Notes

100.28 Mixing angle trisection with Pythagorean triples

A recent post on a question-and-answer website stated that a triangle with integer sides

$$m^3,\ m\left(n^2 - m^2\right),\ n\left(n^2 - 2m^2\right),$$

where m, n are positive integers and $\frac{n}{m} > \sqrt{2}$, will have one of its angles trisecting another. The poster did not explain why the result is true or how it was arrived at, but asked who discovered it first. No one could answer. Of course, once the formula is given the claim can be checked using trigonometry, but it is more fun to figure out how it could be derived.

The problem is of course reminiscent of Pythagorean triples, triples of integers a, b, c that satisfy the Pythagorean theorem $c^2 = a^2 + b^2$, and hence are lengths of the sides of a right triangle. Euclid in Elements X.29 gives a solution without a hint of how it was obtained (trial and error by Pythagoreans seems likely), or why it gives all the triples. A constructive method first appears in Diophantus's *Arithmetica* (c. 250 AD) when solving the famous problem II.8, 'partition a given square into two squares', the one whose margins were too narrow in Fermat's copy of the book, see [1] for a nice commentary.

In modern notation, Diophantus's method is as follows. Set $x = \frac{a}{c}$, $y = \frac{b}{c}$, then we have $x^2 + y^2 = 1$ by the Pythagorean theorem. Write $y^2 = 1 - x^2$ and make the substitution $y = t(1 - x)$. After cancellation, the equation for x becomes linear and gives $x = \frac{t^2 - 1}{t^2 + 1}$, $y = \frac{2t}{t^2 + 1}$. If t is rational then so are x and y, and setting $t = \frac{n}{m}$ we get us the Pythagorean triples $n^2 - m^2$, $2mn$, $n^2 + m^2$. Conversely, if a, b, c are integers then x, y and therefore t, will be rational, so we have an assurance that all triples are accounted for.

The substitution trick can be interpreted geometrically as intersecting the unit circle with lines of slope $-t$ passing through the point $(1, 0)$ on the circle and it works because each line, apart from the tangent, has exactly one other intersection point with the circle, since a circle is a curve of degree two. Diophantus mentions nothing of the sort of course, although II.8 is one problem where he shows awareness of the existence of infinitely many solutions. Remarks in [1] suggest that perhaps Poincaré came up with the geometric interpretation in 1901. It is now called rational parametrisation in algebraic geometry [2, §8.1].

Let us call a triangle *self-trisecting* if one of its angles is one third of another. To extend the above to self-trisecting triangles we first need a 'Pythagorean theorem'. Having a right angle imposes one constraint on a

https://doi.org/10.1017/mag.2016.115 Published online by Cambridge University Press

triangle's angles and, together with the angle-sum being π, it gives us a one-parameter family of possible triples of angles α, $\frac{1}{2}\pi - \alpha$, $\frac{1}{2}\pi$. The self-trisection similarly produces the angle triples 3α, 3α, $\pi - 4\alpha$. Applying the sine rule we obtain

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\left(\frac{1}{2}\pi - \alpha\right)} = \frac{b}{\cos\alpha} = \frac{c}{\sin\frac{1}{2}\pi} = c.$$

We now see that $a = c\sin\alpha$, $b = c\cos\alpha$ so that $a^2 + b^2 = c^2$, which is the Pythagorean theorem. Following the analogy we write

$$\frac{a}{\sin\alpha} = \frac{b}{\sin 3\alpha} = \frac{c}{\sin(\pi - 4\alpha)} = \frac{c}{\sin 4\alpha}.$$

With $x = \dfrac{b}{a}$, $y = \dfrac{c}{a}$ the multiple angle formulas give

$$x = \frac{\sin 3\alpha}{\sin\alpha} = 4\cos^2\alpha - 1 \text{ and } y = \frac{\sin 4\alpha}{\sin\alpha} = 2\cos\alpha\left(4\cos^2\alpha - 2\right). \qquad (1)$$

Squaring the last formula and eliminating the cosine gives us the equation of a curve that replaces the unit circle

$$y^2 = (x + 1)(x - 1)^2.$$

The 'Pythagorean theorem' for self-trisecting triangles is therefore $ac^2 = (a + b)(a - b)^2$.

The curve we obtained is a cubic, a curve of degree three, and generic cubics do not admit rational parametrisations [2, §18.5]. This one however is special, the squared factor means that it has a self-intersection point at $(1, 0)$, called a node. Our nodal cubic looks similar to the rotated folium of Descartes, known from the early history of analytic geometry and calculus, but unlike the folium it does not have an asymptote. Lines passing through the node have exactly one other point of intersection with the cubic, so their slopes will rationally parametrise it like they did the circle. This is seen by making the algebraic substitution $y = t(x - 1)$, which immediately gives $x = t^2 - 1$ and $y = t\left(t^2 - 2\right)$. Setting $t = \dfrac{n}{m}$ we recover the opening formula for the self-trisecting triples. Thus, not only does the formula work, but it gives all of them as in the Pythagorean case.

In hindsight, we did not need to find the algebraic equation to find the rational parametrisation and the triples, indeed setting $t = 2\cos\alpha$ in (1) gives the parametrisation directly. This means that in any integer self-trisecting triangle the cosine of the trisecting angle is rational. Better yet, it leads to a generalisation to *self-k-secting triangles*, the ones where one angle is the kth part of another. Let us illustrate the idea for $k = 5$. As above we have $x = \dfrac{\sin k\alpha}{\sin\alpha}$ and $y = \dfrac{\sin(k + 1)\alpha}{\sin\alpha}$ from the sine rule. Using de Moivre's formula $\cos k\alpha + i\sin k\alpha = (\cos\alpha + i\sin\alpha)^k$ and Pascal's triangle to recover the multiple angle formulas from the binomial theorem

https://doi.org/10.1017/mag.2016.115 Published online by Cambridge University Press

we have:

$$x = \frac{\sin 5\alpha}{\sin \alpha} = 5\cos^4\alpha - 10\cos^2\alpha \sin^2\alpha + \sin^4\alpha$$

$$= 16\cos^4\alpha - 12\cos^2\alpha + 1 \qquad (2a)$$

$$y = \frac{\sin 6\alpha}{\sin \alpha} = 6\cos^5\alpha - 20\cos^3\alpha \sin^2\alpha + 6\cos\alpha \sin^4\alpha$$

$$= 2\cos\alpha\left(16\cos^4\alpha - 16\cos^2\alpha + 3\right). \qquad (2b)$$

Setting as before $t = 2\cos\alpha$ we get a rational parametrisation of a quintic curve, a curve of degree five:

$$x = t^4 - 3t^2 + 1, \qquad y = t\left(t^4 - 4t^2 + 3\right),$$

and the corresponding self-pentasecting triples are

$$a = m^5,\ b = m\left(n^4 - 3n^2m^2 + m^2\right),\ c = n\left(n^4 - 4n^2m^2 + 3m^2\right).$$

The algebraic equation of the quintic, and the self-pentasecting 'Pythagorean theorem', are much messier in this case.

What we used so far is that the righthand sides of (1) and (2) are polynomials in $\cos\alpha$ with integer coefficients (even in $2\cos\alpha$). A simple combinatorial argument shows that this is always the case. The identity $U_k(\cos\alpha) = \dfrac{\sin(k+1)\alpha}{\sin\alpha}$ defines degree k polynomials $U_k(s)$ known as the *Chebyshev polynomials of the second kind* [3]. They were introduced by Chebyshev in 1854, and come up in connection with the four-dimensional spherical harmonics among other things. They can be conveniently recovered directly from the recurrence relation $U_0 = 1$, $U_1 = 2s$, $U_{k+1} = 2sU_k - U_{k-1}$, which also shows that $U_k\left(\frac{1}{2}t\right)$ has integer coefficients for all k. With their help a rational parametrisation generalising the above $k = 3$ and $k = 5$ examples can be written as $x = U_{k-1}\left(\frac{1}{2}t\right)$ and $y = U_k\left(\frac{1}{2}t\right)$. The corresponding self-k-secting triples are given by

$$a = m^k,\ b = m^k U_{k-1}\left(\frac{n}{2m}\right),\ c = m^k U_k\left(\frac{n}{2m}\right). \qquad (3)$$

Note that while we are guaranteed that every integer self-k-secting triangle arises in (3) up to scaling of the sides, not every integer pair m, n necessarily gives rise to one. One obvious restriction follows from the fact that $\dfrac{n}{m} = 2\cos\alpha$. For α to k-sect another angle in a self-k-secting triangle we must have $\alpha + k\alpha < \pi$, that is $0 < \alpha < \dfrac{\pi}{k+1}$. This means that all self-k-secting triangles are already accounted for if we restrict to $n, m > 0$ and $2 > \dfrac{n}{m} > 2\cos\dfrac{\pi}{k+1}$. Let us prove that any such pair of integers does produce a triangle.

https://doi.org/10.1017/mag.2016.115 Published online by Cambridge University Press

Theorem: Let k, m, n be positive integers and $2 > \frac{n}{m} > 2\cos\frac{\pi}{k+1}$. Then a, b, c given by (3) are the sides of an integer triangle with the angle opposite a k-secting the angle opposite c. Every integer self-k-secting triangle arises in this way up to scaling of the sides.

Proof: The last claim follows from the construction, so it remains to prove the converse. Given m, n from the statement of the theorem there is a unique $0 < \alpha < \frac{\pi}{k+1}$ such that $\cos\alpha = \frac{n}{2m}$. It follows from the definition of the Chebyshev polynomials of the second kind (or rather from the theorem that they are well-defined) that

$$\frac{b}{a} = U_{k-1}(\cos\alpha) = \frac{\sin k\alpha}{\sin\alpha} \text{ and } \frac{c}{a} = U_k(\cos\alpha) = \frac{\sin(k+1)\alpha}{\sin\alpha},$$

and therefore that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin k\alpha} = \frac{c}{\sin(\pi - (k+1)\alpha)}.$$

The rest is an application of what can be called the 'converse of the sine rule'. Since α, $k\alpha$ and $\pi - (k+1)\alpha$ add up to π there is a triangle with them as angles, unique up to scaling. Its sides and angles must obey the sine rule, which means that they are as $a : b : c$. Scaling this triangle to make the side opposite α be a we obtain the requisite integer triangle.

References

1. N. Schappacher, *Diophantus of Alexandria: a text and its history*, 1998, available at
   http://www-irma.u-strasbg.fr/~schappa/NSch/Publications.html
2. C. Gibson, *Elementary geometry of algebraic curves* Cambridge University Press, Cambridge (1998).
3. J. Mason, D. Handscomb, *Chebyshev polynomials* CRC Press, Boca Raton, FL (2003).

10.1017/mag.2016.115

SERGIY KOSHKIN
Department of Mathematics and Statistics, University of Houston-Downtown, One Main Street, Houston, TX 77002 USA
e-mail: *koshkins@uhd.edu*

https://doi.org/10.1017/mag.2016.115 Published online by Cambridge University Press