $\mu_1\mu_2$. It corresponds to the repelling fixed point of the quadratic-like map $(Af)^{-1}$.

Example 3. Critical address f_1Af. The pictures for $\mathcal{F}(D)$ and $\mathcal{F}^2(D)$ are the same as those in Example 1. $\mathcal{F}^3(D)$ is shown in Figure 12. From the picture it is clear that the maps corresponding to the inverse branches of the quadratic-like map $(f_1Af)^{-1}$ are $\mu_0\mu_1\mu_0$ and $\mu_0\mu_1\mu_2$. As in Example 2, the set T corresponding to the filled Julia set of $(f_1Af(D))^{-1}$ consists of the projections of points in $\mathbb{R}^+$ that have infinite words in 012 and 010 only. The identification of these follow the same pattern as in the previous examples.

These examples suggest the following approach.

Definition 9.2. Let T be the subset of $\mathbb{R}^+/\sim$ that corresponds (via ϕ) to $\partial K_c \subset \partial\Lambda_+$.

LEMMA 9.2. *There exist two words $\mathcal{A} = a_1 \dots a_m$ and $\mathcal{B} = b_1 \dots b_m$ in 0, 1 and 2 such that the set T consists of projections of those points in $\mathbb{R}^+$ that have base three expansion in $\mathcal{A}$ and $\mathcal{B}$ only. In $\mathbb{R}^+/\sim$ the projections of these points are identified in the familiar way: $\bar{\mathcal{A}} \sim \bar{\mathcal{B}}$, $\mathcal{B}\bar{\mathcal{A}} \sim \mathcal{A}\bar{\mathcal{B}}$ and $W\mathcal{B}\bar{\mathcal{A}} \sim W\mathcal{A}\bar{\mathcal{B}}$, where W is a word in $\mathcal{A}$ and $\mathcal{B}$. The maps $\mu_{a_1} \dots \mu_{a_m}$ and $\mu_{b_1} \dots \mu_{b_m}$ correspond via ϕ to the inverse branches of the quadratic-like map $g = (c_1 \dots c_m)^{-1}$.*

Proof. Recall that the map $g : D^1_{2m} \to D^1_m$ is hybrid-equivalent to $z \to z^2$.

Since the two branches of f are well defined on $D \setminus \Lambda_+$, the two branches g_1^{-1} and g_2^{-1} of g^{-1} are also well defined there and correspond via ϕ^{-1} to two well-defined group elements A and B. These are words of length m in μ_0, μ_1 and μ_2. Let $\mathcal{A}$ and $\mathcal{B}$ denote the corresponding words in 0, 1 and 2. Without loss of generality we may suppose that $\mathcal{A} < \mathcal{B}$ (with the usual ordering of base three expansions).

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From the hybrid equivalence of g to $z \to z^2$, we know that for any $y \in D_m^1 \setminus \Lambda_+$ the set ∂K_c consists of points of the form $\lim_{n\to\infty} t_1 \dots t_n(y)$, where each t_i is either g_1^{-1} or g_2^{-1}. It follows that every point in $\hat{\mathbb{R}}_+$ which is represented by an infinite word in $\mathcal{A}$ and $\mathcal{B}$ is mapped by ϕ to a point of ∂K_c, and every point of ∂K_c is the image under ϕ of at least one such point of $\hat{\mathbb{R}}_+$. The set of points of $\hat{\mathbb{R}}_+$ that are represented by infinite words in $\mathcal{A}$ and $\mathcal{B}$ is a Cantor set, with least point $\bar{\mathcal{A}}$ and greatest point $\bar{\mathcal{B}}$. Since ∂K_c is a circle, we deduce that ϕ identifies $\bar{\mathcal{A}}$ with $\bar{\mathcal{B}}$, that for each of the intervals making up the complement of the Cantor set the map ϕ identifies the endpoints, and that there are no further identifications. □

Lemma 9.2 makes it possible to formalize the definition of $\sim$.

Definition 9.3. We define the critical interval $\mathcal{C}$ in $\mathbb{R}^+$ to be the closed interval with endpoints given by the words $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$.

PROPOSITION 9.1. *The equivalence relation $\sim$ on $\mathbb{R}^+$ is the closure of the following relation on points in the orbit of $\bar{0}$ and $\bar{2}$:*

- $\bar{0} \sim \bar{2}$;
- *if $x \sim y$ and if the interval $[x, y]$ contains the critical interval (or is equal to it), then $wx \sim \tilde{w}y$, for $w \in \{0, 1, 2\}$, where $\tilde{w} = 2 - w$;*
- *$wx \sim wy$ otherwise.*

Proof. We have the following facts.

(1) $\sim$ respects the action of H_4, so if $x \sim y$, then $1x \sim 1y$ and $wx \sim vy$, where $w \in \{0, 2\}$ and $v \in \{w, \tilde{w}\}$, where $\tilde{w} = 2 - w$.

(2) If $x \sim y$ and $z \sim w$, then either $[x, y] \cap [z, w] = \emptyset$, or $[x, y] \subseteq [z, w]$, or $[z, w] \subseteq [x, y]$. This follows from the construction of ϕ.

(3) Since $1 \in K_c$, the map f is $1 : 2$ on K_c. Moreover, the two branches of f correspond to μ_0 and μ_2, and therefore the set $S = \phi^{-1}(f(K_c))$ in $\mathbb{R}^+/\sim$ consists of projections of precisely those points in $\mathbb{R}^+$ whose base three expansions arise from those of points in T with a 0 or 2 inserted at the beginning.

We now construct $\sim$ inductively on points in the orbit of $\bar{0}$ and $\bar{2}$. Firstly, note that all points in $\mathbb{R}^+$ with infinite words only in $\mathcal{A}$ and $\mathcal{B}$ lie in the critical interval. Suppose that x and y are images of $\bar{0}$ and $\bar{2}$, respectively, under words of length n in μ_0, μ_1 and μ_2, and that $x \sim y$. If the interval $[x, y]$ contains the critical interval, then the intervals $[0x, 0y]$ and $[2x, 2y]$ contain the images $[0\bar{\mathcal{A}}, 0\bar{\mathcal{B}}]$ and $[2\bar{\mathcal{A}}, 2\bar{\mathcal{B}}]$, respectively, of the critical interval. By the third condition above, projections of points with words of the form $0t$ or $2t$, where t is an infinite word in $\mathcal{A}$ and $\mathcal{B}$ lie on a set S homeomorphic to a circle in $\mathbb{R}^+/\sim$. Suppose that $\mathcal{A} < \mathcal{B}$. Then $0\bar{\mathcal{A}}$ defines the smallest of the points with such words, and $2\bar{\mathcal{B}}$ defines the greatest. Thus, if these points are to be identified to give a set homeomorphic to a circle, we need $0\bar{\mathcal{A}} \sim 2\bar{\mathcal{B}}$. We have

$$0x < 0\bar{\mathcal{A}} < 0y < 2x < 2\bar{\mathcal{B}} < 2y.$$

So in order not to violate condition (2) above we have to define $0x \sim 2y$ and $2x \sim 0y$. Similarly, one can show that if $[x, y]$ does not contain the critical interval, then we need $0x \sim 0y$ and $2x \sim 2y$.

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To show that $\sim$ does not contain any further identifications, we recall the construction of ϕ: whenever $x \sim y$, i.e. whenever $\phi(x) = \phi(y)$ for a pair x, y, we had that x and y were each specified by a sequence of nested images Σ_n^x and Σ_n^y of D_σ. The boundary of each Σ_n^x and Σ_m^y has endpoints $t_n^x < s_n^x$ and $t_n^y < s_n^y$, respectively, in $\mathbb{R}^+$, which are in the orbit of $\bar{0}$ and $\bar{2}$. Thus, x and y are specified by a nested sequence of intervals $[t_n^x, t_n^x]$ and $[t_n^y, s_n^y]$ in $\mathbb{R}^+$, respectively. One can easily check that each of the t_n^x must be identified by $\sim$ to s_n^y. So the identification $x \sim y$ arises from taking the closure of $\sim$ defined in points in the orbit of $\bar{0}$ and $\bar{2}$. □

Our final task is to develop an algorithm that determines the words $\mathcal{A}$ and $\mathcal{B}$ in 0, 1 and 2, given a specific critical address $c = \overline{c_1 \dots c_m}$. For an arbitrary critical address this is extremely complicated to write down. However, we know the following.

- Since $c_m = f$, and since the branches of f get carried by ϕ^{-1} to one of μ_0 and μ_2 we have $a_m = 0$ and $b_m = 2$. Similarly, whenever c_i is a branch of f, we need each of a_i and b_i to be 0 or 2 (the difficulty is to choose which one).
- Since the map A gets carried to μ_1, we need that $a_i = b_i = 1$ whenever $c_i = A$.

For *real* correspondences, i.e. correspondences in $\mathcal{M} \cap \mathbb{R}$, the algorithm is quite easy, and we restrict ourselves to this case. These correspondences were explored in the previous section and we recall that the critical address c of such a correspondence must be *admissible*, i.e. it must not contain a subsequence which is greater than the original sequence c.

PROPOSITION 9.2. *For an admissible quadratic-like critical address c, the words $\mathcal{A}$ and $\mathcal{B}$ in* 0, 1 *and* 2 *are defined as follows.*

Let $a_m = 0$ and $b_m = 2$. Define $a_i = b_i = 1$ whenever $c_i = A$. Otherwise, define:

- *$a_i = 0$ and $b_i = 2$ if either $c_i = f_1$ and $a_{i+1} \dots a_m < b_{i+1} \dots b_m$, or if $c_i = f_2$ and $a_{i+1} \dots a_m > b_{i+1} \dots b_m$;*
- *otherwise, define $a_i = 2$ and $b_i = 0$.*

Proof. Recall that $\mathcal{A}$ and $\mathcal{B}$ give the words A and B which are conjugate to the two inverse branches of the quadratic-like map $g : D_{2m}^1 \to D_m^1$. Let $k < m$ be the greatest integer such that $c_k \neq A$. So the entries a_k and b_k each are one of 0 or 2. We can label parts of the boundary of $c_k \dots c_m(D_m^1)$ by words in 0, 1 and 2 as in the three examples above: a boundary piece is labelled by $i_1 \dots i_{2m-k+1}$ if its image under ϕ^{-1} is of the form $\mu_{i_1} \dots \mu_{i_{2m-k+1}}(\partial D_\sigma)$. To decide which digit to chose for a_k and b_k, we must determine this labelling of the boundary of $c_k \dots c_m(D_m^1)$. Now the disc $c_{k+1} \dots c_m(D_m^1)$ is a piece of $\mathcal{F}^{2m-k}(D)$. Since $c \in \mathbb{R}$, the piece D_{2m-k}^1 intersects the real line. Moreover, by Lemma 8.2, all of $c_i \dots c_m(D_m^1)$ for $1 \leq i \leq m$ meet the real line, and by Proposition 9.1 they do so in points which are strictly smaller than those in which D_{2m-k}^1 meets the real line. We say that $c_i \dots c_m(D_m^1) < D_{2m-k}^1$ for short.

A relatively simple argument shows that if P_1 and P_2 are pieces of $\mathcal{F}^i(D)$ for some i, which both meet the real line and such that $P_1 < P_2$, then the labelling of the boundaries is as follows. Let W denote the set of words labelling the boundary of P_2. Then the words labelling the boundary of P_1 split into two sets I_1 and I_2, such that if $s \in I_1$, $t \in I_2$ and $w \in W$, then $s < w < t$. The converse of this is also true. In particular, this applies to the

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pieces $c_{k+1} \dots c_m(D^1_m)$ and D^1_{2m-k}. Now since $1 \in D^1_{2m-k}$, $f(D^1_{2m-k})$ is homeomorphic to a disc. Thus the words labelling its boundary are of the form $0w$ or $2w$, where $w \in W$. The image $f(c_{k+1} \dots c_m(D^1_m))$ consists of two components and it is clear that in order not to violate point (2) in the proof of Theorem 9.1 above, one of these components must be labelled by words of the form $0s$ and $2t$, where $s \in I_1$ and $t \in I_2$, while the other must be labelled by words of the form $2s$ and $0t$. Clearly, by the above remark, the former component is less than $f(D^1_{2m-k})$; it lies 'closer' to the point $1/\sqrt{2}$. By definition of the branches of f, this means that the former component is $f_1(c_{k+1} \dots c_m(D^1_m))$. This proves that a_k and b_k must be defined as claimed.

For any other entry $c_j \neq A$, the proof works in exactly the same way. □

Remark 9.2. The results in this section can be extended to the case that the quadratic-like map g in the hypothesis of Theorem 9.1 is hybrid equivalent to any hyperbolic quadratic map q with connected Julia set, in place of the special case $q : z \to z^2$ we have considered so far. The only change to the relation $\sim$, and so to our combinatorial model for Λ, is that the Julia set of q (and hence of g) becomes a quotient of the circle. We can deal with this by imposing on the set T (and its images) the extra identifications that are necessary to obtain the Julia set of the new function q from that of $z \to z^2$.

10. *The connectivity locus $\mathcal{M}$*

Recall that the set $\mathcal{M}$ is defined to be the set of points q in the complex plane for which the limit set of the correspondence $\mathcal{F}_q$ is connected and for which there exists a transversal with the required property. Let $q \in \mathcal{M}$ and $\mathcal{F}_q = \mathcal{F} = f \cup A$. Then the map $f^{-1} \cup A^{-1} : f(D) \cup A(D) \to D$ is a holomorphic map of degree three and, provided that the straightening conjecture for pinched polynomial-like maps is true, it is conformally conjugate on its filled Julia set to a unique cubic of the form μR_3 (see §6). We define

$$\mathcal{R}_{3,0} = \{\mu \in \mathbb{C} : \text{exactly one external ray lands at } 0\}.$$

CONJECTURE 10.1. *The connectivity locus $\mathcal{M}$ is homeomorphic to $\mathcal{R}_{3,0}$.*

Remark 10.1. Note that for matings between the modular group and quadratic polynomials it was conjectured in [**5**] that the connectivity locus is homeomorphic to the Mandelbrot set. The conjecture above is a natural extension of this.

Branner and Douady in [**2**] investigated cubics $Q_a(z) = z^3 - 3a^2z + 2a^3 - 2a$. Observe that Q_a is conjugate to $a^2/4\ R_3$ (both maps are the cubic polynomial that has one critical value a fixed point with multiplier $9a^2$). Let F_+ denote the limb of the connectivity locus of the family Q_a that corresponds to $\mathcal{R}_{3,0}$. Branner and Douady proved that F_+ is homeomorphic to the limb $M_{1/2}$ of the Mandelbrot set. Thus, the conjecture above amounts to the following.

CONJECTURE 10.2. *The connectivity locus $\mathcal{M}$ is homeomorphic to the limb $M_{1/2}$ of the Mandelbrot set.*

We recall Branner and Douady's construction. Given a quadratic P_c with $c \in M_{1/2}$, they construct a cubic Q_a with $a \in F_+$ as follows: the Julia set of P_c can be split into

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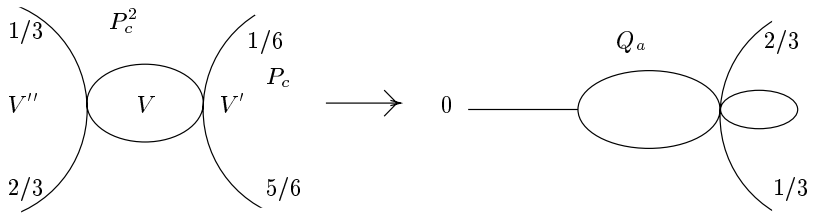


FIGURE 13. Branner and Douady's construction of a cubic by surgery on a quadratic.

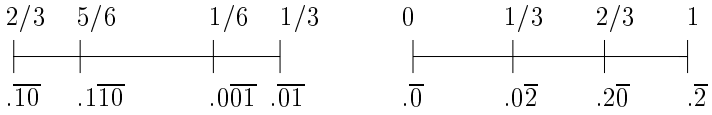


FIGURE 14. Here each 'ruler' is subdivided into three parts: that on the left into one part of length $\frac{1}{2}$ and two equal parts of length $\frac{1}{4}$, and that on the right into three equal parts. This subdivision is repeated on smaller scales.

three regions, the region V lying between external rays with arguments $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{2}{2}$, $\frac{5}{6}$, the region V' lying between external rays with arguments $\frac{1}{6}$ and $\frac{5}{6}$, and the regions V'' lying between external rays with arguments $\frac{1}{3}$ and $\frac{2}{3}$. See Figure 13. They construct a new space X by cutting out the region V'' and glueing together points on the $\frac{1}{3}$ and $\frac{2}{3}$ rays using equipotentials. They then define a degree three map f on X to be $f|_V = P_c^2 : V \to V \cup V'$ and $f|_{V'} = P_c : V' \to V \cup V'$. Using smoothing techniques and surgery it can be shown that f is conjugate to a unique cubic Q_a with $a \in F_+$. The Julia set of Q_a is obtained from that of P_c by the same cut and paste technique. The correspondence between external angles in the dynamical plane of P_c and those in the dynamical plane of Q_a is demonstrated in Figure 14.

In particular, our critical interval (defined in §9), with endpoints in base three notation, translates to a critical interval of P_c with endpoints E_1 and E_2 in binary notation. Inspection of this set up shows that E_1 and E_2 are in the orbit (under doubling) of the arguments of the external rays r_+ and r_- landing at the *root point* of the connected component of the interior of K_c that contains c. The external rays with arguments r_+ and r_- land at the root point of the connected component of the interior of $M_{1/2}$ that contains the parameter c. Thus, we can obtain the pairs of endpoints of all critical intervals relating to our family of correspondences by translating into base three notation the pairs of external angles defining root points of connected components of the interior of $M_{1/2}$.

The correspondence between external angles extends to the connectivity locus: F_+, and hence conjecturally $\mathcal{M}$, is made up of the closure of a countable union of copies of the Mandelbrot set M, one for each quadratic-like critical address. (Branner and Douady remark in [**2**] that their homeomorphism between F_+ and $M_{1/2}$ exhibits the decomposition of $M_{1/2}$ into copies of M, which was first described by Yoccoz.) We conjecture that an explicit combinatorial model for F_+ is given by the construction below, arising from our dynamical consideration in §9. We first define an equivalence relation on the points in the unit interval, represented in base three notation.

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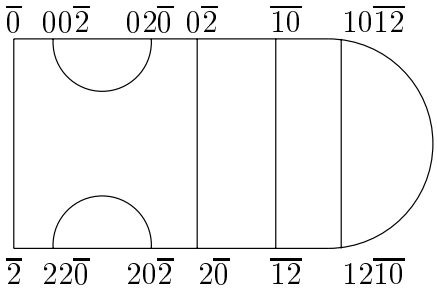


FIGURE 15. The first steps in the construction of a pinched disc model for F_+.

Definition 10.1. Given any integer $n \geq 1$, consider each word $\mathcal{A} = a_1 \dots a_n$ of length n in digits 0, 1, 2 which has $a_n = 0$.

Each such word has a unique partner $\mathcal{B} = b_1 \dots b_n$, as defined in Lemma 9.2. We require that the word $\mathcal{A}$ is not a concatenation of copies of two words X and Y, where X and Y are of shorter length than $\mathcal{A}$ and are partners. Define a relation $\sim$ on I as follows:

- $\bar{\mathcal{A}} \sim \bar{\mathcal{B}}$;
- $\mathcal{A}\bar{\mathcal{B}} \sim \mathcal{B}\bar{\mathcal{A}}$;
- $W\mathcal{A}\bar{\mathcal{B}} \sim W\mathcal{B}\bar{\mathcal{A}}$, where W is any word in $\mathcal{A}$ and $\mathcal{B}$.

We join pairs of equivalent points by geodesics in the unit disc (with the exception of the first such pair, $\bar{0}$ and $\bar{2}$, which are already identified). See Figure 15. For a given $\mathcal{A}, \mathcal{B}$, contracting the geodesics corresponding to pairs of points which are equivalent by the definition above produces a circle parameterized by binary representations of points of a unit interval (think of '$\mathcal{A}$' as '0' and '$\mathcal{B}$' as '1'), and bounding a disc. We claim that as we vary $\mathcal{A}$ and $\mathcal{B}$ no two of these geodesics intersect: thus, contracting all the geodesics in the closure of the relation yields the closure of a countable union of discs M_c^0, each corresponding to a periodic critical address c, and each equipped with a 'binary' parameterization of its boundary circle. Finally, we contract geodesics within each M_c^0 to yield a copy of the combinatorial Mandelbrot set using Lavaur's algorithm [**11**], and so obtain a combinatorial model for the connectivity locus $\mathcal{M}$ as the closure of a countable union of copies of the Mandelbrot set M. A computer plot of $\mathcal{M}$ is exhibited in Figure 16. The analogous locus for the case of H_5 is plotted in Figure 17. We remark that Figures 16 and 17 are different, although both plots are dominated by the primary copy of M (corresponding to the domicile $\bar{f}_1$). The analogous connectivity locus for matings of H_3 with quadratic polynomials [**5**] is conjectured to be homeomorphic to a single copy of M.

We conclude with some remarks about the connectivity locus of the family of matings between the Hecke group H_n and Chebyshev-like maps of degree $n-1$, for higher values of n. Denote this locus by $\mathcal{M}_n$ (so that the sets illustrated in Figures 16 and 17 are $\mathcal{M}_4$ and $\mathcal{M}_5$). As with $n = 4$ we conjecture that in general $\mathcal{M}_n$ is a copy of the principal limb of the connectivity locus of the family μR_{n-1} and that it has a combinatorial description which is given by a generalization of our construction above. However, for $n > 4$, $\mathcal{M}_n$ is no longer a copy of a limb of the Mandelbrot set: while for $n = 4$ the family $\mu R_{n-1}(z)$ is

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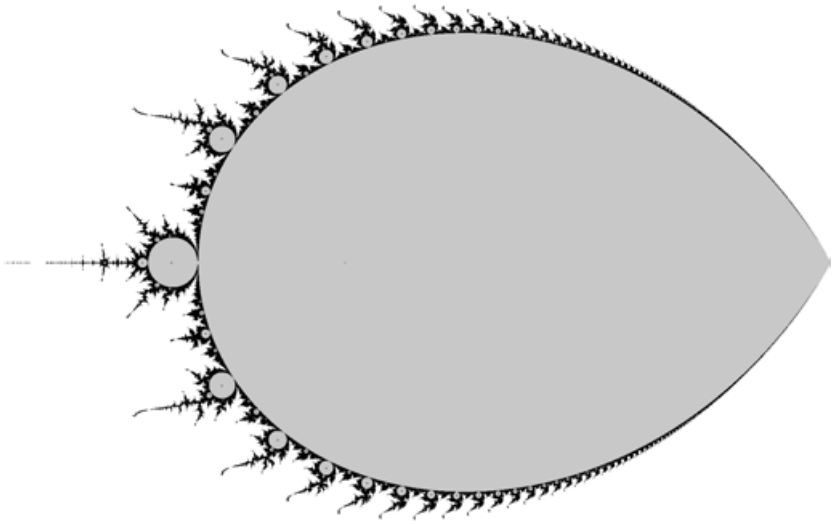


FIGURE 16. The connectivity locus $\mathcal{M}$ for matings of H_4 with polynomials of degree 3 (computer plot by X. Buff).

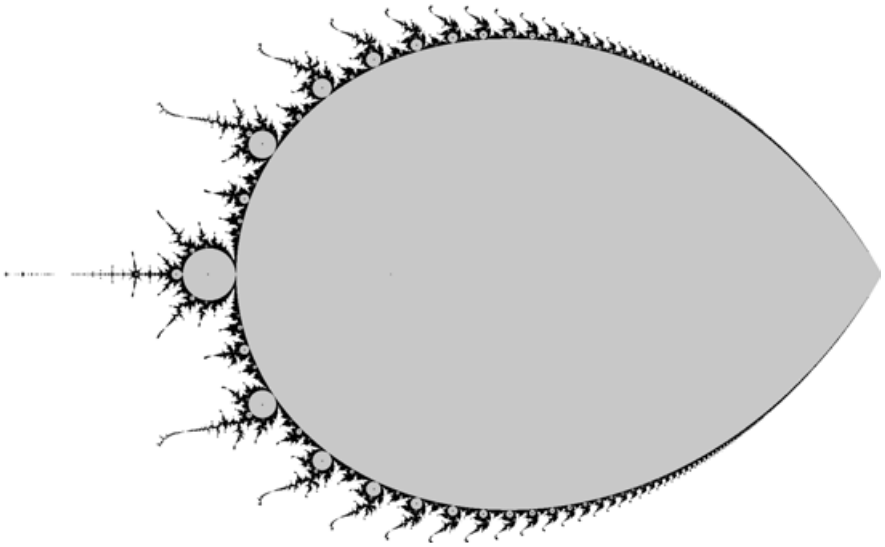


FIGURE 17. The connectivity locus for matings of H_5 with polynomials of degree 4 (computer plot by X. Buff).

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conjugate to the family Q_a considered by Branner and Douady [**2**], and thus to the family $az(1 + z/(n-2))^{n-2}$ considered by Branner and Fagella [**3**], it is no longer the case for larger n that the family $\mu R_{n-1}(z)$ is conjugate to the family $az(1 + z/(n-2))^{n-2}$.

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References

[1] A. F. Beardon. *Iteration of Rational Functions.* Springer, New York, 1991.
[2] B. Branner and A. Douady. *Surgery on Complex Polynomials (Lecture Notes in Mathematics, 1345)*. Springer, Berlin, 1988.
[3] B. Branner and N. Fagella. Homeomorphisms between limbs of the Mandelbrot set. *J. Geom. Anal.* **9** (1999), 327–390.
[4] B. Branner and J. Hubbard. The iteration of cubic polynomials part II: patterns and parapatterns. *Acta Math.* **169** (1992), 229–325.
[5] S. Bullett and C. Penrose. Mating quadratic maps with the modular group. *Invent. Math.* **115** (1994), 483–511.
[6] S. Bullett and C. Penrose. A gallery of iterated correspondences. *Exp. Math.* **3** (1994), 85–105
[7] S. Bullett and C. Penrose. Regular and limit sets for holomorphic correspondences. *Fund. Math.* **167** (2001), 111–171.
[8] A. Douady and J. H. Hubbard. Itération des polynômes quadratiques complexes. *C. R. Acad. Sci. Paris* **294** (1982), 123–126.
[9] A. Douady and J. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Ecole Norm. Sup.* **18** (1985), 263–297.
[10] M. Freiberger. Matings between Hecke groups and polynomials. *PhD Thesis*, Queen Mary University of London, 2001.
[11] P. Lavaurs. Une description combinatoire de l'involution définie par M sur les rationnels à dénominateur inpair. *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 143–146.
[12] B. Maskit. *Kleinian Groups.* Springer, New York, 1987.
[13] Curtis T. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, Princeton, NJ, 1994.
[14] J. Milnor and W. Thurston. On iterated maps of the interval. *Dynamical Systems, Proceedings of the University of Maryland, 1986/1987 (Lecture Notes in Mathematics, 1342)*. Springer, Berlin, 1988, pp. 465–563
[15] C. Pommerenke. *Boundary Behaviour of Conformal Maps*. Springer, New York, 1992.

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