*Conclusions*

There are two useful improved formulae for the period of the simple pendulum (those using $I_2$ and $I_3$) which give

$$T_2 = 2\pi\sqrt{\frac{l}{g\cos\frac{1}{2}\delta}} \qquad \text{and} \qquad T_3 = \frac{\pi}{2}\sqrt{\frac{l}{g}}\left(6\sqrt{\sec\tfrac{1}{2}\delta} - \sec\tfrac{1}{2}\delta - 1\right).$$

Despite $T_3$ giving much greater accuracy, for its simplicity and error less than 1% for acute $\delta$, $T_2$ has to be the 'best buy'.

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## 101.47 Notes on Lagrange's interpolation

Let $a_0, \ldots, a_N$ and $b_0, \ldots, b_N$ be sequences of integers. We know that there exists a unique polynomial $f(x) \in \mathbb{Q}[x]$ of degree not larger than $N$ such that $f(a_j) = b_j$ for $0 \leqslant j \leqslant N$. In fact, such a polynomial can be constructed using Lagrange's interpolation (to be described later), and may be called the Lagrange's polynomial of the table

| | | | | |
|---|---|---|---|---|
| $a_0$ | $a_1$ | … | … | $a_N$ |
| $b_0$ | $b_1$ | … | … | $b_N$ |

TABLE 1

The coefficients of $f$ are not necessarily integers.

A common procedure in designing a test problem in a first course of numerical analysis is to start with a polynomial $f(x)$ in $\mathbb{Z}[x]$ of degree $N$, and a sequence $a_0, \ldots, a_N$ of integers, and to ask students to find the Lagrange polynomial of the table

| | | | | |
|---|---|---|---|---|
| $a_0$ | $a_1$ | … | … | $a_N$ |
| $f(a_0)$ | $f(a_1)$ | … | … | $f(a_N)$ |

TABLE 2

Of course, the answer, being nothing but the polynomial $f$ that we started with, is sure to have integer coefficients. If we delete a column or more of the table above, then it is conceivable that the coefficients of the

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Lagrange polynomial $f^*$ of the resulting table are not integers. Of course, they have to be rational numbers. However, it has been observed over years of teaching the course that the polynomial $f^*$ of any sub-table of Table 2 always has integer coefficients, and that sounded rather mysterious – both for students and instructors.

To remove the mystery, we consider the sub-table of Table 2 obtained by deleting the last column, and we let $g$ be its Lagrange polynomial. We will be done if we prove that $g$ belongs to $\mathbb{Z}[x]$, for then the general result would follow by induction.

To this end, we let $h = f - g$. If $h = 0$, then $g = f$, and hence belongs to $\mathbb{Z}[x]$, as desired. Thus suppose $h \neq 0$, i.e., $f \neq g$.

Since $\deg(f) \leqslant N$ and $\deg(g) \leqslant N - 1$, it follows that $\deg(h) \leqslant N$. But $h$ vanishes at the $N$ distinct points $a_0, \ldots, a_{N-1}$. Therefore $h = k(x - a_0)\ldots(x - a_{N-1})$, where $k$ is a non-zero constant in $\mathbb{Q}$. This shows also that $\deg(h) = N$, and therefore $h$ and $f$ have the same degree and the same leading coefficient, namely $k$. Since $f \in \mathbb{Z}[x]$, it follows that $k \in \mathbb{Z}$. Hence $h \in \mathbb{Z}[x]$, and therefore $g$, being nothing but $f - h$, belongs to $\mathbb{Z}[x]$, as desired.

We have proved that if we delete a column of Table 2, then the Lagrange polynomial of the resulting table belongs to $\mathbb{Z}[x]$. By induction, the same holds if we delete any number of columns, removing the mystery alluded to earlier.

Lagrange's interpolation procedure is fascinating. Besides being extremely useful in the various fields of applied mathematics, it is of a great pedagogical value. It beautifully illustrates the strategy that when faced with a difficult or complicated problem, it is helpful to break the problem into much simpler, albeit many, components, and then assemble the solutions of these parts into a solution of the given problem.

Thus seeking a polynomial $g(x)$ that assumes given values $b_0, \ldots, b_N$ at given points $a_0, \ldots, a_N$, we find, instead, a solution $g_r$, for each $r = 0, \ldots, N$, to the special and clearly much simpler case

$$b_r = 1 \text{ and } b_j = 0 \,\forall\, j \neq r,$$

and then assemble these solutions into the polynomial $g$ given by

$$g = b_0 g_0 + \ldots b_N g_N,$$

which clearly has the required properties. It is obvious that we can take

$$g_r(x) = \frac{G_r(x)}{G_r(a_r)}, \quad \text{where } G_r(x) = \prod_{j=0, j\neq r}^{N} (x - a_j).$$

Thus to find a polynomial $g(x)$ of degree less than or equal to 3 that satisfies the table

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| $x$ | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|
| $g(x)$ | 5 | 1 | $-5$ | $-7$ |

we find polynomials $g_1, g_2, g_3, g_4$ satisfying the tables

| $x$ | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|
| $g_1(x)$ | 1 | 0 | 0 | 0 |
| $g_2(x)$ | 0 | 1 | 0 | 0 |
| $g_3(x)$ | 0 | 0 | 1 | 0 |
| $g_4(x)$ | 0 | 0 | 0 | 0 |

and then we combine these solutions to obtain $g$. It is easy to see that

$$(g_1, g_2, g_3, g_4)$$
$$= \left(\frac{(x+1)x(x-1)}{(-2+1)(-2)(-2-1)}, \frac{(x+2)x(x-1)}{(-1+2)(-1)(-1-1)},\right.$$
$$\left.\frac{(x+2)(x+1)(x-1)}{(0+2)(0+1)(0-1)}, \frac{(x+2)(x+1)x}{(1+2)(1+1)(1)}\right)$$

and hence the desired polynomial is

$$g = 5g_1 + g_2 - 5g_3 - 7g_4 = x^3 + 2x^2 - 5x - 5.$$

This technique is encountered in other courses. For example, in a course in elementary number theory, one meets the celebrated *Chinese remainder theorem*. This states that if $m_1, \ldots, m_n \in \mathbb{N}$ are pairwise coprime, and if $b_1, \ldots, b_n \in \mathbb{N}$, then there is $M \in \mathbb{N}$, unique modulo the product $P = m_1 \ldots m_n$, such that $M$ leaves a remainder of $b_j$ when divided by $m_j$, $1 \leqslant j \leqslant n$; see [1, Theorem 4.12, p. 159]. To find such an $M$, one finds, for every $j = 1, \ldots, n$, an integer $M_j$ that leaves a remainder of 1 when divided by $m_j$ and which is divisible by $m_i$ for every $i \neq j$. The desired $M$ is then obtained by assembling $M_1, \ldots, M_n$ as

$$M = b_1M_1 + \ldots + b_nM_n.$$

It is clear that one can take $M_j$ to be $P_ju_j$, where $P = P/m_j$, and where $u_j$ is the inverse of $P_j \pmod{m_j}$.

As an illustration, to find $M \in N$ satisfying the table

$$\begin{array}{l} M \equiv 4 \pmod 6 \\ M \equiv 2 \pmod 7 \\ M \equiv 1 \pmod{11} \end{array}$$

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one finds $M_1$, $M_2$ and $M_3$ satisfying the tables

| | | |
|---|---|---|
| $M_1 \equiv 1 \pmod 6$ | $M_2 \equiv 0 \pmod 6$ | $M_3 \equiv 0 \pmod 6$ |
| $M_1 \equiv 0 \pmod 7$ | $M_2 \equiv 1 \pmod 7$ | $M_3 \equiv 0 \pmod 7$ |
| $M_1 \equiv 0 \pmod{11}$ | $M_2 \equiv 0 \pmod{11}$ | $M_3 \equiv 1 \pmod{11}$ |

and then takes

$$M = (4)(M_1) + (2)M_2 + (1)M_3.$$

To find $M_1$, $M_2$ and $M_3$, we start with $P = (6)(7)(11) = 462$, and proceed as in the following table:

| $j$ | $m_j$ | $P_j = P/m_j$ | $P_j^{-1} \pmod{m_j}$ | $M_j$ |
|---|---|---|---|---|
| 1 | 6 | 77 | $(-1)^{-1} \pmod 6 \equiv -1$ | $(77)(-1) = -77$ |
| 2 | 7 | 66 | $3^{-1} \pmod 7 \equiv -2$ | $(66)(-2) = -132$ |
| 3 | 11 | 42 | $(-2)^{-1} \pmod{11} \equiv 5$ | $(42)(5) = 210$ |

Thus $M \pmod{462}$ is given by

$$M \equiv (4)(-77) + (2)(-132) + (1)(210) \equiv -308 - 264 + 210 \equiv -362.$$

Hence the smallest positive answer is 100.

Another example where Lagrange's interpolation appears is taken from a first course in mathematical logic. There one is asked to find a statement form $W$ in the statement variables $p_1, \ldots, p_n$ whose truth table is given. Of the $2^n$ rows $R_1, \ldots, R_{2^n}$ of the truth table, let us write only those rows at which the truth value of $W$ is 1, and let us call these rows $r_1, \ldots, r_k$. We assume that there is at least one row at which the truth value of $W$ is 1, i.e., $W$ is not a contradiction. Thus the truth table of $W$ is reduced into the form

| | $p_1$ | $p_2$ | … … … | $p_n$ | $W$ |
|---|---|---|---|---|---|
| $r_1$ | $v_{1,1}$ | $v_{2,1}$ | … … … | $v_{n,1}$ | 1 |
| $r_2$ | $v_{1,2}$ | $v_{2,2}$ | … … … | $v_{n,2}$ | 1 |
| … | … | … | … … … | … | … |
| $r_i$ | $v_{1,i}$ | $v_{2,i}$ | … … … | $v_{n,i}$ | 1 |
| … | … | … | … … … | … | … |
| $r_k$ | $v_{1,k}$ | $v_{2,k}$ | … … … | $v_{n,k}$ | 1 |

Here again, one first finds, for every $i = 1, \ldots, k$, a statement form $W_i$ whose truth value at $r_i$ is 1 and whose truth values at all other rows are 0, and then combines $W_i$'s to obtain $W$. It is not difficult to see that

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$$W = W_1 \vee W_2 \vee \ \ldots \ \vee W_k. \tag{1}$$

It is also not difficult to see that one can choose $W_i$ as

$$W_i = (v_{1,i} * p_1) \wedge (v_{2,i} * p_2) \wedge \ \ldots \ \wedge (v_{n,i} * p_n). \tag{2}$$

where the operation $*$ is defined by

$$1 * q = q \text{ and } 0 * q = \sim q \text{ for all statements } q.$$

The form of $W$ shown in (1) (and (2)) is called the *disjunctive normal form* of $W$; see [2, p. 17]. Similarly, one defines the *conjunctive normal form* of $W$, where $W$ is written as $W = V_1 \wedge \ \ldots \ \wedge V_s$, and where each $V_i$ is written using the connective $\vee$.

For an illustration, consider the problem of finding the statement form $W$ in the statement variables $p$, $q$ and $t$ whose truth table is

| | $p$ | $q$ | $t$ | $W$ |
|---|---|---|---|---|
| $R_1$ | 1 | 1 | 1 | 0 |
| $R_2$ | 1 | 1 | 0 | 1 |
| $R_3$ | 1 | 0 | 1 | 0 |
| $R_4$ | 1 | 0 | 0 | 0 |
| $R_5$ | 0 | 1 | 1 | 1 |
| $R_6$ | 0 | 1 | 0 | 0 |
| $R_7$ | 0 | 0 | 1 | 0 |
| $R_8$ | 0 | 0 | 0 | 1 |

The reduced truth table of W and the tables for $W_1$, $W_2$ and $W_3$ are given by

| | $p$ | $q$ | $t$ | $W$ | $W_1$ | $W_2$ | $W_3$ |
|---|---|---|---|---|---|---|---|
| $r_1 = R_2$ | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| $r_2 = R_5$ | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| $r_3 = R_8$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 |

Then

$$W_1 = p \wedge q \wedge \sim t, \qquad W_2 = \ \sim p \wedge q \wedge t, \qquad W_3 = \ \sim p \wedge \sim q \wedge \sim t,$$

and

$$W = W_1 \vee W_2 \vee W_3.$$

We hope that some readers find it interesting to look for other contexts in which this technique shows up, and that they would agree with us that this

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*divide and conquer* strategy deserves to find a place in books on *Problem Solving Strategies*. Unfortunately, we found no mention of it in the well-known books [3, 4, 5], but we were glad to see it given a place in [6, 4.3.22 (p. 141)].

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