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Some examples of maximal orders

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In ([7], p. 181, problem 7) Maury and Raynaud pose the following question: 'Let J be a ring and G a group such that the group ring $J[G]$ is an order in a quotient ring Q; when is $J[G]$ a maximal order in Q?' The question is interesting for two reasons. In the first place, the analogous question for universal enveloping algebras of finite-dimensional Lie algebras has been settled very satisfactorily by Chamarie ([2], corollaire 2·3·2). Secondly, it has been pointed out by several authors that maximal orders have some very desirable properties – see for example [3], [4], [6], [7] and [12].

We shall consider polycyclic-by-finite groups G, i.e. groups G with a finite subnormal series with successive factors cyclic or finite. By a theorem of Mal'cev such a group G is nilpotent-by-Abelian-by-finite. The main result of this paper is the following:

THEOREM. *Let J be a Noetherian maximal order in a simple Artinian ring and G a nilpotent-by-Abelian group which is polycyclic-by-finite. Then the group ring $J[G]$ is a maximal order in a simple Artinian ring if and only if*

(i) *G has no non-trivial finite normal subgroup, and*

(ii) *G has no infinite dihedral normal subgroup.*

It is perhaps worth making some elementary observations here. First of all, if $J(G)$ is a maximal order in a quotient ring then so is J. For a Noetherian ring J to be an order in a simple Artinian ring it is necessary and sufficient that J be prime. For J prime Noetherian, condition (i) in the Theorem is the necessary and sufficient condition for $J[G]$ to be a prime ring and thus an order in a simple Artinian ring. Thus condition (ii) is the crucial condition determining whether or not $J(G)$ is a maximal order.

The composition of the paper is as follows. Section 1 is concerned with some preliminary matters, most of it well known, but we show that condition (ii) in the Theorem is necessary. The rest of the paper is a proof that conditions (i) and (ii) in the Theorem are sufficient. Let G be a nilpotent-by-Abelian group which is polycyclic-by-finite and which satisfies (i) and (ii). Let N denote the Fitting radical of G, A the centre of N and C the centralizer of A in G. Then $N \leqslant C$ and the group G/C is Abelian. Let T denote the subgroup of G containing C such that T/C is the torsion subgroup of G/C. If J is a Noetherian maximal order in a simple Artinian ring then the aim is to prove that $J[T]$ is a maximal order in a simple Artinian ring, because it is then easy to show that $J[G]$ is a maximal order in a simple Artinian ring. Section 2 contains a short investigation of some conditions on the subgroup A which ensure that if the ring $J[C]$ is a maximal order then so is the ring $J[T]$. This leaves the problem of when $J[C]$ is a maximal order. Note that A is central in C. This case is investigated in § 3. Finally in § 4 we gather together the preliminary material in the previous sections to prove the Theorem.

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1. *Some preliminaries*

A ring Q with identity is a *quotient ring* provided every regular element (i.e. non-zero-divisor) is a unit. For example (left and right) Artinian rings are quotient rings. A ring R is an *order* in a quotient ring Q provided

(i) R is a subring of Q,

(ii) every regular element of R is a unit of Q, and

(iii) for every element q of Q there exist elements $r_i, c_i \in R$ with c_i regular ($i = 1, 2$) such that $q = c_1^{-1} r_1 = r_2 c_2^{-1}$.

Orders R, S in Q are called *equivalent*, written $R \sim S$, provided there exist units q_i of Q ($1 \leqslant i \leqslant 4$) such that

$$q_1 R q_2 \leqslant S \quad \text{and} \quad q_3 S q_4 \leqslant R.$$

An order R in Q is *maximal* if the only order S in Q such that $R \sim S$ and $R \leqslant S$ is $S = R$.

Let R be an order in a quotient ring Q. A left R-submodule A of Q is called a (*fractional*) *left R-ideal* provided

(i) A contains a unit of Q, and

(ii) $Aq \leqslant R$ for some unit q of Q.

Right R-ideals are defined analogously. By an *R-ideal* of Q we mean a left R-ideal which is also a right R-ideal. An *integral* R-ideal is an R-ideal contained in R (i.e. an integral R-ideal is an ideal of R containing a regular element). For any R-ideal A define

$$O_l(A) = \{q \in Q : qA \leqslant A\} \quad \text{and} \quad O_r(A) = \{q \in Q : Aq \leqslant A\}.$$

LEMMA 1·1. *Let R be an order in a quotient ring Q. Then R is a maximal order in Q if and only if $O_l(A) = O_r(A) = R$ for every integral R-ideal A of Q.*

Proof. See ([7], p. 7, proposition 3·1).

It will be convenient to use the shorthand 'R is a maximal order' for 'the ring R is a maximal order in a quotient ring'. Our purpose in this note is to produce a stock of examples of maximal orders by considering group rings. We begin with the following elementary observation which shows how to extend the results about polycyclic-by-finite groups given later in the paper to more general groups.

PROPOSITION 1·2. *Let J be a ring, G a group and R the group ring $J[G]$ such that for any finitely generated subgroup H of G the ring $J[H]$ is a maximal order. Then R is a maximal order.*

Proof. By ([9], lemma 13·3·5) R is an order in a quotient ring Q. Let A be an integral R-ideal of Q. Then A is an ideal of R and there exists a regular element $a \in R$ with $a \in A$. Let $q \in O_l(A)$. Then $q = b^{-1}s$ for some $s, b \in R$ with b regular. There exists a finitely generated subgroup K of G such that the ring $T = J[K]$ contains the elements a, b, s. By hypothesis T is a maximal order in a quotient ring Q'. Note that $A \cap T$ is an integral T-ideal of Q' and $q \in A'$. Now $qA \leqslant A$ implies $sA \leqslant bA$ and hence

$$s(A \cap T) \leqslant (bA) \cap T.$$

Because R is a free left S-module it follows that $(bA) \cap T = b(A \cap T)$ and hence $q \in O_l(A \cap T)$. By Lemma 1·1 $q \in T \leqslant R$. It follows that $O_l(A) \leqslant R$. Similarly $O_r(A) \leqslant R$. By Lemma 1·1 R is a maximal order.

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Let J be a ring. If $G, H, \ldots$ are groups then the augmentation ideals of the group rings $J[G], J[H], \ldots$ will be denoted $\mathfrak{g}, \mathfrak{h}, \ldots$, respectively. Our second elementary observation is the following one.

LEMMA 1·3. *Let J be a ring and G a group such that the group ring $R = J[G]$ is a maximal order. Then G contains no infinite dihedral normal subgroup.*

Proof. Suppose (on the contrary) that G contains an infinite dihedral normal subgroup N. There exist $x, y \in N$ such that

$$N = \langle x, y;\ x^y = x^{-1}, y^2 = 1 \rangle.$$

Let I be the ideal $\mathfrak{n}R$ of R. Note that I is an integral ideal of R because I contains the regular element $x-1$. Let Q denote the quotient ring of R and $q = (x-1)^{-1}(y+1) \in Q$. Since $I = (x-1)R + (y-1)R$, it follows that

$$\begin{aligned} qI &\leqslant (x-1)^{-1}(y+1)(x-1)R = (x-1)^{-1}\{(x^{-1}-1)y + (x-1)\}R \\ &= (-x^{-1}y+1)R = \{(1-x^{-1})y + (1-y)\}R \leqslant I. \end{aligned}$$

Thus $q \in O_l(I)$. If $q \in R$ then $y+1 = (x-1)r$ for some r in R and $y \in \langle x \rangle$, a contradiction. Thus $q \notin R$ and hence $R < O_l(I)$, contradicting Lemma 1·1.

We complete our preliminary remarks by noting the following facts.

LEMMA 1·4. *Let J be a Noetherian ring, G a polycyclic-by-finite group and R the group ring $J[G]$. Then R is a Noetherian ring. Moreover, R is an order in a simple Artinian ring if and only if J is an order in a simple Artinian ring and G contains no non-trivial finite normal subgroup.*

Proof. The fact that R is Noetherian is due to P. Hall (see [9], theorem 10·2·7). The remainder of the lemma is proved by combining ([5], theorem 13) and ([9], theorem 4·2·10).

For any group G, the *finite radical* of G is the product of all finite normal subgroups. If G is polycyclic-by-finite then the finite radical of G is the unique maximal finite normal subgroup. In this note, because of Lemma 1·4, we are principally concerned with polycyclic-by-finite groups with trivial finite radical and we denote the class of such groups by

$$\mathscr{X}$$

and refer to the members of $\mathscr{X}$ as $\mathscr{X}$-groups. Note that $\mathscr{X}$ is closed under taking normal subgroups.

2. *Abelian normal subgroups*

Let A be an Abelian normal subgroup of a group G. For any x in G define

$$[A, x] = \{[a, x] \colon a \in A\}$$

where $[a, x]$ denotes the commutator $a^{-1}x^{-1}ax$. For any a, b in A we have

$$[ab, x] = [a, x]^b [b, x] = [a, x][b, x]$$

and it follows that $[A, x]$ is a subgroup of A.

Before proving the first lemma we note an elementary fact. Let J be a ring, H a normal subgroup of a group G and R, S the group rings $J[G], J[H]$, respectively. If I is an ideal of R then $I \cap S$ is an ideal of S such that $x(I \cap S) = (I \cap S)x$ for all x in G. It follows that $(I \cap S)R$ is an ideal of R.

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LEMMA 2·1. *Let J be a ring and A a non-central Abelian normal subgroup of a group G. Let C denote the centralizer of A in G and S the group ring $J[C]$. Let I be an ideal of the group ring $R = J[G]$ such that I is a finitely generated right ideal. Then there exist a positive integer n and elements x_i $(1 \leqslant i \leqslant n)$ in G but not C such that $\mathfrak{a}_1 \dots \mathfrak{a}_n I \leqslant (I \cap S) R$ where $A_i = [A, x_i]$ $(1 \leqslant i \leqslant n)$.*

Proof. There exists a positive integer m and elements $r_i \in I$ $(1 \leqslant i \leqslant m)$ such that

$$I = r_1 R + \dots + r_m R.$$

Since A is an Abelian normal subgroup of G it is sufficient to prove the following:

$$\left.\begin{array}{c}\text{For any element } r \text{ in } I \text{ there exist a positive integer } k \text{ and elements } y_i \in G \backslash C \\ (1 \leqslant i \leqslant k) \text{ such that } \mathfrak{b}_1 \dots \mathfrak{b}_k r \leqslant (I \cap S) R \text{ where } B_i = [A, y_i]\ (1 \leqslant i \leqslant k).\end{array}\right\} \quad (1)$$

Let T be a transversal to the cosets of C in G and suppose $1 \in T$. Let $r \in I$. Then there exist a positive integer l and distinct elements $t_i \in T$, $s_i \in S$ $(1 \leqslant i \leqslant l)$ such that

$$r = s_1 t_1 + \dots + s_l t_l. \quad (2)$$

By considering the element $r t_1^{-1}$ we can suppose without loss of generality that $t_1 = 1$ in (2). We prove (1) by induction on l. If $l = 1$ then $r = s_1 \in I \cap S$ and there is nothing to prove. Now suppose $l > 1$. Let $a \in A$. Then

$$\begin{aligned} r^a &= s_1 + s_2 t_2^a + \dots + s_l t_l^a \\ &= s_1 + s_2 [a, t_2^{-1}] t_2 + \dots + s_l [a, t_l^{-1}] t_l, \end{aligned}$$

and hence

$$\bar{r} = [a, t_l^{-1}] r - r^a = ([a, t_t^{-1}] - 1) s_1 + s_2' t_2 + \dots + s_{l-1}' t_{l-1},$$

where $s_i' = ([a, t_l^{-1}] - [a, t_i^{-1}]) s_i \in S (2 \leqslant i \leqslant l)$. Note that $\bar{r} \in I$. Repeating this argument,

$$([a_2, t_2^{-1}] - 1) \dots ([a_l, t_l^{-1}] - 1) s_1 \in I$$

for all $a_i \in A$ $(2 \leqslant i \leqslant l)$. Thus, if we define $B_i = [A, t_{i+1}^{-1}]$ $(1 \leqslant i \leqslant l-1)$, then

$$\mathfrak{b}_1 \dots \mathfrak{b}_{l-1} s_1 \leqslant I \cap S \quad (3)$$

because $\mathfrak{b}_i \leqslant S (1 \leqslant i \leqslant l-1)$. Let $b \in \mathfrak{b}_1 \dots \mathfrak{b}_{l-1}$ and consider

$$r^* = br - bs_1 = bs_2 t_2 + \dots + bs_l t_l.$$

By (3), $r^* \in I$ and hence so too does the element $r^* t_2^{-1}$, which has the form

$$r^* t_2^{-1} = bs_2 + \bar{s}_3 \bar{t}_3 + \dots + \bar{s}_l \bar{t}_l$$

where $bs_2, \bar{s}_i \in S, \bar{t}_i \in T$ $(3 \leqslant i \leqslant l)$. By induction on l there exist a positive integer $k \geqslant l$ and elements $y_i \in G \backslash C$ $(l \leqslant i \leqslant k)$ such that if $B_i = [A, y_i]$ $(l \leqslant i \leqslant k)$ then

$$\mathfrak{b}_l \dots \mathfrak{b}_k r^* t_2^{-1} \leqslant (I \cap S) R.$$

It follows that $\mathfrak{b}_l \dots \mathfrak{b}_k r^* \leqslant (I \cap S) R$, and hence

$$\mathfrak{b}_l \dots \mathfrak{b}_k br = \mathfrak{b}_l \dots \mathfrak{b}_k (r^* + bs_1) \leqslant (I \cap S) R$$

by (3). It follows that

$$\mathfrak{b}_1 \dots \mathfrak{b}_k r \leqslant (I \cap S) R,$$

as required. This proves (1) and the result follows.

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LEMMA 2·2. *Let J be a Noetherian order in a simple Artinian ring, G an $\mathfrak{X}$-group and R the group ring $J[G]$. Suppose that G contains a non-central Abelian normal subgroup A with centralizer C in G such that the group ring $S = J[C]$ is a maximal order. Let I be an integral R-ideal of Q, the quotient ring of R. Then there exists a positive integer n and elements x_i $(1 \leqslant i \leqslant n)$ in G but not C such that $\mathfrak{a}_1 \dots \mathfrak{a}_n O_l(I) \leqslant R$, where $A_i = [A, x_i]$ $(1 \leqslant i \leqslant n)$.*

Proof. By Lemma 1·4 R is an order in a simple Artinian ring Q and R is a Noetherian ring. In particular, this means that I is a finitely generated right ideal of R. By Lemma 2·1 there exist a positive integer n and elements $x_i \in G \backslash C$ $(1 \leqslant i \leqslant n)$ such that $\mathfrak{a}_1 \dots \mathfrak{a}_n I \leqslant (I \cap S) R$ where $A_i = [A, x_i]$ $(1 \leqslant i \leqslant n)$. Now consider the group G/C. Because G/C is a polycyclic-by-finite group, there exists a normal subgroup H of finite index in G such that $C \leqslant H$ and H/C is a poly-(infinite cyclic) group (see ([9], lemma 10·2·5)). Let $T = J[H]$. Since S is a maximal order, it follows that the ring T is a maximal order by repeated use of ([2], proposition 2·2·1). Let $r \in \mathfrak{a}_1 \dots \mathfrak{a}_n$ and $q \in O_l(I)$. By ([9], lemma 13·3·5 (iii) and its proof) there exist $a \in R$, $c \in T$ with c regular, such that $rq = c^{-1}a$. Let X be a transversal to the cosets of H in G. Then

$$a = \sum_{x \in X} a_x x$$

for some $a_x \in T$ $(x \in X)$. Now $rI \leqslant (I \cap S) R$ implies

$$rqI \leqslant rI \leqslant (I \cap S) R = \sum_{x \in X} (I \cap S) Tx$$

and hence

$$aI \leqslant c \sum_{x \in X} (I \cap S) Tx.$$

In particular, this gives

$$(\sum_{x \in X} a_x x)(I \cap S) T \leqslant c \sum_{x \in X} (I \cap S) Tx,$$

and hence

$$a_x (I \cap S) T \leqslant c(I \cap S) T \quad (x \in X). \tag{4}$$

Since G is an $\mathfrak{X}$-group it follows that A is a non-trivial free Abelian group and hence each subgroup A_i $(1 \leqslant i \leqslant n)$ is infinite. By ([9], lemma 3·1·2) $\mathfrak{a}_1 \dots \mathfrak{a}_n I \neq 0$. Thus $I \cap S \neq 0$ and it follows that $(I \cap S) T$ is a non-zero ideal of the ring T. Since H is an $\mathfrak{X}$-group, it follows that T is an order in a simple Artinian ring Q' (Lemma 1·4) and hence $(I \cap S) T$ is an integral T-ideal of Q'. But T is a maximal order. Hence (4) gives

$$c^{-1} a_x \in T \quad (x \in X)$$

and hence

$$rq = c^{-1}a = \sum_{x \in X} c^{-1} a_x x \in \sum_{x \in X} Tx = R.$$

The result follows.

LEMMA 2·3. *Let J be a ring and G a group such that the group ring $R = J[G]$ is an order in a quotient ring Q. Let A be an Abelian subgroup of G of rank > 1. Then*

$$R = \{q \in Q : \mathfrak{a} q \leqslant R\}.$$

Proof. Let $a, b \in A$ have infinite order and satisfy $\langle a \rangle \cap \langle b \rangle = 1$. Let $q \in Q$ and $\mathfrak{a} q \leqslant R$. Then $(a-1) q \in R$ implies $q = (a-1)^{-1} r$ for some r in R. Similarly $q = (b-1)^{-1} s$ for some s in R. Since $ab = ba$ it follows that

$$(b-1) r = (a-1)(b-1) q = (a-1) s. \tag{5}$$

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Let T be a transversal to the right cosets of A in G. Then there exists a finite subset T' of T and elements $r_t, s_t \in J[A]$ $(t \in T')$ such that

$$r = \sum_{t \in T'} r_t t \quad \text{and} \quad s = \sum_{t \in T'} s_t t.$$

Then (5) gives

$$(b-1) r_t = (a-1) s_t \quad (t \in T').$$

Since $b-1$ is a regular element of the ring $J[A]$ modulo the ideal $(a-1)J[A]$ it follows that $r_t = (a-1)\bar{r}_t$ for some $\bar{r}_t \in J[A]$ for each t in T'. Then

$$r = (a-1) \sum_{t \in T'} \bar{r}_t t$$

and $q = (a-1)^{-1} r = \Sigma_{t \in T'} \bar{r}_t t \in R$, as required.

COROLLARY 2·4. *Let J be a Noetherian order in a simple Artinian ring, G an $\mathfrak{X}$-group and R the group ring $J[G]$. Suppose that G contains an Abelian normal subgroup A with centralizer C in G such that the group ring $J[C]$ is a maximal order and rank $[A, x] \neq 1$ $(x \in G)$. Then R is a maximal order in a simple Artinian ring.*

Proof. By Lemma 1·4 R is an order in a simple Artinian ring Q. If $C = G$ then there is nothing to prove. If $C \neq G$ then by Lemmas 2·2 and 2·3 $O_l(I) \leqslant R$ for any integral R-ideal I of Q. Similarly $O_r(I) \leqslant R$. By Lemma 1·1 R is a maximal order.

COROLLARY 2·5. *Let J be a Noetherian order in a simple Artinian ring, G an $\mathfrak{X}$-group and R the group ring $J[G]$. Suppose that G contains Abelian normal subgroups A_i $(i = 1, 2)$ with centralizers C_i $(i = 1, 2)$ in G such that $A_1 \cap A_2 = 1$ and $J[C_i]$ is a maximal order $(i = 1, 2)$. Then R is a maximal order in a simple Artinian ring.*

Proof. Again R is an order in a simple Artinian ring Q. If $C_1 = G$ or $C_2 = G$ then there is nothing to prove. Suppose $C_i \neq G$ $(i = 1, 2)$. Let I be an integral R-ideal of Q. By Lemma 2·2, there exist positive integers m, n and elements $x_i \in G \backslash C_1$ $(1 \leqslant i \leqslant m)$, $y_i \in G \backslash C_2$ $(1 \leqslant i \leqslant n)$, such that if $X_i = [A_1, x_i]$ $(1 \leqslant i \leqslant m)$ and $Y_j = [A_2, y_j]$ $(1 \leqslant j \leqslant n)$, then

$$\mathfrak{x}_1 \dots \mathfrak{x}_m O_l(I) \leqslant R \quad \text{and} \quad \mathfrak{y}_1 \dots \mathfrak{y}_n O_l(I) \leqslant R.$$

Let $1 \neq a_i \in X_i$ $(1 \leqslant i \leqslant m)$, $1 \neq b_j \in Y_j$ $(1 \leqslant j \leqslant n)$. It follows that

$$\mathfrak{O}_l(I) \leqslant (1-a_1)^{-1} \dots (1-a_m)^{-1} R \cap (1-b_1)^{-1} \dots (1-b_n)^{-1} R.$$

Adapting the proof of Lemma 2·3 we have $O_l(I) \leqslant R$. Similarly $O_r(I) \leqslant R$. By Lemma 1·1 R is a maximal order.

LEMMA 2·6. *Let J be a Noetherian order in a simple Artinian ring and G an $\mathfrak{X}$-group which contains an infinite cyclic normal subgroup A with centralizer C in G such that G/A is an $\mathfrak{X}$-group and the ring $J[C]$ is a maximal order. Then the group ring $R = J[G]$ is a maximal order in a simple Artinian ring.*

Proof. Let $A = \langle x \rangle$. Then $|G:C| \leqslant 2$. If $G = C$ then there is nothing to prove. Suppose $|G:C| = 2$ and $t \in G \backslash C$. Let I be an integral R-ideal of Q, the quotient ring of R (Lemma 1·4). By Lemma 2·2 there exists a positive integer n such that

$$(x^2-1)^n O_l(I) \leqslant R.$$

Let $q \in O_l(I)$. Then $q = (x^2-1)^{-n} r$ for some r in R and $qI \leqslant I$. Hence

$$rI \leqslant (x^2-1)^n I. \tag{6}$$

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Let $P = \mathfrak{a}R = R\mathfrak{a}$. Note that

$$\bigcap_{k=1}^{\infty} P^k = \left[\bigcap_{k=1}^{\infty} \mathfrak{a}^k\right] R = 0.$$

Thus there exists an integer $k \geqslant 0$ such that $I \leqslant P^k, I \not\leqslant P^{k+1}$. Let $I_1 = I(x-1)^{-k}$. Then I_1 is an ideal of R, $I_1 \not\leqslant P$ and $I = I_1(x-1)^k$. Thus (6) gives

$$rI_1 \leqslant (x^2-1)^n I_1 \leqslant P^n. \tag{7}$$

By ([9], theorem 4·2·10) P is a prime ideal of R and hence $r \in P$. Thus $r = (x-1)r_1$ for some $r_1 \in R$, and it follows by (7) that $(x-1)r_1 I_1 \leqslant (x-1)^n R$. Thus

$$r_1 I_1 \leqslant P^{n-1}$$

and, by induction on n, $r_1 \in P^{n-1}$. Thus $r \in P^n$. Suppose $r = (x-1)^n r_2$ for some $r_2 \in R$. Then (6) gives

$$r_2 I \leqslant (x+1)^n I.$$

Let $P' = (x+1)R = R(x+1)$. By ([8], theorem 1·9) P' is a prime ideal of R and we can repeat the above argument to obtain $r_2 \in (P')^n$, i.e. $r_2 = (x+1)^n r_3$ for some $r_3 \in R$. Then

$$r = (x-1)^n (x+1)^n r_3$$

and hence $q = r_3 \in R$. It follows that $O_l(I) \leqslant R$. Similarly $\mathcal{O}_r(I) \leqslant R$. By Lemma 1·1 R is a maximal order.

3. $\mathfrak{X}$-groups with essential centre

In § 2 our concern on the whole was with non-central Abelian normal subgroups of an $\mathfrak{X}$-group G. This leads us to question what happens if we consider $\mathfrak{X}$-groups with the following property:

every Abelian normal subgroup of G is central in G. . . (8)

We shall show that if J is a Noetherian maximal order in a simple Artinian ring then $J[G]$ is a maximal order for any $\mathfrak{X}$-group G satisfying (8). In fact we can weaken (8) somewhat. We shall call a normal subgroup N of an $\mathfrak{X}$-group G *essential* provided

$$N \cap H \neq 1$$

for every non-trivial normal subgroup H of G. In particular we shall consider $\mathfrak{X}$-groups G with essential centre. Note that any $\mathfrak{X}$-group G satisfying (8) has essential centre because for every normal subgroup H of G there exists a non-trivial soluble normal subgroup N of G such that N is a subgroup of finite index in H.

Let G be an $\mathfrak{X}$-group. Let F denote the Fitting radical of G; i.e. F is the product of all nilpotent normal subgroups of G. Because G satisfies the maximal condition on subgroups, F is the unique maximal nilpotent normal subgroup of G. Following Passman [9] and Roseblade ([10], § 1·5) the *Zalesskii subgroup* $\mathfrak{Z}(G)$ of G is defined to be the centre of F. The *FC-centre* $\Delta(G)$ of G is defined by

$$\Delta(G) = \{x \in G: |G : \mathbb{C}_G(x)| < \infty\}.$$

As Roseblade ([10], § 1·5) points out, two facts can be noted, namely

$$\Delta(G) \leqslant \mathfrak{Z}(G), \tag{9}$$

and

$$\mathfrak{Z}(G) \text{ is essential in } G. \tag{10}$$

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In particular (10) shows that if $\mathfrak{I}(G)$ is central in G then G has essential centre. The first result of this section is rather elementary.

LEMMA 3·1. *Let G be an $\mathfrak{X}$-group with Fitting radical F. Let N be a normal subgroup of G such that $F \leqslant N$ or N has finite index in G. Then $\mathfrak{I}(N) = N \cap \mathfrak{I}(G)$.*

Proof. Let E denote the Fitting radical of N. Then E is a nilpotent normal subgroup of G and hence $E \leqslant F$. On the other hand $N \cap F$ is a nilpotent normal subgroup of N and hence $N \cap F \leqslant E$. Thus $E = N \cap F$. If $F \leqslant N$ then $E = F$ and hence

$$\mathfrak{I}(N) = \mathfrak{I}(G) = N \cap \mathfrak{I}(G).$$

Now suppose that N has finite index in G. Then E has finite index in F. Since G is an $\mathfrak{X}$-group it follows that F has torsion-free centre and hence, by ([9], lemmas 11·1·3 and 11·1·4), the centre $\mathfrak{I}(N)$ of E centralizes F. It follows that $\mathfrak{I}(N) = N \cap \mathfrak{I}(G)$.

LEMMA 3·2. *Let G be an $\mathfrak{X}$-group with centre Z. Then Z is essential in G if and only if*

(i) $Z = \Delta(G)$, *and*

(ii) *G contains a normal subgroup H of finite index such that H acts nilpotently on $\mathfrak{I}(G)$.*

In this case G/Z is an $\mathfrak{X}$-group.

Before proving Lemma 3·2 we make a simple observation. Let A be an Abelian normal subgroup of an $\mathfrak{X}$-group G. Let $C = \mathbb{C}_G(A)$. Then A is a left module over the integral group ring $\mathbb{Z}[G/C]$ where the action of G/C on A is just conjugation. Because G has trivial finite radical it follows that the torsion subgroup of A is trivial and hence A is torsion-free. It is then natural to consider the Abelian group $\mathbb{Q} \otimes_{\mathbb{Z}} A$ as a left $\mathbb{Q}[G/C]$-module. In particular, if G/C is a finite group then $\mathbb{Q}[G/C]$ is a semiprime Artinian ring by Maschke's Theorem and hence the module $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is completely reducible. In case $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is irreducible we say that *G acts rationally irreducibly on A*. In any case if G/C is finite there exist a positive integer m and normal subgroups A_i $(1 \leqslant i \leqslant m)$ of G contained in A such that G acts rationally irreducibly on A_i for each $1 \leqslant i \leqslant m$, $B = \langle A_1, \ldots, A_m \rangle = A_1 \times \ldots \times A_m$ and B has finite index in A. We shall require this fact in the proof of Lemma 3·2 and also in § 4.

Proof of Lemma 3·2. Suppose Z is essential in G. We prove first of all that G/Z is an $\mathfrak{X}$-group. Let K be a normal subgroup of G containing Z such that K/Z is a finite group. Then K is centre-by-finite and, by Schur's Lemma (see [9], lemma 4·1·4)), K' is a finite normal subgroup of G. Thus K is an Abelian group. Let $x \in G$. If $y \in K$, then there exists a positive integer n such that $y^n \in Z$ and hence $[y^n, x] = 1$. It follows that $[y, x]^n = 1$ and hence $[y, x] = 1$. Thus $[K, x] = 1$ and it follows that $[K, G] = 1$. Hence $K = Z$. It follows that G/Z is an $\mathfrak{X}$-group.

Let $A = \Delta(G)$. By ([9], lemma 4·1·5) A is a torsion-free Abelian normal subgroup of G. As we remarked above, there exist a positive integer m and normal subgroups A_i $(1 \leqslant i \leqslant m)$ of G contrained in A such that G acts rationally irreducibly on A_i $(1 \leqslant i \leqslant m)$ and $A_1 \times \ldots \times A_m$ has finite index in A. For each $1 \leqslant i \leqslant m$, $A_i \cap Z \neq 1$ and hence $|A_i : A_i \cap Z| < \infty$. It follows that $|A : Z| < \infty$. But G/Z is an $\mathfrak{X}$-group. Hence $A = Z$.

By a theorem of Mal'cev (see ([9], theorem 12·1·5)) G is nilpotent-by-Abelian-by-finite and hence G/C is Abelian-by-finite, where $C = \mathbb{C}_G(\mathfrak{I}(G))$. Let H be a normal subgroup of finite index in G such that $C \leqslant H$ and H/C is Abelian. Let $D = \mathfrak{I}(G)$ and $\bar{G} = G/C$. Then D is a finitely generated left $\mathbb{Z}[\bar{G}]$-module such that Z is an essential

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submodule of D. By a result of Roseblade (see ([9], corollary 11·2·4)), it follows that there exists a positive integer k such that $\bar{\mathfrak{h}}^k D$ is the zero submodule of D, where $\bar{\mathfrak{h}}$ denotes the augmentation ideal of $\bar{H} = H/C$. It follows that H acts nilpotently on D.

Conversely, suppose G satisfies (i) and (ii). Let N be a non-trivial normal subgroup of G. Then $N \cap D \neq 1$ by (10). Because H acts nilpotently on D, there exists a positive integer t such that

$$[N \cap D, {}_t H] = [[N \cap D, {}_{t-1} H], H] = 1. \tag{11}$$

By choosing t as small as possible, we infer from (11) that there exists a non-trivial subgroup M of $N \cap D$ such that

$$[M, H] = 1.$$

Then $M \leqslant \Delta(G) = Z$ by (i). It follows that $1 \neq M \leqslant N \cap Z$. Hence Z is essential in G.

COROLLARY 3·3. *Let G be an $\mathfrak{X}$-group with essential centre Z and let N be a normal subgroup of finite index in G. Then $Z \cap N$ is the centre of N and $Z \cap N$ is essential in N.*

Proof. Clearly $\Delta(N) = N \cap \Delta(G)$ as $|G:N| < \infty$. By Lemma 3·2, $\Delta(G) = Z$ and hence

$$Z(N) \leqslant \Delta(N) = N \cap \Delta(G) = N \cap Z \leqslant Z(N),$$

where $Z(N)$ denotes the centre of N. Thus $Z(N) = Z \cap N = \Delta(N)$. By Lemma 3·1, $\mathfrak{I}(N) = N \cap \mathfrak{I}(G)$ and by Lemma 3·2 there exists a normal subgroup H of finite index in G such that H acts nilpotently on $\mathfrak{I}(G)$. Thus $H \cap N$ acts nilpotently on $\mathfrak{I}(N)$. By Lemma 3·2, $Z \cap N$ is essential in N.

Having dealt with these preliminaries, we now address ourselves to the main result of this section.

THEOREM 3·4. *Let J be a Noetherian maximal order in a simple Artinian ring and G an $\mathfrak{X}$-group with essential centre. Then the group ring $R = J[G]$ is a Noetherian maximal order in a simple Artinian ring.*

The remainder of this section is devoted to the proof of Theorem 3·4. Note that Lemma 1·4 applies to show that R is a Noetherian order in a simple Artinian ring Q. It remains to prove that R is a maximal order in Q. There are two ingredients in the proof, the first being the characterization of maximal orders given by Chamarie [3] and the second the description of prime ideals given by Roseblade [10].

First we consider Chamarie's work. Let S be a maximal order in a quotient ring Q' and A an S-ideal of Q'. Define

$$A^* = \{q \in Q' : qA \leqslant S\}.$$

Note that, for q in Q',

$$q \in A^* \Leftrightarrow qA \leqslant S \Leftrightarrow AqA \leqslant A \Leftrightarrow Aq \leqslant S$$

because $O_l(A) = O_r(A) = S$ (Lemma 1·1). Moreover A^* is an S-ideal of Q'. Define $A^{**} = (A^*)^*$ and note that $A \leqslant A^{**}$. The S-ideal A is called *reflexive* provided $A^{**} = A$.

A multiplicatively closed subset T of a ring S is called an *Ore set* provided for any $s \in S$, $t \in T$ there exist $s_i \in S$, $t_i \in T$ $(i = 1, 2)$ such that

$$t_1 s = s_1 t \quad \text{and} \quad s t_2 = t s_2.$$

Let P be a semiprime ideal of a Noetherian prime ring S and define

$$\mathbb{C}_S(P) = \{c \in S : c + P \text{ is a regular element of the ring } S/P\}.$$

The ideal P is called *localizable* provided $\mathscr{C}_S(P)$ is an Ore set. In this case every element

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of $\mathscr{C}_S(P)$ is regular in S (see [11], lemmas 1·2 and 1·3) and we denote the partial (classical) quotient ring of S with respect of $\mathscr{C}_S(P)$ by S_P. Chamarie ([3], propositions 1·9 and 1·10) proved:

LEMMA 3·5. *Let S be a Noetherian maximal order in a simple Artinian ring Q'. Let $\mathscr{P}$ denote the collection of reflexive prime ideals of S. Then each P in $\mathscr{P}$ has height 1, is localizable and the ring S_P is a local Asano order. Moreover,*

$$S = (\bigcap_{P \in \mathscr{P}} S_P) \cap S_0,$$

where $S_0 = \{q \in Q' : qI \leqslant S$ for some non-zero ideal I of $S\}$.

Now consider Roseblade's work on prime ideals in [10]. Let G be a polycyclic-by-finite group. Let J be a ring and I an ideal of the group ring $J[G]$. Define

$$I^\dagger = G \cap (1 + I)$$

and call I *almost faithful* if $I^\dagger$ is finite. A subgroup H of G is called *orbital* if it has only finitely many conjugates in G and *isolated* if, for all x in G and positive integers n, $x^n \in H$ implies $x \in H$. The group G is called *orbitally sound* if all its isolated orbital subgroups are normal. Note that if G is orbitally sound and N is a normal subgroup of finite index in G then N is orbitally sound. Roseblade ([10], theorems C 1 and C 2) proved:

LEMMA 3·6. *Let J be a ring and G a polycyclic-by-finite group. Then*

(i) *G contains an orbitally sound characteristic subgroup N of finite index.*

(ii) *If P is an almost faithful prime ideal of the ring $J[N]$ then*

$$P = (P \cap J[\Delta(N)])\, J[N].$$

Now let J be a Noetherian maximal order in a simple Artinian ring and G an $\mathscr{X}$-group with essential centre Z. As we have already observed, $R = J[G]$ is an order in a simple Artinian ring Q. By ([9], lemma 10·2·5) and Lemma 3·6 there exists an orbitally sound characteristic subgroup N of finite index in G such that N is a poly-(infinite cyclic) group. Moreover, we can assume that N contains the Fitting radical F of G because F is a poly-(infinite cyclic) group by ([9], lemma 11·1·3) and F normalizes all isolated orbital subgroups of G by ([10], § 1·3). By [9] $Z \leqslant N$ and hence Z is the centre of N (Corollary 3·3).

Let $S = J[N]$. By ([2], proposition 2·2·1) the ring S is a Noetherian maximal order in a simple Artinian ring Q'. Let P be a reflexive prime ideal of S. Combining Lemma 3·2, Corollary 3·3 and Lemma 3·6(ii) we have:

$$P \cap J[Z] \neq 0. \tag{12}$$

LEMMA 3·7. *Let P be a reflexive prime ideal of S. Then $\mathscr{C}_S(P)$ is an Ore set of R and the partial quotient ring R_P of R with respect to $\mathscr{C}_S(P)$ is a local Asano order.*

Proof. Let $P_1 = P \cap J[Z]$. By (12) P_1 is a non-zero prime ideal of $J(Z)$. Now $RP_1 = P_1 R$ is an ideal of R and $SP_1 = P_1 S$ is an ideal of S. Moreover, the ring $S/(SP_1)$ is a crossed product of the ring $J[Z]/P_1$ with the group N/Z. But N/Z is an $\mathscr{X}$-group (Lemma 3·2) and hence by ([8], theorem 1·9) SP_1 is a non-zero prime ideal of S. But $P \geqslant SP_1$ and P has height 1 (Lemma 3·5). Thus

$$P = SP_1.$$

Similarly $RP = RP_1$ is a prime ideal of R.

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By Lemma 3·5 $\mathscr{C}_S(P)$ is an Ore set in S and by ([9], lemma 13·3·5) $\mathscr{C}_S(P)$ is an Ore set in R. Let S_P and R_P denote the corresponding partial quotient rings of S and R with respect to $\mathscr{C}_S(P)$. Note that

$$R_P P = R_P(RP)$$

and hence $R_P P$ is a prime ideal of R_P. Moreover, the ring $R_P/(R_P P)$ is a finitely generated module over the simple Artinian ring $S_P/(S_P P)$ and hence $R_P/(R_P P)$ is an Artinian ring. Also by ([9], theorem 7·2·5) the ideal $R_P P$ is contained in the Jacobson radical of R_P. It follows that $R_P P$ is the unique maximal ideal of R_P. Because S_P is a local Asano order there exists an element p in P such that

$$S_P P = S_P p = pS_P = PS_P.$$

It follows that

$$R_P p = R_P P = PR_P = pR_P.$$

Hence R_P is a local Asano order.

Proof of Theorem 3·4. We continue to use the same notation. Let $\mathscr{P}$ denote the collection of reflexive prime ideals of the ring S and

$$S_0 = \{q \in Q' : qI \leqslant S \text{ for some non-zero ideal } I \text{ of } S\}.$$

By Lemma 3·5,

$$S = \left(\bigcap_{P \in \mathscr{P}} S_P\right) \cap S_0. \tag{13}$$

Let A be an integral R-ideal of Q and let $q \in Q$ satisfy $qA \leqslant A$. Let T be a transversal to the cosets of N in G. Then

$$q = \sum_{t \in T} q_t t$$

for some elements $q_t \in Q'$ by ([9], lemma 13·3·5). By ([9], lemma 13·3·5), $A \cap S$ is an integral S-ideal of Q'. Moreover, $qA \leqslant R$ implies

$$q_t(A \cap S) \leqslant S \quad (t \in T)$$

and hence

$$q_t \in S_0 \quad (t \in T). \tag{14}$$

Now, for any P in $\mathscr{P}$, the partial quotient ring R_P of R with respect to $\mathscr{C}_S(P)$ is a local Asano order by Lemma 3·7. But $qA \leqslant A$ implies

$$qAR_P \leqslant AR_P$$

and hence $q \in R_P$. Clearly

$$R_P = \bigoplus_{t \in T} S_P t,$$

and hence

$$q_t \in S_P \quad (t \in T). \tag{15}$$

Since (15) holds for all P in $\mathscr{P}$, we can combine (13), (14), (15) to obtain

$$q_t \in S \quad (t \in T).$$

Thus $q = \Sigma_{t \in T} q_t t \in R$. It follows that $O_l(A) \leqslant R$. Similarly $O_r(A) \leqslant R$. By Lemma 1·1, R is a maximal order in Q. This completes the proof of Theorem 3·4.

We complete this section with two applications of Theorem 3·4.

COROLLARY 3·8. *Let J be a Noetherian maximal order in a simple Artinian ring and G an $\mathscr{X}$-group such that rank $[\Im(G), x] \neq 1$ for all x in G. Then the group ring $J[G]$ is a Noetherian maximal order in a simple Artinian ring.*

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Proof. Let $C = \mathbb{C}_G(\mathfrak{H}(G))$. By Lemma 3·1 we have $\mathfrak{H}(C) = \mathfrak{H}(G)$ and thus the $\mathfrak{X}$-group C has central Zalesskii subgroup. By (10) G has essential centre and by Theorem 3·4 the ring $J[C]$ is a maximal order. The result now follows by Corollary 2·4.

COROLLARY 3·9. *Let J be a Noetherian maximal order in a simple Artinian ring and G an $\mathfrak{X}$-group such that $G/\mathbb{C}_G(\mathfrak{H}(G))$ is a finite group of odd order. Then $R = J[G]$ is a Noetherian maximal order in a simple Artinian ring.*

Proof. Let $C = \mathbb{C}_G(\mathfrak{H}(G))$. By the proof of Corollary 3·8, $J[C]$ is a maximal order. Let $x \in G$ satisfy rank $[A, x] = 1$, where $A = \mathfrak{H}(G)$. Since x normalizes the infinite cyclic subgroup $[A, x]$ it follows that x^2 centralizes $[A, x]$. Because G/C has odd order we deduce that x centralizes $[A, x]$. There exists a positive integer k such that $x^k \in C$. Then, for any a in A,

$$1 = [a, x^k] = [a, x^{k-1}][a, x]^{x^{k-1}} = [a, x^{k-1}][a, x],$$

and by induction

$$1 = [a, x]^k.$$

Since A is torsion-free it follows that $[a, x] = 1$. Hence $[A, x] = 1$, a contradiction. Thus rank $[A, x] \neq 1$ for all x in G. By Corollary 3·8, $J[G]$ is a maximal order, as required.

4. *The main theorem*

In this section we combine the results of §§ 2 and 3 to prove Theorem 4·1, which clearly gives the Theorem in the Introduction as an immediate consequence. To highlight these results, note that if G is an $\mathfrak{X}$-group with Fitting radical F then G/F is Abelian-by-finite by a theorem of Mal'cev (see ([9], theorem 12·1·5)). If $A = \mathfrak{H}(G)$ then this means in particular that $G/\mathfrak{H}_G(A)$ is Abelian-by-finite.

THEOREM 4·1. *Let J be a Noetherian maximal order in a simple Artinian ring and G a polycyclic-by-finite group such that $G/\mathbb{C}_G(\mathfrak{H}(G))$ is Abelian. Then the group ring $J[G]$ is a Noetherian maximal order in a simple Artinian ring if and only if*

(i) *G contains no non-trivial finite normal subgroup, and*

(ii) *G contains no infinite dihedral normal subgroup.*

Proof. Suppose that $R = J[G]$ is a Noetherian maximal order in a simple Artinian ring. Then R is a prime ring ([5], theorem 13) and hence (i) holds by ([9], theorem 4·2·10). Also, by Lemma 1·3, (ii) holds.

Conversely, suppose that G satisfies (i) and (ii). Let $A = \mathfrak{H}(G)$ and $C = \mathbb{C}_G(A)$. Let T denote the normal subgroup of G containing C such that T/C is the torsion subgroup of the Abelian group G/C. Note that T/C is a finite group. Hence $\mathbb{Q} \otimes A$ is a completely reducible $\mathbb{Q}[T/C]$-module, and there exist a positive integer k and a direct product $B = A_1 \times \ldots \times A_k$ of normal subgroups A_i of T contained in A, such that T acts rationally irreducibly on A_i $(1 \leqslant i \leqslant k)$ and B has finite index in A. We claim:

$$\text{for any subgroup } C \leqslant H \leqslant T, \text{ the ring } J[H] \text{ is a maximal order in a simple Artinian ring.} \tag{16}$$

The proof of (16) is by induction on $|H:C|$. Suppose first that $|H:C| = 1$; that is $H = C$. By the proof of Corollary 3·8, $J[H]$ is a maximal order in a simple Artinian ring. Now suppose $|H:C| > 1$. Let $C_i = \mathbb{C}_H(A_i)$ $(1 \leqslant i \leqslant k)$. Note that $C \leqslant C_i \leqslant H$ $(1 \leqslant i \leqslant k)$. Suppose $1 \leqslant i < j \leqslant k$ and $C_i \neq H$, $C_j \neq H$. Then $|C_i:C| < |H:C|$ implies by induction

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that $J[C_i]$ is a maximal order. Similarly $J[C_j]$ is a maximal order. By Corollary 2·5 $J[H]$ is a maximal order in a simple Artinian ring. Thus we can suppose without loss of generality that $H = C_i$ $(2 \leqslant i \leqslant k)$. In addition we can suppose that rank $[A, x] = 1$ for some $x \in H$ by Corollary 3·8. We next prove that $C = C_1$. Let $h \in C_1$. Then $h \in \mathbb{C}_H(B)$. Because A/B is finite and A is torsion-free Abelian it follows that $h \in C$. Hence $C_1 \leqslant C$. It follows that $C = C_1$. Thus rank $[A_1, x] = 1$. Because T/C is an Abelian group it follows that $D = [A_1, x]$ is an infinite cyclic normal subgroup of T. But T acts rationally irreducibly on A_1, which means that A_1/D is a finite group. Since A_1 is free Abelian it follows that

$$A_1 \text{ is infinite cyclic.} \tag{17}$$

Let $g \in G$. Then A_1^g is a normal subgroup of T and $A_1^g \leqslant A$. If $A_1 \cap A_1^g = 1$, then by the theory of completely reducible modules we can suppose without loss of generality that $A_2 = A_1^g$. Then $H = C_2 = C_1^g$ and hence $H = C_1$ which implies $H = C$, a contradiction. Thus we can suppose

$$A_1 \cap A_1^g \neq 1 \quad (g \in G). \tag{18}$$

Let $A_1^G = \langle A_1^y; y \in G \rangle$. Note that A_1^G is a normal subgroup of G contained in A. Since G satisfies Max it follows that there exist a positive integer n and elements $y_i \in G$ $(1 \leqslant i \leqslant n)$ such that

$$A_1^G = A_1^{y_1} \dots A_1^{y_n}.$$

Consider the chain

$$A_1^G = A_1^{y_1} \dots A_1^{y_n} \geqslant A_1^{y_2} \dots A_1^{y_n} \geqslant \dots \geqslant A_1^{y_n}.$$

For each $1 \leqslant i \leqslant n-1$

$$(A_1^{y_i} \dots A_1^{y_n})/(A_1^{y_{i+1}} \dots A_1^{y_n}) \cong A_1^{y_i}/(A_1^{y_i} \cap (A_1^{y_{i+1}} \dots A_n^{y_n}))$$

which is finite by (17) and (18). Thus A_1^G/A_1 is finite. But A_1^G is a free Abelian group and hence by (17) A_1^G is cyclic. Because $A_1^G \cap (A_2 \times \dots \times A_n) = 1$, it follows that we can suppose without loss of generality that

$$A_1 \lhd G. \tag{19}$$

Clearly, we can further suppose that there is no infinite cyclic normal subgroup X of G such that $A_1 \leqslant X \leqslant A$.

The next step is to prove that G/A_1 has trivial finite radical. Let K be a normal subgroup of G such that $A_1 \leqslant K$ and K/A_1 is a finite group. By (17) and (19) we know that $|G : \mathbb{C}_G(A_1)| = 2$ and hence $|K : K \cap \mathbb{C}_G(A_1)| \leqslant 2$. Now $K \cap \mathbb{C}_G(A_1)$ is centre-by-finite, so that by Schur's Lemma ([9], lemma 4·1·4) and (i) it follows that $K \cap \mathbb{C}_G(A_1)$ is Abelian and, by (17), $K \cap \mathbb{C}_G(A_1)$ is infinite cyclic. There are two cases. Suppose first that K is torsion-free. Then K is infinite cyclic by a lemma of Formanek ([9], lemma 13·3·1). Thus K is contained in the Fitting radical F of G. Moevover, F centralizes A_1 and K/A_1 is finite together give $K \leqslant A$. Thus $K = A_1$. Now suppose that K is not torsion-free. Then there exists $y \in K$ such that $y^2 = 1$. In this case K is an infinite dihedral normal subgroup of G, a possibility excluded by (ii). Thus G/A_1 has trivial finite radical. It follows that H/A_1 has trivial finite radical. Moreover, $C_1 = \mathbb{C}_H(A_1) \neq H$, so that $J[C_1]$ is a maximal order by induction. By Lemma 2·6 $J[H]$ is a maximal order in a simple Artinian ring. This proves (16). In particular, $J[T]$ is a maximal order in a simple Artinian ring. By ([2], proposition 2·2·1) $J[G]$ is a maximal order in a simple Artinian ring. This completes the proof of Theorem 4·1.

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COROLLARY 4·2. *Let J be a Noetherian maximal order in a simple Artinian ring and G a polycyclic group which is nilpotent-by-Abelian. Then the group ring $J[G]$ is a maximal order in a simple Artinian ring if and only if*

(i) *G has no non-trivial finite normal subgroup, and*

(ii) *G has no infinite dihedral normal subgroup.*

The next two corollaries deal with torsion-free groups. The first is immediate from Theorem 4·1.

COROLLARY 4·3. *Let J be a Noetherian maximal order in a simple Artinian ring and G a torsion-free polycyclic-by-finite group such that $G/\mathbb{C}_G(\mathfrak{I}(G))$ is Abelian. Then the group ring $J[G]$ is a Noetherian maximal order in a simple Artinian ring.*

COROLLARY 4·4. *Let J be a Noetherian maximal order in a simple Artinian ring and G a nilpotent-by-Abelian group which is torsion-free. Then the group ring $J[G]$ is a maximal order.*

Proof. Note that we do *not* assume that G satisfies the maximal condition on subgroups. By Proposition 1·2, we can suppose without loss of generality that G is finitely generated. Let N be a nilpotent normal subgroup of G such that G/N is an Abelian group. Let H be the normal subgroup of G containing N such that H/N is the torsion subgroup of G/N. Because G is finitely generated, the group G/N is finitely generated and hence H/N is finite and G/H free Abelian of finite rank. By the proof of ([2], proposition 2·2·1), it is sufficient to prove that $J[H]$ is a maximal order in a simple Artinian ring. By ([9], lemma 11·1·3) and ([1], theorem A) $J[H]$ is an order in an Artinian ring Q and by ([5], theorem 13) and ([9] theorem 4·2·10) Q is simple. We again apply Proposition 1·2, so that we can assume without loss of generality that H is finitely generated. Now ([9], lemma 4·1·7) implies that N is finitely generated in this case and hence H is polycyclic-by-finite. By Corollary 4·3, $J[H]$ is a maximal order in Q.

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