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Betweenness and order in semigroups

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Introduction. A betweenness semigroup is a semigroup possessing a ternary relation, 'b lies between a and c', which is invariant under the semigroup operation. We take as our axioms of betweenness those suggested by Shepperd in (2) and given below in 1·01–1·04. It was shown in (2) that these axioms lead to just two types of betweenness; namely a linear betweenness corresponding to the order of points on an undirected straight line or a cyclic betweenness of just four points on a circle. In (3), Shepperd characterized betweenness groups and we here generalize his results to betweenness semigroups.

Semigroups whose betweenness is linear are investigated in section 2 by considering the effect of multiplication upon an underlying order. This naturally defines three separate classes of linear betweenness semigroup, one of which is the class of ordered semigroups. The other two classes are studied in sections 3 and 4 using the betweenness relations that can exist on an element and its powers. The possibility of a member of one of these classes being an ordered semigroup under a suitable reordering is considered at the end of the relevant sections. The Abelian members of one of these classes were studied by Clifford in (1) and his notation is adopted.

In section 5 all the distinct betweenness semigroups whose betweenness is cyclic are calculated and displayed.

1. *Preliminaries.* The semigroups are written multiplicatively and the two-sided unit, where it exists, is denoted by 1. The relation 'b lies between a and c' is written $[a, b, c]$ and satisfies the axioms of betweenness:

1·01. For any a, b, c, either $[a, b, c]$ or $[b, c, a]$ or $[c, a, b]$.

1·02. $[a, b, c]$ and $[a, c, b]$ if, and only if, $b = c$.

1·03. $[a, b, c]$ implies $[c, b, a]$.

1·04. $[a, b, c]$ and $[a, c, d]$ implies $[b, c, d]$.

It follows that

1·05. $[a, b, a]$ if, and only if, $a = b$.

A set possessing a ternary relation satisfying 1·01–1·04 is a *betweenness set*. If in addition the set has a semi-group structure and satisfies:

1·06. $[a, b, c]$ implies $[ga, gb, gc]$ and $[ag, bg, cg]$ for all g in the set; then the set is a *betweenness semigroup*.

The order relation 'a is less than or equal to b' is written $a < b$ and the axioms of order are:

1·11. For any a, b either $a < b$ or $b < a$.

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1·12. $a < b$ and $b < a$ if, and only if, $a = b$.

1·13. $a < b$ and $b < c$ implies $a < c$.

A set possessing a binary relation satisfying 1·11–1·13 is an *ordered set*. If in addition the set has a semigroup structure and satisfies:

1·14. $a < b$ implies $ga < gb$ and $ag < bg$ for all g in the set; then the set is an *ordered semigroup*.

As shown in (2) an ordered set can be made into a betweenness set by defining:

1·21. $[a, b, c]$ if, and only if, either $a < b < c$ or $c < b < a$.

If the ordered set is an ordered semigroup then 1·21 constructs a betweenness semigroup as then 1·06 follows from 1·14. A betweenness relation arising in this way from an order relation is said to be *linear* and is *induced by the order relation*.

The ${}_nC_3$ induced betweenness relations on the ordered set $(a_1, a_2, \ldots, a_n)$ are summarized by the notation $[a_1, a_2, \ldots, a_n]$ and the notation $\mathrm{Cy}[a, b, c, d]$ is used for the cyclic betweenness on four elements, $[a, b, c]$, $[b, c, d]$, $[c, d, a]$ and $[d, a, b]$.

It has been shown (2) that:

1·3. A betweenness set either has linear betweenness or has cyclic betweenness and contains just four elements.

It can be shown that a linear betweenness set is characterized by axioms 1·01–1·04, together with

1·31. $[a, b, c]$, $[b, c, d]$ and $b \neq c$ implies $[a, b, d]$.

In the corresponding characterization of a cyclic betweenness set, 1·31 is replaced by:

1·32. $[a, b, c]$, $[b, c, d]$ and $b \neq c$ implies $[d, a, b]$.

A betweenness semigroup in which the betweenness is linear (cyclic) will be called a *linear* (*cyclic*) *betweenness semigroup*.

2. *Linear betweenness semigroups.* Let T be a linear betweenness semigroup in which the order relation $<$ induces the betweenness. For such semigroups it is the properties associated with the order relation that are to be investigated. However, at such points where $a < b < c$ and $c < b < a$ would need to be discussed as separate but similar cases, we will use the betweenness notation $[a, b, c]$. We make the following definitions:

2·01. *If $ga < gb$ for all $a, b \in T$ such that $a < b$ then g is a left conserver.*

2·02. *If $ga < gb$ for all $a, b \in T$ such that $b < a$ then g is a left inverter.*

2·03. *If $gx = g^2$ for all $x \in T$ then g is a left null.*

If g is both a left conserver and a left inverter then by 1·12 g is a left null.

2·04. *A left conserver* (*inverter*) *that is not a left null is a proper left conserver* (*inverter*).

2·05. *If g is a left null and $g^2 = g$ then g is a left zero.*

Five more terms are defined upon substituting right for left and commuting all products throughout 2·01–2·05.

In the following lemmas proof will be offered only for left multiplication.

2·1. LEMMA. *Any element of T is either a proper left* (*right*) *conserver, a proper left* (*right*) *inverter or a left* (*right*) *null.*

Proof. If g is not a left null there exists $a, b \in T$ such that $a < b$ and $ga \neq gb$. If $x, y \in T$ and $x < y$, then by considering all possible orders on the set $\{a, b, x, y\}$

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consistent with $a < b$ and $x < y$, and their respective induced betweennesses, either

$$[a,b,x,y] \quad \text{implying} \quad [ga,gb,gx,gy],$$

or $$[a,x,b,y] \quad \text{implying} \quad [ga,gx,gb,gy],$$

or $$[a,x,y,b] \quad \text{implying} \quad [ga,gx,gy,gb],$$

or $$[x,a,b,y] \quad \text{implying} \quad [gx,ga,gb,gy],$$

or $$[x,a,y,b] \quad \text{implying} \quad [gx,ga,gy,gb],$$

or $$[x,y,a,b] \quad \text{implying} \quad [gx,gy,ga,gb],$$

the implication following from 1·06. Thus in each case, $ga < gb$ implies $gx < gy$ and $ga > gb$ implies $gx > gy$ by 1·21. Thus, as x, y are any elements of T such that $x < y$, $ga < gb$ implies g is a proper left conserver and $ga > gb$ implies g is a proper left inverter.

By considering the effect of successive multiplications upon the order relation between any two elements and appealing to the associative law it is easily seen that:

2·2. The set of left (right) conservers is a subsemigroup containing the subsemigroup of all left (right) nulls.

2·21. The set of all two-sided conservers is an ordered subsemigroup containing the ordered subsemigroup of all two-sided nulls.

2·3. The product of two left (right) inverters is a left (right) conserver and the product of a left (right) conserver and a left (right) inverter is a left (right) inverter.

We observe that an ordered semigroup is a semigroup on a linearly ordered set such that all elements are two-sided conservers, and following Clifford (1) we define:

2·41. *An ordered semigroup of the second kind as a semigroup on a linearly ordered set such that each element is either a two-sided conserver or a two-sided inverter and which contains an element that properly inverts on at least one side* (a generalization of Clifford's Abelian semigroups of the same name studied in (1)), and we define:

2·42. *An ordered semigroup of the third kind as a linear betweenness semigroup which contains an element that properly inverts an order inducing the betweenness on one side and properly conserves it on the other.*

We have now shown that:

2·5. THEOREM. *A linear betweenness semigroup must either be*

(1) *an ordered semigroup,*

or (2) *an ordered semigroup of the second kind,*

or (3) *an ordered semigroup of the third kind.*

In order to study the last two classes above we introduce the concept of convexity and two preparatory lemmas.

2·6. *A subset X of a betweenness set S is convex if $a, b \in X$, $s \in S$ and $[a,s,b]$ implies $s \in X$.*

It follows immediately that

2·61. The intersection of two convex sets is convex.

2·7. LEMMA. *If a linear betweenness set S consists of the non-intersecting subsets A, B, C where B is convex or empty and A, C are non-empty and if, for $a \in A$, $c \in C$ and $x, y \in S$, $[x,a,c]$ implies $x \in A$ and $[a,c,y]$ implies $y \in C$, then A and C are convex and B lies between them.*

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Proof. Let $c \in C, x, y \in A$ and $s \in S$ such that $[x, s, y]$, then either $[c, x, s, y]$ or $[x, s, y, c]$ implying $s \in A$ and A is convex. Similarly, C is convex.

Clearly $[a, b, c]$ for any $a \in A, b \in B$ and $c \in C$ as otherwise $b \in B$ is contradicted. Hence B lies between A and C.

Henceforth, for the phrase 'the order relation of a to b' we will write $a:b$ and in the phrase 'y inverts (or conserves) $a:b$ by multiplication on the right (or left)' the words 'by multiplication' will be omitted.

2·8. LEMMA. *If a linear betweenness semigroup T contains a left (right) zero, then the set of right (left) nulls is convex. If also both the set of proper right (left) conservers and the set of proper right (left) inverters are not empty, then they are both convex and the set of right (left) nulls lies between them.*

Proof. Let w be a left zero and u, v be right nulls. Then $bu = wu = w$, $bv = wv = w$ for all $b \in T$. Now let x be such that $[u, x, v]$. Then $[bu, bx, bv]$, implying $bx = w$ for all $b \in T$. Hence the set of right nulls is convex.

Let y be a proper right conserver and z a proper right inverter. Then there exists $a, b \in T$ such that $a:w$ is properly conserved by y on the right and $b:w$ is properly inverted by z on the right.

We may assume that either: (a) $a < b < w$, or (b) $b < a < w$, or (c) $a < w < b$. Case (a) gives $az > bz > w$ and as $bz \neq w$, z properly inverts $a:w$ on the right and similarly in case (b), y properly conserves $b:w$ on the right. Thus in (a) and (b) we assume $a = b$ and so $ay < w < az$. If $c \in T$ and $[c, y, z]$ then $[ac, ay, az]$ whence $ac < ay < w$ and as $ay \neq w$, c is a proper right conserver. Similarly, if $d \in T$ and $[y, z, d]$ then d is a proper right inverter.

In case (c) we may assume that $az = by = w$ as otherwise we have the previous case. If $c \in T$ and $[c, y, z]$ then $[ac, ay, w]$, that is $ac < ay < w$ and as $ay \neq w$, c is a proper right conserver. Similarly, if $d \in T$ and $[y, z, d]$ then d is a proper right inverter.

The result now follows by 2·7 in all cases.

This result is a generalization of Lemma 2 in (1) and as remarked therein does not necessarily hold if either the set of proper right inverters or the set of proper right conservers is empty.

3. *Ordered semigroups of the second kind.* We let S be an ordered semigroup of the second kind and define the following subsets of S:

3·01. P, *the set of elements that properly conserve on at least one side, called conservers,* and

3·02. Q, *the set of elements that properly invert on at least one side, called inverters,* and

3·03. N, *the set of two-sided nulls.*

Now Q is not empty by 2·41 and by 2·2, 2·21 and 2·3:

$$P + N \text{ and } N \text{ are ordered subsemigroups,}$$

$$PQ \subseteq Q + N, \quad QP \subseteq Q + N \text{ and}$$

$$Q^2 \subseteq P + N, \quad \text{thus} \quad P + N \text{ is not empty.}$$

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The following results are easily verified:

3·11. *N is not empty if, and only if, S contains a two-sided zero w.*

3·12. *N is not empty if, and only if, S contains a right null and a left null.*

3·13. *$n \in N$ if, and only if, $nS = Sn = \{w\}$.*

3·14. *If N is not empty, the set of right nulls, the set of left nulls and N are all convex* (by 3·11, 2·8 and 2·61).

We now consider the convexity of P and Q.

3·2. LEMMA. *If P is not empty, P and Q are convex and N lies between them.*

Proof. Let $p \in P$, $q \in Q$ and, without loss of generality, $q < p$. We will now show that $a \in S$, $a < q < p$ implies $a \in Q$.

Now $ap < qp < q^2 < aq$ and $pa < pq < q^2 < qa$ and so if either $ap \neq aq$ or $pa \neq qa$ then a will properly invert $q:p$ on one side and thus be in Q. Otherwise

$$ap = pa = pq = qp = aq = qa = q^2.$$

In this case let us assume, without loss of generality, that q properly inverts on the left. Thus either

(a) there exists $r < a$ such that $q^2 < rq$ and $q^2 \neq rq$;
or (b) there exists $s > p$ such that $q^2 > sq$ and $q^2 \neq sq$;
as $tq = q^2$ for all t such that $a < t < p$.

In case (a) $rp < ap = q^2 < rq$, $q^2 \neq rq$ and so $r \in Q$ as r properly inverts $p:q$ on the left. Thus $ra > rq > q^2 = qa$, $q^2 \neq rq$ and so $a \in Q$ as it properly inverts $r:q$ on the right.

In case (b) $sp > p^2 > qp = q^2 > sq$, $q^2 \neq sq$ and so $s \in P$ as s properly conserves $p:q$ on the left. Thus $sa < sq < q^2 = qa$, $sq \neq q^2$ and so $a \in Q$ as it properly inverts $s:q$ on the right.

Similarly, it can be shown that, if $b \in S$ and $q < p < b$, then $b \in P$. Hence by 2·7 P and Q are convex and N if it exists, lies between them.

The following example shows an ordered semigroup of the second kind in which Q is not convex and P is, necessarily, empty

	f	g^2	w	f^2	g
f	f^2	w	w	w	w
g^2	w	w	w	w	w
w	w	w	w	w	w
f^2	w	w	w	w	w
g	w	w	w	w	g^2

where $[f, g^2, w, f^2, g]$, $N = \{g^2, w, f^2\}$ and $Q = \{f, g\}$. However, when P is empty it is possible to make S an ordered semigroup as follows:

3·3. LEMMA. *If P is empty, then S is an ordered semigroup under the order α defined by $a < b$ implies $a \alpha b$ if a and $b \in N$, otherwise $a < b$ implies $b \alpha a$.*

Proof. By 2·3 $Q^2 \subseteq N$ and so N is not empty and $NS = SN = \{w\}$ where w is the two-sided zero. Thus all products are in N.

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If $a\alpha b$ and $b\alpha c$ then $a\alpha c$ follows from the transitivity of $<$ if either two or none of a, b and c lie in Q. Otherwise either

(a) $a, b \in N$ and $c \in Q$;

or (b) $b, c \in N$ and $a \in Q$, as N is convex.

In case (a), $a < b$ and $c < b$ and so $c < a < b$ as N is convex, hence $a\alpha c$. In case (b), $b < a$ and $b < c$ and so $b < c < a$ as N is convex, hence $a\alpha c$. Thus α satisfies the order transitivity 1·13 and also satisfies 1·11 and 1·12. Also $a\alpha b$ implies $ag\alpha bg$ and $ga\alpha gb$ for all $g \in S$. For if either $g \in N$ or a and $b \in N$ then $ag = bg = ga = gb = w$, otherwise $g \in Q$ and either $a \in Q$ or $b \in Q$ whence $b < a$ and so $ag < bg$ and $ga < gb$, thus $ag\alpha bg$ and $ga\alpha gb$ as all products are in N.

Corresponding to the three subclasses of linear betweenness semigroups we may define subclasses of linear betweenness groups and deduce from (3) that, for a linear betweenness group G:

3·41. G is either an ordered group or an ordered group of the second kind.

3·42. In G every element properly inverts or properly conserves the order relation between all pairs of elements on both sides.

3·43. If G is an ordered group of the second kind then G is locally infinite except possibly for an element of order two in Q. The subset P is a convex normal subgroup of index two in G and is an ordered group under an order inducing the betweenness of G. The other coset Q is also convex.

3·44. If G is a locally infinite ordered group of the second kind and $<$ is an ordering inducing the betweenness, then under the order α defined by $g\alpha k$ if, and only if, $(gk^{-1})^2 < 1$, G is an ordered group.

In (3) the sets P and Q are labelled H and F, respectively, and are defined in terms of the betweenness relation on the set $\{1, x, x^2\}$ for each $x \in G$. These definitions are extended to semigroups in 3·5. As will be shown in 3·63 $F = Q$ and $H = P$ for groups but for semigroups $F \subseteq Q$ and $H \supseteq P + N$ is all that can be said.

In a linear betweenness semigroup T, we define

3·5.
$$H = \{h \in T \mid [1, h, h^2]\} \quad \text{if} \quad 1 \in T$$
$$= \{h \in T \mid [h, h^2, h^3]\} \quad \text{if} \quad 1 \notin T,$$
$$F = T - H.$$

For any $x \in T$ either:

$[x, x^2, x^3]$ implying $[x, x^2, x^3, x^4, \ldots]$,

or $[x, x^3, x^2]$ implying $[x, x^3, x^5, \ldots, x^6, x^4, x^2]$,

or $[x^3, x, x^2]$ implying $[\ldots, x^5, x^3, x, x^2, x^4, x^6, \ldots]$

and if $1 \in T$ then either

$[1, x, x^2]$ implying $[1, x, x^2, x^3, x^4, \ldots]$,

or $[x, x^2, 1]$ implying $[x, x^3, x^5, \ldots, x^4, x^2, 1]$,

or $[x, 1, x^2]$ implying $[\ldots, x^5, x^3, x, 1, x^2, x^4, \ldots]$.

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Thus

3·51. If 1 exists $x = x^2$ implies $x \in H$ and if 1 does not exist $x^2 = x^3$ implies $x \in H$.

3·52. For all $x \in T$, $x^2 \in H$.

3·53. If $[y, y^2, y^3]$ and 1 exists then y is not necessarily in H. For if $[1, y^2, y]$ and $y \neq y^2 = y^3$ then $y \in F$ and $[y, y^2, y^3]$.

We now prove some results concerning the sets F, Q and $H \cap Q$.

3·61. LEMMA. *The set F consists of two-sided inverters.*

Proof. If $f \in F$ then either $[f, f^3, f^2]$ or $[f^3, f, f^2]$ and $f^2 \neq f^3$ if $1 \notin T$. Thus either $f > f^2$, $f^3 > f^2$ or $f < f^2$, $f^3 > f^2$ in both cases and so f is a proper two-sided inverter. The proof when T contains 1 follows similarly. Thus $F \subseteq Q$ and so $H \supseteq P + N$.

3·62. LEMMA. *If $H \cap Q$ is not empty, then $1 \notin T$ and there exists a two-sided zero w say and $(H \cap Q)^2 = \{w\}$.*

Proof. If $1 \in T$ and $h \in H \cap Q$, we assume, without loss of generality, that $1 < h$ hence $h^2 < h$ as $h \in Q$. Thus either $[1, h^2, h]$ or $[h^2, 1, h]$; but $[1, h, h^2]$ as $h \in H$, and so, in the latter case, $h = 1$ by 1·02 contradicting $h \in Q$, and in the former case, $h = h^2$ by 1·02. Now as h is a two-sided inverter, h^2 is a two-sided conserver by 3·03, thus $h^2 = h$ implies $h \in N$ contradicting $h \in Q$. Thus $H \cap Q$ is empty if $1 \in T$.

If $1 \notin T$ and $h \in H \cap Q$ then we can show that $h^2 = h^3$ and therefore $h^2 \in N$, $(h^2)^2 = h^2$ and so h^2 is a two-sided zero. As a two-sided zero is unique, $h^2 = w$ for all $h \in H \cap Q$.

If $x, y \in H \cap Q$ then without loss of generality either $x < y < w$ or $x < w < y$. In the first case multiplication by x gives $xy = yx = w$ as $x^2 = wx = xw = w$ and in the second case $x < w$ implies $w < xy$ and $w < y$ implies $xy < w$ as $x, y \in Q$, hence $xy = w$. Thus $(H \cap Q)^2 = \{w\}$.

3·63. LEMMA. *If $x \in H$ and $y \in S$ such that $[x, y, x^2]$ then $y \in H$; if $z \in S$ such that $[1, z, x]$ then $z \in H$.*

Proof. From $[x, y, x^2]$ we have $[x^2, xy, x^3]$, $[x, x^2y, x^4]$ and $[xy, y^2, x^2y]$ and so $[x, y, x^2, y^2, x^4]$. If $[x^2, y^2, x^3]$ then $[x^3, xy^2, x^4]$, $[x^4, x^2, y^2, x^5]$ and if $[x^3, y^2, x^4]$ then $[x^4, xy^2, x^5]$, $[x^5, x^2y^2, x^6]$. Now as $[xy^2, y^3, x^2y^2]$ we have, in both cases, $[x, y, x^2, y^2, y^3]$, and $[1, x, y, x^2, y^2, y^3]$ if $1 \in S$, thus $y \in H$.

From $[1, z, x]$ we have $[1, z, x, x^2]$, $[x, xz, x^2]$ and $[z, z^2, xz]$ and so $[1, z, z^2]$ and $z \in H$.

3·64. LEMMA. *The subset H is convex.*

Proof. Let $g, h \in H$ and $x \in S$ such that $[g, x, h]$. If $1 \in S$ then either $[h, x, 1]$ or $[1, x, g]$ and $x \in H$ by 3·63. Thus we assume $1 \notin S$, $x \neq x^2$ and, without loss of generality, $[h, x, x^2]$, as otherwise $[x^2, x, g]$. Now $[h, x, x^2]$ implies $[xh, x^2, x]$ and either

(*a*) $[xh, x, x^2]$,
or (*b*) $[x, xh, x^2]$, $xh \neq x^2$,
or (*c*) $[x, x^2, xh]$.

In cases (*a*) and (*b*) $[x, x^2, x^3]$ and $x \in H$. In case (*c*) we have $[h, x, x^2, xh]$. Now $[h, x, h^2]$ implies $x \in H$ by 3·63 and so we assume either

(*d*) $[h, h^2, x, x^2, xh]$,
or (*e*) $[h^2, h, x, x^2, xh]$.

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In case (d), $[xh, xh^2, x^2, x^3, x^2h]$; but $[x, x^2, xh]$ implies $[xh, x^2h, xh^2]$ and so

$$x^2h = x^3 = x^2 = xh^2,$$

thus $x \in H$. In case (e), $[xh^2, xh, x^2, x^3, x^2h]$; but again $[xh, x^2h, xh^2]$ and so

$$xh = x^2 = x^3 = x^2h$$

and so $x \in H$.

3·65. LEMMA. *The set H is a subsemigroup.*

Proof. By 3·61 $F \subseteq Q$ and so $H = P + N + (H \cap Q)$. As $P + N$ is a subsemigroup by 2·2 we assume that $H \cap Q$ is not empty and so, by 3·62, N is not empty and contains a two-sided zero, w say, $(H \cap Q)(H \cap Q) = \{w\}$ and $1 \notin S$. By 3·13 $N(H \cap Q) = \{w\}$ and $(H \cap Q)N = \{w\}$ and so it is sufficient to show that $P(H \cap Q) \subseteq H$ and $(H \cap Q)P \subseteq H$.

Let $x \in P$ and $y \in H \cap Q$, then $y^2 = w$ by 3·62. As $y \in Q$ it inverts $x : w$ on both sides and as $yw = wy = w$, we have $[x, w, xy]$ and $[x, w, yx]$. Therefore either $[w, xy, yx]$ or $[w, yx, xy]$ as $x \neq w$. If $[w, xy, yx]$ then $[w, yxy, y^2x]$ and so $yxy = w$ as $y^2 = w$. Thus $(xy)^2 = (xy)^3 = w = (yx)^2 = (yx)^3$ and so $xy, yx \in H$. Similarly, $xy, yx \in H$ if $[w, yx, xy]$.

Summarizing some of the above results:

3·66. THEOREM. *In an ordered semigroup S of the second kind,*

(a) the set H is a convex subsemigroup containing the ordered subsemigroups $P + N$ and N,

(b) if Q (or F) is not convex, P is empty and S can be made an ordered semigroup,

(c) in general $H \supseteq P + N$, $F \subseteq Q$ and we have equality if either $1 \in S$ or $q^2 \neq q^3$ for all $q \in Q$.

We will now examine the problem of re-ordering S in such a way as to make it an ordered semigroup.

If G is an Abelian locally infinite betweenness group with an order $<$ inducing the betweenness then by 3·44 G is an ordered group under an order α where, for any $g, k \in G$, $g \alpha k$ if, and only if, $g^2 < k^2$.

The condition that G should be a locally infinite group can be relaxed.

3·71. THEOREM. *If S is a cancellative Abelian ordered semigroup of the second kind, then S can be re-ordered to form an ordered semigroup if, and only if, the squares of distinct elements are distinct.*

Proof. Let the squares of distinct elements be distinct. If $<$ is an order inducing the betweenness of S we define a relation α by $g \alpha k$ if, and only if, $g^2 < k^2$ for all $g, k \in S$. It is easily shown that α is an order relation, that is it satisfies 1·11–1·13. Also 1·14 is satisfied for if $a \alpha b$ then $a^2 < b^2$ and so $g^2a^2 < g^2b^2$ for all $g \in S$ as all squares are conservers by 2·2 and 2·3. Therefore if $a \neq b$, $g^2a^2 \neq g^2b^2$ as S is cancellative and so $ga \alpha gb$ for all $g \in S$. Hence under α, S is an ordered semigroup.

It is necessary that distinct elements have distinct squares for if β is an order making S an ordered semigroup and there exist $a, b \in S$ such that $a \neq b$, $a^2 = b^2$ and $a \beta b$ then $a^2 \beta ab$ and $ab \beta b^2$ and so $a^2 = ab$ contradicting S cancellative.

An example is the set of positive and negative integers excluding 0. Under multiplication the natural order makes it an ordered semigroup of the second kind and it

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cannot be made an ordered semigroup as $(-1)^2 = 1^2$. Under addition with 0 added it forms an ordered semigroup.

As in the proof of necessity above it is easily seen that:

3·72. If S can be made an ordered semigroup and there exist $a, b \in S$ such that $a^2 = b^2$ then $a^2 = ab = ba$.

It is conjectured that:

3·73. The condition that S should be Abelian in Theorem 3·71 can be dropped.

4. *Ordered semigroups of the third kind.* By Definition 2·43, for any ordered semigroup of the third kind either it, or an anti-isomorph of it, possesses an element that is a proper left inverter and a proper right conserver. Therefore throughout this chapter we define

4·0. DEFINITION. *S is an ordered semigroup of the third kind containing an element x that is a proper left inverter and a proper right conserver.*

4·1. LEMMA. *No two-sided unit exists in S, $x \neq x^2$ and x^2 is a left zero.*

Proof. By 2·3 x^2 is a left conserver and so $x \neq x^2$ as x is a proper left inverter. If $1 \in S$, without loss of generality, let $1 < x$ giving $x < x^2$ and $x > x^2$ on right and left multiplication by x, respectively, contradicting $x \neq x^2$, thus $1 \notin S$. Similarly, by considering $x : x^2$ it can be shown that $x^2 = x^3$. As x^2 is a left conserver and x a left inverter by 2·3, x^2 is a left null and as $x^2 x = x^2$ it is a left zero.

4·2. LEMMA. *There exists $a \in S$ such that either $[a, x^2, x, xa]$ or $[a, x^2, xa, x]$.*

Proof. As x is a proper left inverter there exists $a \in S$ such that $a : x$ is properly inverted by x on the left. Now either $[a, x^2, x]$ or $[a, x, x^2]$ as $[x, a, x^2]$ implies $xa = x^2$. If $[a, x, x^2]$ then $[a, x, x^2, xa]$ and $x^2 a = x^2 \neq xa$ and so a is a proper right inverter. Thus xa is a proper right inverter by 2·3 contradicting 2·8 as x^2 is a left zero and yet the set of right inverters is not convex. Thus $[a, x^2, x]$ and so either $[a, x^2, xa, x]$ or $[a, x^2, x, xa]$. An example of each is shown below.

4·21. *Example.*

	a	ax	x^2	xa	x
a	a	ax	ax	ax	ax
ax	ax	ax	ax	ax	ax
x^2	x^2	x^2	x^2	x^2	x^2
xa	xa	x^2	x^2	x^2	x^2
x	xa	x^2	x^2	x^2	x^2

where $[a, ax, x^2, xa, x]$.

4·22. *Example.*

	a	x^2	x	xa
a	a	a	a	a
x^2	x^2	x^2	x^2	x^2
x	xa	x^2	x^2	x^2
xa	xa	xa	xa	xa

where $[a, x^2, x, xa]$.

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4·3. LEMMA. *All elements of S are right conservers.*

Proof. By 4·2 there exists $a \in S$ such that either $[a, x^2, x, xa]$ or $[a, x^2, xa, x]$. As $x^2a = x^2 \neq xa$, a is a proper right conserver as is x, thus by 2·8, x^2 is a proper right conserver if a proper right inverter exists. Let f be a proper right inverter, then as x inverts $f:x^2$ on the left, $[xf, x^2, f]$. However, x^2 is a proper right conserver and xf is a right inverter by 2·3, thus the set of right inverters is not convex, contradicting 2·8. Thus a proper right inverter cannot exist. Thus each element of S is a right conserver, either proper or null.

The set of proper left inverters (conservers) is not necessarily convex as is shown by omitting d, e, f and j from example 4·31 below and defining the betweenness as $[a, b, c, h, g, i]$ so that c and i are left inverters and a and h are left conservers.

Also if $y \in S$, $y \neq x$ and y is a proper left inverter and a proper right conserver then there are just five betweennesses possible on $\{x, x^2, y, y^2\}$ that are distinct up to interchanging x and y. For, of the twelve possible betweennesses, only eight are distinct up to interchanging x and y. Also, as $[y^2, x^2, y]$ gives $x^2 = y^2$ on multiplying on the right by y, two of the eight betweennesses contradict $x^2 \neq x$ and two more are found to be the same on interchanging x and y. The five remaining betweennesses are $[y^2, y, x^2, x]$, $[y, y^2, x^2, x]$, $[x^2, x, y, y^2]$, where all elements are distinct, and $[x^2, y, x]$ and $[y, x^2, x]$, where $x^2 = y^2$. The simultaneous existence of these five possibilities is exhibited in example 4·31 where c, d, f, h and i are left inverters and the betweenness is $[a, b, c, d, e, f, g, h, i, j]$.

4·31. *Example.*

	a	b	c	d	e	f	g	h	i	j
a	a	b	b	b	b	b	b	b	b	b
b	b	b	b	b	b	b	b	b	b	b
c	c	b	b	b	b	b	b	b	b	b
d	e	e	e	e	e	e	e	e	e	b
e	e	e	e	e	e	e	e	e	e	e
f	g	g	g	g	g	g	g	g	g	e
g	g	g	g	g	g	g	g	g	g	g
h	h	g	g	g	g	g	g	g	g	g
i	i	g	g	g	g	g	g	g	g	g
j	j	j	j	j	j	j	j	j	j	j

4·4. THEOREM. *If $a \in S$ and $a:x$ is properly inverted by x on the left, and if α is an order under which S is an ordered semigroup such that $x\alpha x^2$, then either $a\alpha x\alpha x\alpha a\alpha x^2$ or $a\alpha x a\alpha x\alpha x^2$.*

Proof. As $xa\alpha a x^2$ implies $xa = x^2$, and as $xax^2\alpha a$ implies $x^2\alpha xa$ and $xa\alpha x^2$, that is $xa = x^2$, then $a\alpha x\alpha x^2$ as $xa \neq x^2$. As $xa\alpha a$ implies $x^2\alpha xa$ implying $xa = x^2$, then $a\alpha x a$. Hence the alternatives stated in the theorem.

4·41. Any order on S that induces a linear betweenness must retain the convexity of the sets $\{y|by = by_0\}$, $\{y|yb = y_0b\}$, for all b and y_0, by 1·05.

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4·42. Thus a necessary condition that S can be made an ordered semigroup is that under at least one of the orders of 4·4 the sets of 4·41 shall retain their convexity.

A procedure to find, if possible, an order under which S is an ordered semigroup would therefore be first to list all the convex sets of 4·41 and so find all possible orders preserving the convexity of these sets. We then examine these to find if any make S an ordered semigroup.

Applying this procedure to example 4·21 we find that the necessarily convex sets are $\{ax, x^2, xa, x\}$, $\{xa, x\}$, $\{a, ax\}$ and $\{x^2, xa, x\}$. Thus the possible new orders are a, ax, x^2, x, xa, a, ax, xa, x, x^2, and a, ax, x, xa, x^2. The first order makes the example an ordered semigroup of the third kind, while the other orders make it an ordered semigroup and illustrate the two possibilities of 4·4. Similarly, the only possible new order of example 4·22 is x^2, x, xa, a which makes the example an ordered semigroup. The addition of just one element to this last example produces an ordered semigroup of the third kind for which no other order conserves the convexity of the sets 4·41 and so it cannot be made an ordered semigroup. The semigroup is

	a	x^2	x	xa	c
a	a	a	a	a	a
x^2	x^2	x^2	x^2	x^2	x^2
x	xa	x^2	x^2	x^2	x^2
xa	xa	xa	xa	xa	xa
c	xa	xa	xa	xa	c

This concludes the discussion of ordered semigroups of the third kind.

5. *Cyclic betweenness semigroups.* As described in section 1 a cyclic betweenness semigroup consists of just four elements with a cyclic betweenness relation preserved under left and right multiplication. Throughout this section we let S be a cyclic betweenness semigroup with elements a, b, c, d and betweenness $\mathrm{Cy}[a, b, c, d]$.

5·1. THEOREM. *Up to anti-isomorphism, there are nineteen distinct cyclic betweenness semigroups, their multiplication tables* (1–19) *being*:

(1)

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(2)

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	d	c
d	d	c	c	d

(3)

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	b	a	d	c
d	a	b	c	d

(4)

	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	b	a	d
d	d	b	b	d

(5)

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	d	a	b
d	c	d	a	b

(6)

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

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(7)

	a	b	c	d
a	a	b	c	d
b	b	d	d	b
c	c	d	a	b
d	d	b	b	d

(8)

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	a	b	c	d
d	a	b	c	d

(9)

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	c
d	c	c	c	c

(10)

	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

(11)

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	a	a

(12)

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	b	b

(13)

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	c	c
d	a	a	c	c

(14)

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	c	c	c	c
d	c	c	c	c

(15)

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	d	d
d	a	a	d	d

(16)

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	d	d	d	d
d	d	d	d	d

(17)

	a	b	c	d
a	a	a	d	d
b	b	b	c	c
c	b	b	c	c
d	a	a	d	d

(18)

	a	b	c	d
a	a	a	c	c
b	a	a	c	c
c	c	c	a	a
d	c	c	a	a

(19)

	a	b	c	d
a	a	a	d	d
b	a	a	d	d
c	d	d	a	a
d	d	d	a	a

the betweenness being Cy $[a, b, c, d]$ *in each semigroup.*

We begin the proof with some definitions and lemmas.

5·11. *Two betweenness semigroups isomorphic under a betweenness preserving isomorphism are said to be B-isomorphic.*

5·12. *An element which can always be cancelled on the left (right) will be called left (right) cancellable.*

5·13. *A left (right) cancellable element that is not a left (right) unit will be called proper left (right) cancellable.*

5·14. *If x, y and z are different elements of S and $[x, y, z]$ then we will say x and z are separate and x and y are adjacent.*

We define zeros and nulls as in 2·03 and 2·05, respectively.

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5·2. LEMMA. *If $x \in S$ is neither left cancellable nor left null, then either $xa = xb \neq xc = xd$ or $xd = xa \neq xb = xc$. If x is neither right cancellable nor right null, then either $ax = bx \neq xc = xd$ or $dx = ax \neq bx = cx$.*

Proof. As Cy $[xa, xb, xc, xd]$ and x is not left null $xa \neq xc$ and $xb \neq xd$. As x is not left cancellable either $xa = xb$, $xa = xd$, $xb = xc$ or $xc = xd$. Now $xa = xb$ implies $[xa, xc, xd]$ and $[xc, xd, xa]$ giving $xc = xd$. Similarly, $xa = xd$ gives $xb = xc$, $xb = xc$ gives $xa = xd$ and $xc = xd$ gives $xa = xb$. The proof for right multiplication follows similarly.

5·3. LEMMA. *If $x \in S$ is proper left cancellable then there exists a proper left cancellable element $y \in S$ such that y^2 is a left unit.*

Proof. The mapping $(a, b, c, d) \to (xa, xb, xc, xd)$ is a permutation and not the identity permutation. Thus it is of order 2, 3 or 4. It cannot be of order 3 as such a permutation will leave one element fixed and move the other three and so will not be betweenness preserving. Thus either x^2 is a left unit or it is proper left cancellable and x^4 is a left unit whence $y = x$ or $y = x^2$, respectively.

Thus we can divide the set of all cyclic betweenness semigroups into the non-intersecting subsets A, B, C, D and E, where:

5·31. $S \in A$ if, and only if, S contains a proper left (right) cancellable element x such that x^2 is a left (right) unit adjacent to x.

5·32. $S \in B$ if, and only if, $S \notin A$ and S contains a proper left (right) cancellable element x such that x^2 is a left (right) unit separate from x.

5·33. $S \in C$ if, and only if, $S \notin A \cup B$ and S contains a left (right) unit.

5·34. $S \in D$ if, and only if, $S \notin A \cup B \cup C$ and S contains a left (right) null.

5·35. $S \in E$ if, and only if, $S \notin A \cup B \cup C \cup D$.

As the calculation is up to anti-isomorphism only those members of the above sets having the defining property on the left are calculated.

During the calculation partially complete multiplication tables are considered. It is therefore useful to extend the concept of B-isomorphism to partial tables.

5·4. DEFINITION. *Let U and V be (partially complete) multiplication tables on four elements possessing cyclic betweenness such that no entry contradicts the axioms of a cyclic betweenness semigroup. Then if there exists a betweenness preserving mapping μ of the elements of U onto the elements of V such that all entries of U are mapped isomorphically (anti-isomorphically) onto all entries in V, U and V are said to be CB-similar.*

If U is part of the table of a cyclic betweenness semigroup W then $W\mu$ is a B-isomorph (anti-isomorph) of W containing V as part of its table. Thus any cyclic betweenness semigroup having U as part of its table is the same, up to anti-isomorphism, as some cyclic betweenness semigroup having V as part of its table.

Those calculations giving non-associative tables will be stopped when this is apparent. The associativity of the remaining tables will not, however, be demonstrated. Lemma 5·2 will be used throughout and reference to it will be omitted.

5·5. *Set A.* Let a be a left unit, b be proper left cancellable and $b^2 = a$. Thus $ba = ab = b$ and as b is proper left cancellable, $bc = d$ and $bd = c$. Now either (a) $cb = d$, $db = c$ or (b) $cb = a$, $db = b$.

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In case (a), $ca = cbb = db = c$ giving $da = d$ and as $cc = cbd = d^2, cd = dbd = dc$ either $c^2 = a$, $cd = b$, or $c^2 = d$, $cd = c$ giving tables (1) and (2), respectively.

In case (b), $cc = cbd = ad = d$, $ca = cbb = ab = b$ and thus $da = d$, $dc = c$, $cd = c$ and $d^2 = d$, that is, table (3).

Now tables (1), (2) and (3) are distinct semigroups and they complete set A up to anti-isomorphism.

5·6. *Set B.* Let a be a left unit, c be proper left cancellable and $c^2 = a$. Thus $ca = ac = c$ and, as c is proper left cancellable, either (a) $cb = b$, $cd = d$, or (b) $cb = d$, $cd = b$.

In case (a), b and d are right zeros, thus ba is a right zero and either $ba = b$ or $ba = d$. If $ba = d$ then $bc = b$ and $ba = bca = bc$ giving a contradiction. Thus $ba = b$, implying $bc = da = d$ and $dc = b$ giving table (4).

In case (b), either $ba = a$ or $da = a$ or a is right cancellable. The first two alternatives are found to be CB-similar upon interchanging b and d in either. Thus we neglect the case $da = a$.

Now $ba = a$ implies $bx = bax = ax$ for all x and thus $cx = dx$ for all x giving table (5).

If a is right cancellable then it must be a right unit as it is idempotent. As

$$(bc)\,c = bc^2 = ba = b,$$

c must be right cancellable and either $bc = b$ or $bc = d$. If $bc = b$ then $dc = d$ and b and d are left zeros and we have an anti-isomorph of table (4). Otherwise $bc = d$ and $dc = b$. Now $bd \neq c$ as then $b^2 = a$ and any such semigroup is in set A. Thus either $bd = a$ or $bd = b$ or $bd = d$. As $db = bcb = bd$, the last two alternatives are found to be CB-similar upon interchanging b and d in either. The first case implies $b^2 = d^2 = c$ and gives table (6) while the second case implies $b^2 = d^2 = d$ and gives table (7).

Now the tables (4) to (7) are distinct semigroups and as cyclic betweenness semigroups they are distinct from the members of set A. They complete set B up to anti-isomorphism.

Permuting the elements of (5) according to $(a, b, c, d) \to (a, d, b, c)$ shows it to be (non-betweennness) isomorphic to (3). This isomorphism exists as the distinction between sets A and B is solely one of different betweenness relations on the elements. However, with this exception, all the sets are distinguished by mutually exclusive multiplicative relations and so no isomorphism can exist between members of different sets; but (non-betweenness) isomorphisms can exist between different members of the same set.

5·7. *Set C.* The only cancellable elements of this set are units. Let a be a left unit. Either a is a right unit or there exists $y \neq a$ such that $ya = a$ implying y is a left unit. Hence either

(a) there exists a left unit separate from a,

or (b) there exists a left unit adjacent to a,

or (c) a is a right unit.

In case (a), c is the left unit separate from a and so all elements are right nulls giving table (8).

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In case (b) we let b be the left unit adjacent to a. Now $ca = ca^2 = (ca)a$ and so $ca \neq b$. Also $ca \neq a$ as this gives table (8). Hence either $ca = da = c$ or $ca = da = d$, alternatives that are found to be CB-similar upon interchanging a with b and c with d in either. Thus we only consider $ca = da = c$ giving $c^2 = c(ca) = c^2a$ and so $c^2 = c$ as $c^2 = a$ implies $c(cb) = c^2b = b$, contradicting the requirement that c should not be proper left cancellable. Thus c is a two-sided zero making d a left null giving table (9).

In case (c) either $cb = c$ or $cd = c$, but these are found to be CB-similar upon interchanging b and d in either. Thus letting $cb = c$ implies $b^2 = b$, $db = c$ as $b^2 = c$, $db = b$ gives $dc = dbb = b^2 = c$ which contradicts the requirement that d should not be proper left cancellable. Hence $dc = c$, $d^2 = d$ and $bc = bd = c^2 = cd$. Now $c^2 = (dc)c = dc^2$ and so $c^2 = c$ or $c^2 = d$; but the latter implies $dc = cd$ giving a contradiction, while $c^2 = c$ gives table (10).

The tables (8) to (10) are distinct semigroups and complete set C up to anti-isomorphism.

5·8. *Set D.* The members of this set contain no cancellable elements but contain nulls. As the square of a left null is a left zero we shall assume that a is a left zero. As c is not right cancellable either $bc = a$ or $c^2 = a$ or $dc = a$. As $c^2 = a$ implies $bc = dc = a$ and the other two cases are found to be CB-similar upon interchanging b and d in either we shall assume $bc = a$ and so $c^2 = dc$. We now consider the four possible values of c^2.

If $c^2 = a$ then c is a right null and so a is a right zero. Thus b and d are left nulls and we have table (11).

If $c^2 = dc = b$ then $cb = bc = a$. Thus b is a right null making a a right zero and so $bd = a$ and $cd = d^2 = b$ giving table (12).

If $c^2 = dc = c$ then $d^2c = d(dc) = dc = c$ and so either $d^2 = c$ or $d^2 = d$. However, $d^2 = d$ implies $db = c$, $da = d$ which contradicts the associativity of dbc. Thus $d^2 = c$ implying $cd = c$ and $bd = a$. Now for any x, $(xa)c = x(ac) = xa$ and so xa is a or c. Putting $ca = a$ makes a a right zero and hence b a right null giving table (13). Putting $ca = c$ makes c a left null. If $ba = c$ then $b^2 = c$ contradicting $b^2b = bb^2$. Hence $ba = b^2 = a$ implying $da = db = c$ and giving table (14).

If $c^2 = dc = d$ then $cd = dc = d$. Hence for any x, $(xa)c = x(ac) = xa$ and so xa is a or d. Putting $ca = a$ gives table (15) by repeated application of Lemma 5·2. Putting $ca = d$ makes c a left null and so d is a left zero, hence b is a left null and we have table (16).

The tables (11) to (16) are distinct cyclic betweenness semigroups and complete the set D up to anti-isomorphism.

5·9. *Set E.* The members of this set contain no cancellable or null elements. The calculation is assisted by the following lemma.

5·91. Lemma. *If $S \in E$, $x, y \in S$ and $x \neq y$ then $x^2 = xy$ if, and only if, $y^2 = yx$ and $x^2 = yx$ if, and only if, $y^2 = xy$.*

Proof. If $x^2 = xy$, then as x is not a left null, x and y are adjacent. Thus we let v, w, x and y be the elements of S such that Cy $[v, w, x, y]$. Thus either $yv = yw \neq yx = y^2$ or $y^2 = yv \neq yw = yx$. Assuming the latter we have $xy^2 = xyv$ and $xyw = xyx$; but

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$x^2 = xy$ implies $xyx = xy^2$ making xy a left zero, contradicting $S \in E$. Hence $x^2 = xy$ implies $yx = y^2$ and interchanging x and y gives the reverse implication. Similarly, $x^2 = yx$ if, and only if, $y^2 = xy$.

If $S \in E$ either $a^2 = ab \neq ac = ad$ or $ad = a^2 \neq ab = ac$, alternatives that are found to be CB-similar upon interchanging b and d in either. Assuming the former, $a^2 = ab$ implies $b^2 = ba$ and so $ba = b^2 \neq bc = bd$ and we put $w = a^2$, $x = ac$, $y = b^2$ and $z = bc$. Now either (a) $c^2 = bc$ or (b) $c^2 \neq bc$.

In case (a) $c^2 = bc$ implies $cb = b^2 = y$ and so $y = ca = cb$, $z = c^2 = cd$, $w = da = db$, $x = dc = d^2$, $x \neq w \neq y$ and $x \neq z \neq y$. As an element must commute with its square, either

$$w = a \quad \text{or} \quad w = c, \quad y = x;$$

and either

$$y = b \quad \text{or} \quad y = d, \quad w = z;$$

and either

$$z = c \quad \text{or} \quad z = a, \quad y = x;$$

and either

$$x = d \quad \text{or} \quad x = b, \quad w = z.$$

Clearly the conjunction of the right-hand alternatives is false, and so we first assume $w = a$. Then $y = b$, as otherwise $b(aa) \neq (ba)a$, and $x = d$ as otherwise $(aa)d \neq a(ad)$. Hence $z = c$ as $y \neq x$ giving table (17). The same result is obtained on assuming the idempotency of any element.

In case (b), $c^2 \neq bc$ implies $b^2 \neq cb$ and so $z = x$ and $w = y$. As $b^2 \neq bc$, $c^2 \neq cb$ and so $ca = cb = da = db = u$ say and $c^2 = cd = dc = d^2 = v$ say.

If $y \in \{a, b\}$, $x \in \{c, d\}$ as a^2 is not a left zero. Thus $v \in \{a, b\}$ as c^2 is not a right zero and as $c^2 = v$, $cv = vc$ and so $u = x$. The associativity of bcc gives $y = v$.

Similarly, if $y \in \{c, d\}$ then $x \in \{a, b\}$, $v = y$ and $u = x$.

The CB-similarity between the cases $y = a$ and $y = b$ and also between the cases $y = c$ and $y = d$ is shown by interchanging a with b and c with d in any one. The cases $y = a$ and $y = d$ are found to be CB-similar upon interchanging a with d and b with c in either. Thus we need only consider $y = v = a$ with $u = x = c$ or $u = x = d$, giving tables (18) and (19) respectively.

The tables (17) to (19) are distinct cyclic betweenness semigroups and complete set E up to anti-isomorphism.

Thus Theorem 5·1 is proved. We note that (1) is the four group, (6) is the cyclic group of order four and the following pairs are (non-betweenness) isomorphic, (2) and (7), (3) and (5), (13) and (15), (14) and (16) and (18) and (19). Of the non-Abelian semigroups only (17) is (betweenness) isomorphic to its own anti-isomorph.

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