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Optimal control of a class of piecewise deterministic processes

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A new control strategy for a class of piecewise deterministic processes (PDP) is presented. In this class, PDP stochastic processes consist of ordinary differential equations that are subject to random switches corresponding to a discrete Markov process. The proposed strategy aims at controlling the probability density function (PDF) of the PDP. The optimal control formulation is based on the hyperbolic Fokker–Planck system that governs the time evolution of the PDF of the PDP and on tracking objectives of terminal configuration with a target PDF. The corresponding optimization problems are formulated as a sequence of open-loop hyperbolic optimality systems following a model predictive control framework. These systems are discretized by first-order schemes that guarantee positivity and conservativeness of the numerical PDF solution. The effectiveness of the proposed computational control framework is validated considering PDP with dichotomic noise.

Key words: Optimal control problems involving partial differential equations; Hybrid systems; Fokker–Planck equations; Stability and convergence of numerical methods; Markov renewal processes

1 Introduction

Stochastic modelling is one of the most active research field of applied sciences, engineering and computational finance. Randomness can be included into modelling equations in two different ways. On the one hand, noise is added to deterministic evolution equations to model random perturbations as in, e.g. the Langevin equation [34]. On the other hand, random perturbations are being modelled by jump (point) processes, where a stochastic action affects the deterministic motion only at some instants of time as in, e.g. queueing and renewal processes [20]. In the former case, a vast literature exists on the theory and application of stochastic processes that is well grounded on the Itō calculus and the Fokker–Planck (FP) equation. In the latter case, the theory and application of point processes is less investigated. Our purpose is to discuss a special class of point processes and introduce a new control strategy for these models.

In the framework of dynamical systems, an early work on a set of differential equations that change their deterministic structure at random points in time, following a Markov

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process, can be found in [40]. A first general formulation of systems that switch randomly within a certain number of deterministic states at random times is given in [21]. In this reference, the name piecewise deterministic processes (PDP) appears for the first time.

The PDP appear in probability calculus and operation research, stochastic hybrid systems, reliability analysis, statistical physics and financial mathematics; see, e.g. [1, 9, 13, 14, 18, 19, 26], for recent works and additional references. We deal with a class of PDP models described by a state function that is continuous in time and driven by a discrete state Markov process. These models are used to describe phenomena where a deterministic motion is subject to a sudden interaction that produces randomization of the motion for a short time compared to the timescale where the process is observed. In this framework, applications include dichotomic noise, random telegraph processes, transport processes and binary noise. Further applications include reacting–diffusing systems [32], biological dispersal [31], non-Maxwellian equilibrium [2, 8, 27, 42] and filtered telegraph signal analysis [46].

In many applications, the control and optimization of PDP is of particular interest; see, e.g. [19] for a review. In this context, control problems are usually formulated considering average functionals of cost and utility, and involve dynamic programming principles as well as the solution of Hamilton–Jacobi–Bellmann equations. The control mechanisms considered in this framework can be impulsive or continuous; see, e.g. [22]. In particular, for the latter case we quote [1, 19], where a reduction technique is proposed to transform the control problem to a discrete time Markov decision process [11].

The definition of PDP suggests that these systems are mostly locally deterministic, but the whole evolution is not, in the sense that it represents a random sample path of a probability space, so that the meaningful information can be described by the marginal distribution functions for each state of the system. The model for such distributions is the analogous to the FP equation for the stochastic Itō equation [10]. Specifically, it is a forward Kolmogorov equation that in our case plays the role of a Liouville–Master equation, because it corresponds to the Liouville equation for the deterministic part and to a Master equation for the Markov chain part. Sometime this equation is also referred to as a generalized FP equation. For a numerical investigation of the Liouville–Master equation see [3, 4, 16, 25].

Our purpose is to develop and investigate an extension of a new control strategy for stochastic Itō processes, proposed in [6, 7], to control PDP. In our approach, a statistical point of view is taken based on the fact that often the state of a stochastic process can be completely characterized by the shape of its statistical distribution that is represented by the probability density function (PDF) of the process. The other essential component of our strategy is to recognize that the evolution of the PDF associated to a PDP is modelled by a FP partial differential equation (PDE). The third component of our new control methodology is to formulate the control objectives in terms of the PDF. Therefore, the PDP control problem is formulated as a PDE-constrained optimal control problem with a control objective defined with the PDF and the FP equation represents the differential constraint. We remark that one of the purposes of this work is to provide a bridge between the world of stochastic PDP processes and that of PDE optimization. For this reason, we present our work on PDP using a PDE formalism.

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In the next section, we introduce a class of PDP and provide their mathematical characterization. Further, we consider the PDF of these processes and present a derivation of the FP equation that describes the time evolution of the PDF of a PDP system. While our derivation is not rigourous, it provides a clear insight into our FP model. Note that FP equations usually are PDEs of parabolic type that are related to Itō stochastic processes, while in our case we have a FP model that is a system of first-order hyperbolic PDEs with coupling through zero-order stochastic transition matrix terms.

In Section 3, we discuss our hyperbolic FP system with Cauchy data and introduce our FP control strategy. Further, we introduce a control mechanism in the PDP (resp. FP) model and define a tracking objective based on the PDF of the PDP. To characterize the solution to our optimization problem, we introduce the Lagrange formalism and formulate the first-order optimality conditions. The resulting optimality system consists of a forward hyperbolic FP system, a backward adjoint FP system and an integral optimality-condition equation. In Section 4, we focus on a specific PDP with dichotomic noise that allows us to specifically address important aspects of PDP and of the related FP and adjoint FP systems. In Section 5, all details concerning the numerical approximation of the FP optimality system are presented. It is emphasized that the discretization of the FP equation must guarantee positiveness of the numerical PDF and its conservativeness. Furthermore, we discuss the discretization of the adjoint FP system, showing that the resulting scheme corresponds to the discrete adjoint of the forward FP model. This fact guarantees an accurate gradient given by the optimality-condition equation. In Section 6, we introduce a model predictive control (MPC) strategy that requires to solve an open-loop FP optimality system for each time window of a sequence of time horizons. In Section 7, we report results of numerical investigation of our FP control strategy applied to a linear-filter PDP and to a non-linear-filter PDP with dichotomic noise. The results reported in this section demonstrate the ability of our control framework to steer the entire PDF of the PDP towards a target configuration. We also present Monte Carlo simulations of PDP models obtained with our FP-based optimal control framework. A section of conclusion completes the exposition of our work.

2 Piecewise deterministic processes

A PDP is a model governed by a set of differential equations that change their deterministic structure at random points in time. PDP models define the largest class of Markov processes including most of the non-diffusion models of applied probability. This general class of deterministic-stochastic models was proposed in [21]. It has many important applications in mathematical biology, finance and physics.

We focus on processes that switch randomly between deterministic states driven by a renewal process, denoted with $\mathscr{S}(t)$, that is a discrete stochastic jump process where the influence of the past is erased at the epochs of jumps.

We consider, the PDP equation model that is a first-order system of ordinary differential equations where the known driving function of the dynamics is affected by a renewal process. The d-components state function $X(t)$, $X : [t_0, \infty) \to \Omega$, $\Omega \subseteq \mathbb{R}^d$, is defined by the following properties. We have

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(a) The state function satisfies the following equation

$$\frac{d}{dt}X(t) = A_{\mathscr{S}(t)}(X), \quad t \in [t_0, \infty), \tag{2.1}$$

where $\mathscr{S}(t) : [t_0, \infty[\to \mathbb{S}$ is a Markov process (defined below in (c) and (d)) with discrete states $\mathbb{S} = \{1, \ldots, S\}$. Correspondingly, given $s \in \mathbb{S}$, we say that the dynamics is in the (deterministic) state s, and it is driven by the function $A_s : \Omega \to \mathbb{R}^d$, that belongs to the set of functions $\{A_1, \ldots, A_S\}$. We require that all $A_s(\cdot), s \in \mathbb{S}$, be Lipschitz continuous, so that for fixed s, the solution $X(t)$ exists and is unique and bounded.

(b) The state function satisfies the initial condition $X(t_0) = X_0 \in \Omega$, being in the initial state $s_0 = \mathscr{S}(t_0)$.

(c) The process $\mathscr{S}(t)$ is characterized by an exponential PDF $\psi_s : \mathbb{R}^+ \to \mathbb{R}^+$, of transition events, as follows

$$\psi_s(t) = \mu_s e^{-\mu_s t}, \quad \text{with} \int_0^\infty \psi_s(t)\, dt = 1, \tag{2.2}$$

for each state $s \in \mathbb{S}$. In other words, it is the PDF for the time that the system stays in the state s, that is the time between consecutive events of a Poisson process.

(d) The process $\mathscr{S}(t)$ is modelled by a stochastic transition probability matrix, $\hat{q} := \{q_{ij}\}$, with the following properties

$$0 \leqslant q_{ij} \leqslant 1, \qquad \sum_{i=1}^{S} q_{ij} = 1, \qquad \forall i, j \in \mathbb{S}. \tag{2.3}$$

When a transition event occurs, the PDP system switches instantaneously from a state $j \in \mathbb{S}$, with the driving function A_j, randomly to a new state $i \in \mathbb{S}$, driven by the function A_i. Virtual transitions from the state j to itself are allowed for this model, that is, $q_{jj} > 0$.

Both (c) and (d) define the Markov renewal process $\mathscr{S}(t)$, that generates a temporal sequence of transition events $(t_0, t_1, \ldots, t_k, t_{k+1}, \ldots)$ and states $(s_0, s_1, \ldots, s_k, s_{k+1}, \ldots)$. It is said that (d) defines the embedded discrete Markov chain of the process $\mathscr{S}(t)$, while (c) defines a continuous time process for the epochs where the Markov chain changes its state, i.e. a renewal process.

The state space of the PDP is the union of S disjoint copies of $\mathbb{R}^d$. In the jth copy, the vector field is A_j and, at a jump time, a change to another component A_i, with i randomly chosen, can occur. Note that the state function $X(t)$ is continuous through the jumps of the renewal process. We consider PDP without barriers.

3 The PDP Fokker–Planck equation

From a statistical point of view, the state of a PDP can be characterized by the shape of its statistical distribution which is represented by the marginal PDFs. It is one of the greatest achievements of probability theory and statistical physics to have recognized that the evolution of the PDF associated to a stochastic process can be modelled by

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PDEs with Cauchy data given by the initial PDF distribution; see, e.g. [23,47,48]. These PDEs have been investigated in many works and are named after many famous scientists including Kolmogorov, Fokker, Planck and Smoluchowski. We use the term FP equation for convenience and refer to [23] for an historical account of the subject.

Since the derivation of the FP system for PDP processes is less common in the scientific literature, we provide a short derivation in the following. For this purpose, we consider the process (2.1) in one dimension, $d = 1$. For the derivation of the FP system for the d-dimensional case, see [5,17].

Consider small time steps of size Δt. At each time step there is the probability $(\mu_s \Delta t)$ that a transition event occurs and $(1 - \mu_s \Delta t)$ that the jump in the process does not occur. In the latter case, the motion is deterministic. Let $\mathbb{P}$ be the probability measure on the hybrid space state of the system; then for a small interval of time Δt and displacement Δx, the change of the probability distribution $F_s(x,t) := \mathbb{P}(X(t) \leqslant x, \mathscr{S}(t) = s)$ for the state s is as follows

$$F_s(x + \Delta x, t + \Delta t) \simeq (1 - \mu_s \Delta t) F_s(x,t).$$

Note that the right-hand side of this expression represents the probability that the jump in time does not occur and the evolution of the probability distribution follows a deterministic law. Next, we assume enough regularity of F_s and A_s and use Taylor expansion to obtain the following approximation

$$F_s + A_s(x)\, \Delta t\, \partial_x F_s + \partial_t F_s \Delta t \simeq (1 - \mu_s \Delta t) F_s,$$

where ∂_t and ∂_x denote partial derivatives with respect to the variables time t and state x, respectively, and we used the approximation $\Delta x \approx A_s(x) \Delta t$.

Now, by considering the first-order terms, we obtain the following

$$\partial_t F_s(x,t) = -A_s(x)\, \partial_x F_s(x,t) - \mu_s\, F_s(x,t). \tag{3.1}$$

This equation is valid as long as no switching event occurs. To take into account a switching event within the time interval $(t, t + \Delta t)$, we must include on the right-hand side the amount of ingoing probability from the other states. This is given by the joint probability described by the stochastic matrix q_{sj}, with $\sum_{s=1}^{S} q_{sj} = 1$, multiplied by $\mu_j\, F_j(x,t) \Delta t$. Therefore, (3.1) becomes the following

$$\partial_t F_s(x,t) + A_s(x)\, \partial_x F_s(x,t) = -\mu_s\, F_s(x,t) + \sum_{j=1}^{S} q_{sj} \mu_j F_j(x,t). \tag{3.2}$$

We refer to this equation as the Liouville–Master equation for the probability distribution function.

The marginal PDF f_s for the state s is related to the corresponding probability distribution as follows

$$F_s(x,t) = \int_{-\infty}^{x} f_s(z,t)\, dz, \qquad s = 1, \ldots, S.$$

Hence, by differentiating (3.2) with respect to x, we obtain the following FP system for

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the PDFs

$$\partial_t f_s(x,t) + \partial_x \left(A_s(x)\, f_s(x,t)\right) = \sum_{j=1}^{S} Q_{sj} f_j(x,t), \qquad s = 1,\dots,S, \tag{3.3}$$

where

$$Q_{sj} = \begin{cases} \mu_j\, q_{sj} & \text{if } j \neq s, \\ \mu_s\,(q_{ss} - 1), \end{cases} \tag{3.4}$$

for $s \in \mathbb{S}$, $x \in \Omega \subset \mathbb{R}$, for the scalar process $X(t)$ in the state s. We have $\sum_{s=1}^{S} Q_{sj} = 0$.

Note that we consider PDP whose PDFs represent the absolutely continuous part of the Lebesgue measure. For a rigourous derivation of the FP system for a the general PDP see [17], where an existence and uniqueness theorem is proved for the weak formulation solution of the FP system. In this reference, the PDP is defined by a space-depending transition rate matrix $a(i,j,x)$ for jumps from i to j of the Markov chain, and a transition measure $d\mu(i,j,x)(dy)$ for random jumps on the deterministic dynamics. In our case, the transition rate is constant, i.e. $a(i,j,x) = \mu_i q_{ji}$, and discontinuous jumps in the values of $X(t)$ are not allowed, hence $d\mu(i,j,x)(dy) = \delta(x-y)\,dy$, where $\delta(x)$ is the δ-Dirac function.

We consider PDP in a finite time horizon $t \in [0,T]$, and we require that

$$\sum_{s=1}^{S} \int_{\Omega} f_s(x,t)\, dx = 1. \tag{3.5}$$

The initial conditions for the solution of the FP system are given as follows

$$f_s(x,0) = f_s^0(x), \qquad s = 1,\dots,S, \tag{3.6}$$

where $f_s^0(x) \geqslant 0$, $x \in \Omega$, $\sum_{s=1}^{S} \int_{\Omega} f_s^0(x)\, dx = 1$.

In general, we consider the case where Ω is the real line and we do not prescribe boundary conditions [24]. If (c) is non-Poisson, then $\mathscr{S}(t)$ is a semi-Markov process, and the FP system would require appropriate non-local boundary conditions; see, e.g. [3,5]. We assume that initial conditions have compact support in Ω. In our numerical experiments, we shall consider functions $A_s(x)$, such that the domain Ω is bounded for all times, and zero boundary conditions can apply.

For the purpose of the control, we introduce in (3.3) a control mechanism in the deterministic dynamics. That is, we consider $A_s(x,u_s)$, in (Ω, U_s), where $U_s \subset \mathbb{R}^l$, $s \in \mathbb{S}$, are closed compact sets. Then we assume that for a given state (x,s) of the system, an admissible open-loop control function $u_s(t) : [0,T) \to U_s$ exists and is continuous in the interval $[0,T)$ [41]. We denote with $\mathscr{U}_s$ the set of all control functions u_s in the discrete state s. The control mechanism depends on the state s of the system, hence it changes according to the Markov process $\mathscr{S}(t)$. Note that with our FP control framework, it is possible to determine an optimal control bypassing the underlying stochastic process. Therefore, for our control function we do not use the concept of function adapted to the filtration generated by the solution of the stochastic process. Further, we assume that $A_s(x,u_s), s \in \mathbb{S}$ is Lipschitz continuous in the set (Ω, U_s) so that the solution $X(t)$ to

$$\frac{d}{dt} X(t) = A_{\mathscr{S}(t)}(X, u_{\mathscr{S}(t)}(t)), \quad t \in [0,T), \tag{3.7}$$

exists and is unique and bounded.

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We remark that the FP system given by (3.3) can be written in the following form

$$\partial_t f_s(x,t) + A_s(x,u_s)\,\partial_x f_s(x,t) = \sum_{j=1}^{S} \tilde{Q}_{sj}(x,u_s)\, f_j(x,t), \qquad s = 1,\dots,S, \tag{3.8}$$

where $\tilde{Q}_{sj}(x,u_s) = Q_{sj} - \delta_{sj}\partial_x A_s(x,u_s)$, with Q_{sj} depending on μ_j as defined in equation (3.4), and δ_{sj} denotes the Kronecker's delta. In the following, we denote with $f = (f_s)_{s=1}^S$ and $u = (u_s)_{s=1}^S$.

The model (3.8) is a first-order hyperbolic system in diagonal form, with coupling given through zero-order terms. This is a *strictly hyperbolic* model provided that the functions A_s are distinct for all $(x,u_s) \in (\Omega, U_s)$. It is well known that for first-order strictly hyperbolic PDEs, the solution to the initial value problem exists and is unique with the same regularity as the initial data, see, e.g. [35].

The purpose of this work is to formulate an optimal control strategy for the PDFs of a PDP based on the FP system. Specifically, we consider the problem to find optimal controls u_s, $s = 1,\dots,S$, such that the solution to the FP model (3.3)–(3.6) minimizes the following cost functional

$$J(f,u) := \frac{1}{2}\sum_{s=1}^{S} \|f_s(\cdot,T) - f_s^T(\cdot)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\sum_{s=1}^{S} \|u_s\|^2_{L^2(0,T;\mathbb{R}^l)}, \tag{3.9}$$

where $f^T = (f_1^T,\cdots,f_S^T) \in C_0^\infty(\Omega,\mathbb{R}^S)$ is a vector of analytic functions with trace zero, that represents a desired target PDF at time T. This objective models the requirement that the PDF of the PDP at final time $f_s(\cdot,T)$ approaches as close as possible (in the $L^2(\Omega)$ norm) the desired target f^T, while minimizing the cost of the controls defined in the square integrable norm in $(0,T;\mathbb{R}^l)$, i.e. $\|\cdot\|_{L^2(0,T;\mathbb{R}^l)}$. It is assumed that $J(f,u)$ is twice Frechét-differentiable and that the second Frechét derivative J'' is locally Lipschitz-continuous. In compact form, we have the following optimal control problem

$$\min_{u\in\mathscr{U}} J(f,u), \;\; (f,u) \text{ subject to } (3.8),(3.5),(3.6), \tag{3.10}$$

where $\mathscr{U} = \mathscr{U}_1 \times \mathscr{U}_2 \times \ldots \times \mathscr{U}_S$. This is an infinite-dimensional constrained minimization problem [12, 36], whose solution can be characterized by the corresponding first-order optimality conditions. In order to formulate these conditions, we introduce the Lagrange function

$$\begin{aligned} L(f,u,p) = {} & J(f,u) \\ & + \sum_{s=1}^{S}\int_0^T\int_\Omega \Big(\partial_t f_s(x,t) + \partial_x\left(A_s(x,u_s)\, f_s(x,t)\right) - \sum_{j=1}^{S} Q_{sj} f_j(x,t)\Big) p_s(x,t)\,dx\,dt, \end{aligned} \tag{3.11}$$

where $p = (p_s)_{s=1}^S$ is the vector of adjoint PDF variables. The first-order optimality conditions for the optimal control problem (3.10) are formally derived by equating to zero the Frechét derivatives of the Lagrange function with respect to the set of variables

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(f_s, u_s, p_s); see, e.g. [12,36,38]. We obtain the following optimality system. For $s = 1, \ldots, S$, we have

$$\partial_t f_s(x,t) + \partial_x \left(A_s(x,u_s)\, f_s(x,t)\right) = \sum_{j=1}^{S} Q_{sj} f_j(x,t), \tag{3.12}$$

$$f_s(x,0) = f_s^0(x), \tag{3.13}$$

$$-\partial_t p_s(x,t) - A_s(x,u_s)\, \partial_x p_s(x,t) = \sum_{j=1}^{S} Q_{js} p_j(x,t), \tag{3.14}$$

$$p_s(x,T) = -\left(f_s(x,T) - f_s^T(x)\right), \tag{3.15}$$

$$\nu\, u_s(t) - \int_\Omega \partial_x p_s(x,t)\, \partial_{u_s} A_s(x,u_s)\, f_s(x,t)\, dx = 0. \tag{3.16}$$

We remark that the adjoint FP system (3.14) has a similar structure as the state model (3.12) written in the form (3.8). Note that the adjoint system describes backward evolution of the adjoint variables, starting with the terminal condition (3.15) at $t = T$, and it is strictly hyperbolic as the forward equation. Hence, assuming $f \in C^\infty(\mathbb{R} \times (0,T), \mathbb{R}^S)$ and a desired smooth target $f^T \in C^\infty(\mathbb{R}, \mathbb{R}^S)$, we have $p \in C^\infty(\mathbb{R} \times (0,T), \mathbb{R}^S)$. Based on these regularity results, on the convexity and differentiability of the cost functional and on the form of the optimality condition (3.12), standard arguments apply to prove existence of optimal solutions; see, e.g. [12, 15, 36, 38]. Note that in the case $S = 1$, the zero-order coupling term in the forward and adjoint systems disappears. However, in this case the problem becomes deterministic and the FP formulation is meaningless.

Assuming piecewise constant controls in the subinterval (t_k, t_{k+1}), we obtain the following optimality condition

$$\nu\, u_s - \int_{t_k}^{t_{k+1}} \int_\Omega \partial_x p_s(x,t)\, \partial_{u_s} A_s(x,u_s)\, f_s(x,t)\, dx\, dt = 0. \tag{3.17}$$

Note that the piecewise control setting motivates a MPC approach [29, 37, 39], where the optimality system (3.12)–(3.15) and (3.17), is considered in the sequence of subintervals (t_k, t_{k+1}), $k = 0, 1, \ldots$, where $t_k = k\,\Delta t$. In this case, the initial condition (3.13) is given at $t_k = 0$, $k = 0$, while in the subsequent subintervals it results from the solution of the previous time window. This approach, allows us to choose a target density function that is changing in time, thus defining a desired density trajectory $f^T(\cdot, t_{k+1})$, $k = 0, \ldots, N-1$. We discuss this setting in detail in the section dedicated to the MPC approach.

4 A PDP with dichotomic noise

In the following, we focus on a particular PDP. This choice is motivated by the wish to provide a detailed discussion and implementation for a specific problem.

We consider the case of a dissipative process subject to dichotomic noise; see, e.g. [2,4,46]. Let $X(t)$ be a process whose evolution is described by the following equation

$$\frac{dX(t)}{dt} = -\gamma\, X + (W + u(t))\, \xi(t), \tag{4.1}$$

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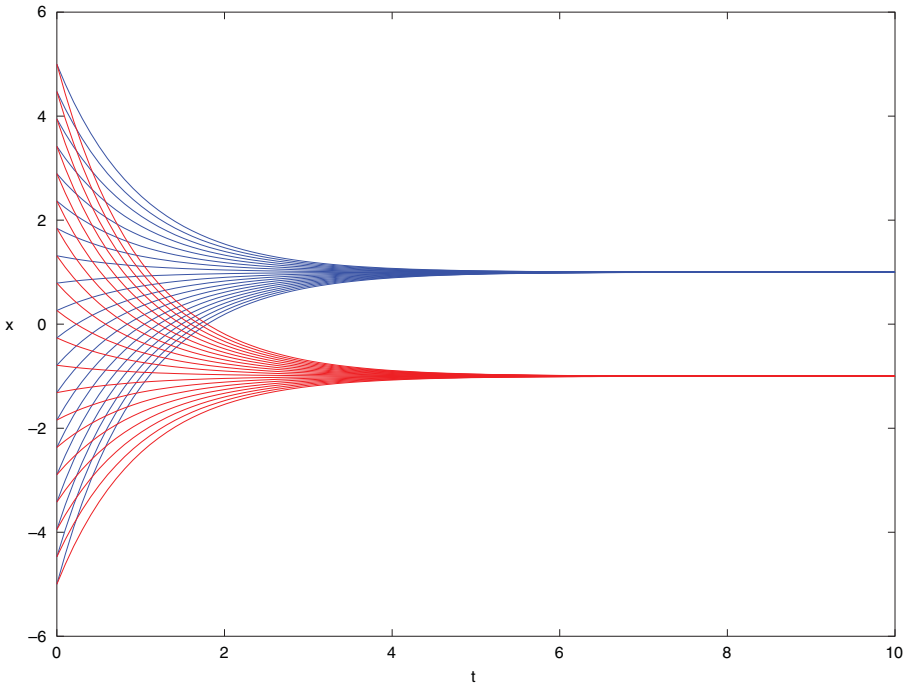


FIGURE 1. (Colour online) Characteristics of the FP equation for a PDP with dichotomic noise.

where the noised input $\xi(t)$ represents a dichotomic noise (or random telegraph signal), that takes values ± 1, with exponential statistics of the switching time given by the PDF $\mu \exp(-\mu t)$. The control function is assumed to be two-component piecewise constant in time, i.e. $u = (u_1, u_2)$. The solution to (4.1) is composed of pieces of increasing and decreasing exponentials, but the whole process $X(t)$ is not deterministic. In fact, it represents a random sample path in a probability space. For simplicity, we take $\gamma = 1$ and $W = 1$. Therefore, we have the following controlled dynamics

$$A_1(x, u_1) = 1 - x + u_1, \qquad A_2(x, u_2) = -(1 + x + u_2). \tag{4.2}$$

For the purpose of illustration, we depict in Figure 1 the characteristics of the hyperbolic FP system (3.12) corresponding to the setting (4.2) and $u_1 = 0$ and $u_2 = 0$. There are two families of curves, one for each state of the system. It is clear that if the initial non-zero PDF data is contained in the interval $(-1, 1)$, then the PDF will never escape from this interval during the time evolution. On the other hand, the PDF outside $(-1, 1)$ remains equal to zero. Within $(-1, 1)$, the PDF can switch between characteristics belonging to the two different families, but it will never cross the points $x = \pm 1$. We name the interval $(-1, 1)$ the invariant set, while the set $\mathbb{R}\backslash[-1, 1]$ is named transient, since an initial PDF located in this region will soon be transported and trapped in the invariant set. We denote with Ω_u the invariant set corresponding to the values of the controls $u = (u_1, u_2)$, that is $\Omega_u = (-(1 + u_2), 1 + u_1)$. To avoid both degeneracy of Ω_u, and that A_1 and A_2 become

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equal, we require the following constraints on the controls, $u_1, u_2 > -1$. In our numerical experiments, these bounds are never active.

Next, we focus on the controlled FP system for the PDP dichotomic model given by (3.12) with (4.2). We have

$$\partial_t f_1(x,t) + \partial_x\left((1 - x + u_1) f_1(x,t)\right) = -\mu f_1(x,t) + \mu f_2(x,t), \tag{4.3}$$

$$\partial_t f_2(x,t) - \partial_x\left((1 + x + u_2) f_2(x,t)\right) = +\mu f_1(x,t) - \mu f_2(x,t), \tag{4.4}$$

where u_1 and u_2 are assumed to be constant. Further, we assume appropriate initial conditions

$$f_s(x,0) = f_s^0(x), \qquad s = 1,2, \tag{4.5}$$

where $f_s^0(x) \geqslant 0$, $\sum_{s=1}^2 \int_\Omega f_s^0(x) = 1$.

Combining (4.3) and (4.4), and using integration by parts, we obtain

$$\partial_t \left(\int_\Omega f_1(x,t)\,dx + \int_\Omega f_2(x,t)\,dx \right) = 0.$$

This proves conservativeness of the FP system. In particular, we have

$$\int_\Omega f_1(x,t)\,dx + \int_\Omega f_2(x,t)\,dx = \int_\Omega f_1^0(x) + \int_\Omega f_2^0(x) = 1. \tag{4.6}$$

Next, we consider the adjoint problem. We have

$$-\partial_t p_1(x,t) - (1 - x + u_1)\,\partial_x p_1(x,t) = -\mu\, p_1(x,t) + \mu\, p_2(x,t), \tag{4.7}$$

$$-\partial_t p_2(x,t) + (1 + x + u_2)\,\partial_x p_2(x,t) = +\mu\, p_1(x,t) - \mu\, p_2(x,t), \tag{4.8}$$

with terminal condition given as follows

$$p_s(x,T) = -\left(f_s(x,T) - f_s^T(x)\right), \qquad s = 1,2.$$

Note that we cannot argue positiveness of the adjoint variable. On the other hand, since $\sum_{s=1}^2 \int_\Omega f_s^T(x) = 1$, from (4.7) and (4.8), we obtain the following

$$\partial_t \int_\Omega (p_1(x,t) + p_2(x,t))dx + \int_\Omega (p_1(x,T) + p_2(x,T))dx = 0, \tag{4.9}$$

since $\int_\Omega (p_1(x,T) + p_2(x,T))dx = 0$, we have $\int_\Omega (p_1(x,t) + p_2(x,t))dx = 0$ for all times.

5 Numerical solution

We consider the PDP with dichotomic noise to implement our control strategy. For this purpose, in this section we discuss a suitable approximation of the FP optimality system.

Consider the dichotomic FP problem in the time interval $(0,T)$ and a bounded spatial domain $\Omega = (-L, L)$. We choose L sufficiently large to include the invariant set and the space–time support of the forward and adjoint solutions. Furthermore, we assume that the control functions u_1 and u_2 are constant in $(0,T)$.

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For the time discretization, we define the time step size $\delta t = T/N$, in which N is a positive integer, and $I_{\delta t} = \{t_n = n\,\delta t, n = 0, 1, \ldots, N\}$ is the time mesh. Moreover, we consider a uniform mesh on the state space with mesh size $h = 2L/M$, and the mesh-point coordinates are denoted with $x_j = jh - L$, $j = 0, \ldots, M$. We have the following mesh

$$\Omega_h = \{x \in \mathbb{R} : x_j = j\,h,\ j \in \mathbb{Z}\} \cap \Omega.$$

For grid functions w and v defined on $\Omega_h \times I_{\delta t}$, with values $w_j^n = w(x_j, t_n)$, we introduce the discrete L^2-scalar product

$$(w^n, v^n)_h = h \sum_{j=1}^{M-1} w_j^n v_j^n,$$

with associated L^2-norm $\|v^n\| = (v^n, v^n)^{1/2}$. Further, we introduce the following inner product

$$\langle w, v \rangle_{h,\delta t} = \delta t \sum_{n=0}^{N-1} (w^n, v^n)_h.$$

Here and below, we denote the approximation to $f(x_j, t_n)$ with f_j^n.

Next, we consider the following explicit discretization of the time derivative

$$D_t f_j^n = \frac{1}{\delta t}\left(f_j^{n+1} - f_j^n\right), \tag{5.1}$$

and the first-order forward and backward space derivatives are given by

$$D_x^+ f_j^n = \frac{1}{h}\left(f_{j+1}^n - f_j^n\right), \qquad D_x^- f_j^n = \frac{1}{h}\left(f_j^n - f_{j-1}^n\right).$$

For ease of notation, let us denote with $f = f_1$ and $g = f_2$. According to upwind discretization, the discrete FP equations are as follows

$$D_t f_j^n + D_x^-\left((1 + u_1 - x_j)\, f_j^n\right)^+ + D_x^+\left((1 + u_1 - x_j)\, f_j^n\right)^- = -\mu\, f_j^n + \mu\, g_j^n, \tag{5.2}$$

$$D_t g_j^n - D_x^+\left((1 + u_2 + x_j)\, g_j^n\right)^+ - D_x^-\left((1 + u_2 + x_j)\, g_j^n\right)^- = +\mu\, f_j^n - \mu\, g_j^n, \tag{5.3}$$

where the symbols $(f)^+$ and $(f)^-$, denote $\max(f, 0)$ and $\min(f, 0)$, respectively, and we assume the boundary conditions $f_0^n = 0$ and $g_M^n = 0$ for $n = 0, \ldots, N$. Further, we have the following initial conditions

$$f_j^0 = f_{0,j}, \qquad g_j^0 = g_{0,j}. \tag{5.4}$$

We have the following theorem.

Theorem 1 *The discretization scheme (5.2)–(5.3) is stable, positive preserving, conservative and first-order accurate with respect to the solution of equations (4.3) and (4.4), provided that the following Courant–Friedrichs–Lewy condition on the time step size is satisfied*

$$\delta t \leqslant \min\left\{\frac{h}{2 + u_1 + \mu h}, \frac{h}{2 + u_2 + \mu h}\right\}. \tag{5.5}$$

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Proof Let us define the grid functions

$$b_1(x_j) = (1 + u_1 - x_j)\frac{\delta t}{h}, \qquad b_2(x_j) = -(1 + u_2 + x_j)\frac{\delta t}{h}.$$

The discrete FP system (5.2) and (5.3) becomes

$$\begin{aligned} f_j^{n+1} &= f_j^n(1 - |b_1(x_j)|) + (b_1(x_{j-1})f_{j-1}^n)^+ - (b_1(x_{j+1})f_{j+1}^n)^- \\ &\quad -\delta t\mu f_j^n + \delta t\mu g_j^n, \end{aligned} \tag{5.6}$$

$$\begin{aligned} g_j^{n+1} &= g_j^n(1 - |b_2(x_j)|) + (b_2(x_{j+1})g_{j+1}^n)^+ - (b_2(x_{j-1})g_{j-1}^n)^- \\ &\quad +\delta t\mu f_j^n - \delta t\mu g_j^n. \end{aligned} \tag{5.7}$$

Further, we have

$$\begin{aligned} f_j^{n+1} + g_j^{n+1} &= f_j^n(1 - |b_1(x_j)|) + (b_1(x_{j-1}))^+ f_{j-1}^n - (b_1(x_{j+1}))^- f_{j+1}^n \\ &\quad +g_j^n(1 - |b_2(x_j)|) + (b_2(x_{j+1}))^+ g_{j+1}^n - (b_2(x_{j-1}))^- g_{j-1}^n. \end{aligned} \tag{5.8}$$

By inspection of (5.6)–(5.7) and (5.8), we conclude that positivity of f_j^n, g_j^n and of $f_j^n + g_j^n$, $n = 1, \ldots, N$ and $j = 0, \ldots, M$, is guaranteed if the following requirements are satisfied:

$$\max_{x\in\Omega} \big(|b_1(x)|, |b_2(x)|\big) \leqslant 1$$

and

$$\min_j(1 - b_1(x_j)) - \delta t\mu \geqslant 0, \qquad \min_j(1 - b_2(x_j)) - \delta t\mu \geqslant 0.$$

Note that we assume that $f_j^0 + g_j^0 \geqslant 0$ for $j = 0, \ldots, M$. Now, it is immediate to see that these two conditions hold if (5.5) is true; see also [4].

Next, to prove conservativeness, we take the sum of (5.8) on all interior grid points. We obtain

$$\sum_{j=1}^{M-1} \left(f_j^{n+1} + g_j^{n+1}\right) = \sum_{j=1}^{M-1} \left(f_j^n + g_j^n\right) - \delta t\, f_{M-1}^n - \delta t\, g_1^n, \tag{5.9}$$

where we use the fact $b_1(x_{N-1}) = \delta t$ and $b_2(x_1) = \delta t$ (for $u_1 = u_2 = 0$). Note that in the case of a compactly supported solution such that $f_{M-1}^n = 0$ and $g_1^n = 0$, our first-order scheme is conservative and positive preserving. In addition, the same result proves that the scheme is stable.

Next, we discuss the accuracy of the scheme (5.2) and (5.3). We assume sufficient regularity of the data such that the solution of the continuous problem is twice continuously differentiable in space and time. Denote with ϕ and ψ the local truncation error for the discretization of (5.2) and (5.3), respectively. It is a standard calculation [4] to show that the following holds

$$|\phi_j^n| \leqslant C\,(\delta t + h), \qquad |\psi_j^n| \leqslant C\,(\delta t + h), \tag{5.10}$$

where C depends on the maximum of the second derivatives of f and g in the space–time domain. Therefore, it is a consequence of the Lax–Richtmyer equivalence theorem [43] that the estimate (5.10) and the stability of the scheme prove that the scheme is first-order accurate. □

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Now, we focus on the discretization of the adjoint system. For optimization purposes, it is desirable that the discrete adjoint system corresponds to the discrete adjoint (transpose) of the forward system. For this reason, we derive the discrete adjoint system directly from the discrete Lagrange function.

The discrete Lagrange function is as follows. We write the discrete cost functional

$$J_h = \frac{h}{2}\sum_{j=1}^{M-1}\left[\left(f_j^N - f_j^T\right)^2 + \left(g_j^N - g_j^T\right)^2\right] + \frac{v}{2}\delta t\sum_{n=0}^{N-1}\left[\left(u_1^n\right)^2 + \left(u_2^n\right)^2\right],$$

and the Lagrange function is given by

$$\begin{aligned} L_h &= J_h + \left\langle D_t f + D_x^+(Af)^- + D_x^-(Af)^+ + \mu f - \mu g, p\right\rangle_{h,\delta t} \\ &\quad + \left\langle D_t g + D_x^+(Ag)^- + D_x^-(Ag)^+ - \mu f + \mu g, q\right\rangle_{h,\delta t} \\ &= J_h + h\delta t\sum_{j=1}^{M-1}\sum_{n=0}^{N-1} p_j^n\left[\frac{f_j^{n+1} - f_j^n}{\delta t} + D_x^+(A_j f_j^n)^- + D_x^-(A_j f_j^n)^+ + \mu f_j^n - \mu g_j^n\right] \\ &\quad + h\delta t\sum_{j=1}^{M-1}\sum_{n=0}^{N-1} q_j^n\left[\frac{g_j^{n+1} - g_j^n}{\delta t} + D_x^+(A_j g_j^n)^- + D_x^-(A_j g_j^n)^+ - \mu f_j^n + \mu g_j^n\right], \quad (5.11) \end{aligned}$$

where we denote with p the adjoint variable corresponding to the FP equation for f, and with q the adjoint variable corresponding to the FP equation for g.

For the purpose of clear illustration, in the derivation of the adjoint system we consider only summation over the space index of the f component. We have

$$\sum_{j=1}^{M-1} p_j^n\left[\frac{(A_j f_j^n)^+ - (A_{j-1} f_{j-1}^n)^+}{h} + \frac{(A_{j+1} f_{j+1}^n)^- - (A_j f_j^n)^-}{h} - (-\mu f_i^n + \mu g_i^n)\right].$$

Since f is non-negative, we have $(A_j f_j)^+ = (A_j)^+ f_j$ and $(A_j f_j)^- = (A_j)^- f_j$. Now, we shift the spatial index in the following summations

$$\sum_{j=1}^{M-1} p_j^n (A_{j-1} f_{j-1}^n)^+ = \sum_{j=1}^{M-1} p_{j+1}(A_j)^+ f_j^n \quad \text{and} \quad \sum_{j=1}^{M-1} p_j^n (A_{j+1} f_{j+1}^n)^- = \sum_{j=1}^{M-1} p_{j-1}(A_j)^- f_j^n,$$

where we use the fact that f has compact support. Now, the summation reads as follows

$$\sum_{j=1}^{M-1} f_j^n\left[\frac{(A_j)^+(p_j^n - p_{j+1}^n)}{h} + \frac{(A_j)^-(p_{j-1}^n - p_j^n)}{h} + \mu p_j^n\right] - \mu f_j^n q_j^n.$$

Next, we consider the discrete Lagrange function rewritten using the transformation just described and collect all terms that multiply f_j^n. We consider the derivative of the Lagrange function with respect to this variable and equate to zero the resulting gradient. We obtain

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the following discrete adjoint system

$$-\frac{1}{\delta t}(p_j^n - p_j^{n-1}) = (A_j)^+ D_x^+(p_j^n) + (A_j)^- D_x^-(p_j^n) - \mu p_j^n + \mu q_j^n, \tag{5.12}$$

$$-\frac{1}{\delta t}(q_j^n - q_j^{n-1}) = (A_j)^+ D_x^+(q_j^n) + (A_j)^- D_x^-(q_j^n) + \mu p_j^n - \mu q_j^n, \tag{5.13}$$

with the following discrete terminal conditions

$$\begin{aligned} p_j^N &= f_j^T - f_j^N, \\ q_j^N &= g_j^T - g_j^N. \end{aligned} \tag{5.14}$$

Equations (5.12) and (5.13) correspond to the discrete version of the system (4.7) and (4.8). Note that in (5.12) and (5.13), the upwind sign of the discrete derivative is applied in reverse mode. This is correct because the adjoint equation is solved backward in time, so that the usual upwind condition has to be inverted. Note that the discussion above is valid also for the case $S > 2$ and a general non-symmetric matrix Q. By inspection we find that the evolution matrix representing (5.12) and (5.13) corresponds to the transpose of the evolution matrix representing (5.6) and (5.7), that is what we expected from the discretize then optimize approach [12].

To complete the discussion on the discretization of the optimality system, we consider the following two components of the gradient

$$(\nabla \hat{J})_1 = \nu u_1 - \int_0^T \int_\Omega \partial_x p(x,t)\, \partial_{u_1} A_1(x,u_1)\, f(x,t)\, dx\, dt, \tag{5.15}$$

$$(\nabla \hat{J})_2 = \nu u_2 - \int_0^T \int_\Omega \partial_x q(x,t)\, \partial_{u_2} A_2(x,u_2)\, g(x,t)\, dx\, dt. \tag{5.16}$$

We approximate these optimality conditions by using for the space derivatives the centred difference scheme in the interior of the domain and the forward/backward scheme next to the boundary. We use the rectangle scheme for the quadrature of integrals.

6 A model predictive control scheme

Our purpose is to define a control strategy for the PDF of a PDP to track a given sequence of desired PDFs in time. For this reason, we consider the minimization of the tracking objective (3.9) at given time intervals. For our discussion, it is again convenient to denote $f = (f_s)_{s=1}^S$ and $u = (u_s)_{s=1}^S$, with $u_s \in U_s \subset \mathbb{R}$, so that $u \in U \subset \mathbb{R}^S$.

Let $(0, T)$ be the time interval where the process is considered. We choose time windows of size $\Delta t = T/K$ with K a positive integer. Let $t_k = k\Delta t$, $k = 0, 1, \ldots, K$. At time t_0, we have a given initial vector of PDFs denoted with $\rho = (\rho_s)_{s=1}^S$, and with $f^T(\cdot, t_k)$ we denote the sequence of desired vector PDFs.

Our scheme starts at time t_0 and solves the open-loop FP optimal control problem defined in the interval (t_0, t_1). Then, with the PDF f resulting at $t = t_1$, we define the initial PDFs for the subsequent optimization problem (3.10) defined in the interval (t_1, t_2). This procedure is repeated by receding the time horizon until the last time window is reached. This is an instance of the class of MPC schemes [29, 37, 39] that is widely used

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in engineering applications to design closed-loop algorithms. In fact, we implement a MPC scheme where the time horizon used to evaluate the control coincides with the time horizon where the control is used. One important aspect of this approach is that it can be applied to infinite- dimensional evolution systems [33], that is the case of the FP model. We refer to [44] to show that the closed-loop system with the MPC scheme is nominally asymptotically stable.

The MPC procedure is summarized in the following algorithm.

Algorithm 2 (MPC Control) Set $k = 0$, set $\rho_0 = \rho$;

(1) Assign the initial PDFs, $f(x, t_k) = \rho_k(x)$, and the targets $f^T(\cdot, t_{k+1})$;

(2) In (t_k, t_{k+1}), apply an optimization to solve (3.10), i.e. $\min_{u\in U} J(f(u), u)$, thus obtain the optimal pair (f, u);

(3) If $t_{k+1} < T$, set $k := k + 1$, $\rho_k = f(\cdot, t_k)$, go to (1);

(4) End.

Next, we discuss the first step of Algorithm 2, that consists in solving $\min_{u\in U} J(f(u), u)$. In fact, the solution of the FP control problem gives the mapping $u \to f(u)$ that allows to transform the constrained optimization problem in an unconstrained one as follows

$$\min_{u\in U} \hat{J}(u). \tag{6.1}$$

Thus, the solution to the FP system is included into the objective at $u \in U$. Further, to compute $\nabla_u \hat{J}(u)$ for a given u, we have to solve first the forward FP system and then the adjoint FP system. This procedure is summarized in the following

Algorithm 3 (Evaluation of the gradient at u)

(1) Solve the discrete FP equations (5.2) and (5.3) with given initial condition (5.4);

(2) Solve the discrete adjoint FP equations (5.12) and (5.13) with terminal condition (5.14);

(3) Compute the gradient $\nabla_u \hat{J}(u)$ using (5.15) and (5.16);

(4) End.

Based on the gradient, we solve the optimization problem (6.1) implementing a nonlinear conjugate gradient scheme; see, e.g. [28, 49].

It is clear that the solution of the optimality system may become prohibitive when high-dimensional stochastic processes are considered. In this case, special techniques for solving high-dimensional PDEs should be considered; see, e.g. [30, 50].

7 Numerical experiments

In this section, we present results of numerical experiments to validate our control framework for PDP. We first consider a PDP model where the dynamics corresponds to

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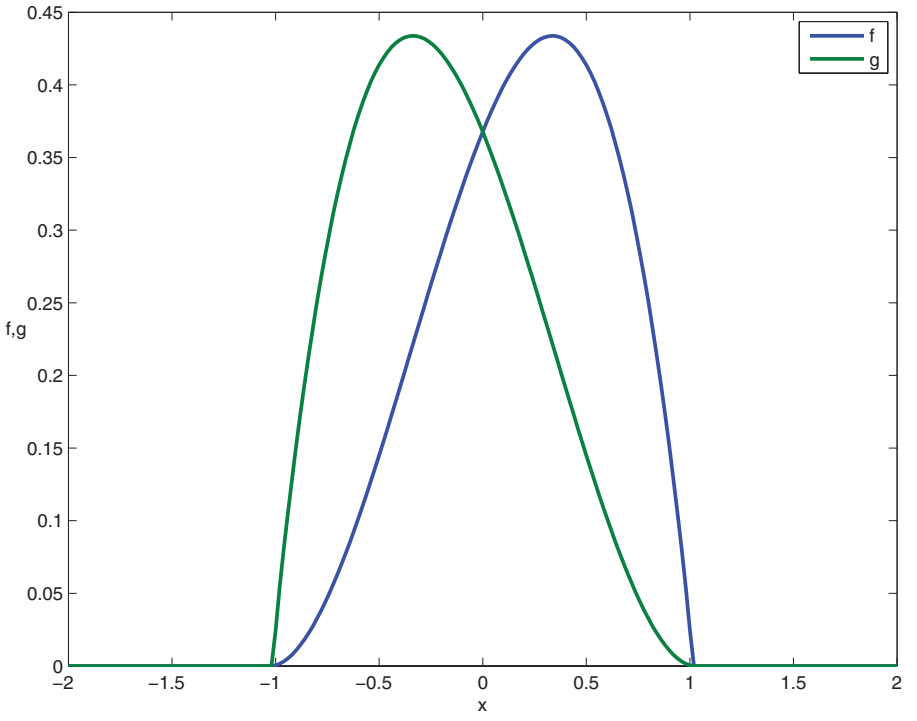


FIGURE 2. (Colour online) The PDFs at time $t = 10$ resulting from the FP evolution without control.

a linear filter. This is the case discussed in Section 4, where the following dynamics is considered $A_1(x, u_1) = 1 - x + u_1$ and $A_2(x, u_2) = -(1 + x + u_2)$. We remark that in this case the control in the differential model enters linearly. For the uncontrolled process an analytical solution has been obtained by using a spectral method [42].

Further, we discuss the optimal control of a non-linear filtering PDP. This process was investigated in [45], and corresponds to the following choice $A_1(x, u_1) = -(\beta + u_1)(\sin(x) - 1)$ and $A_2(x, u_2) = -(\beta + u_2)(\sin(x) + 1)$. In this case, the control defines a bilinear structure with the state variable. In both cases, the purpose of the control is to drive the PDP to attain a given configuration in a short time.

Consider a PDP with linear-filter dynamics and with dichotomic noise. We assume that initial distribution of the PDP system is given by two Gaussian distributions, defined in $\Omega = (-2, 2)$, centred in $x = 0$ and variance $\sigma^2 = 10^{-2}$. The transition rate of the underling Markov process is $\mu = 2$. See Figure 2 for a snapshot of the FP solution at $t = 10$ without control.

We define a PDF target given by two Gaussian densities, f^T and g^T, travelling in opposite directions and with increasing variances as follows

$$\begin{aligned} f^T(x,t) &= \frac{1}{2\sqrt{2\pi\sigma^2(t)}} e^{-\frac{(x-m_1(t))^2}{2\sigma^2(t)}}, \\ g^T(x,t) &= \frac{1}{2\sqrt{2\pi\sigma^2(t)}} e^{-\frac{(x-m_2(t))^2}{2\sigma^2(t)}}. \end{aligned} \tag{7.1}$$

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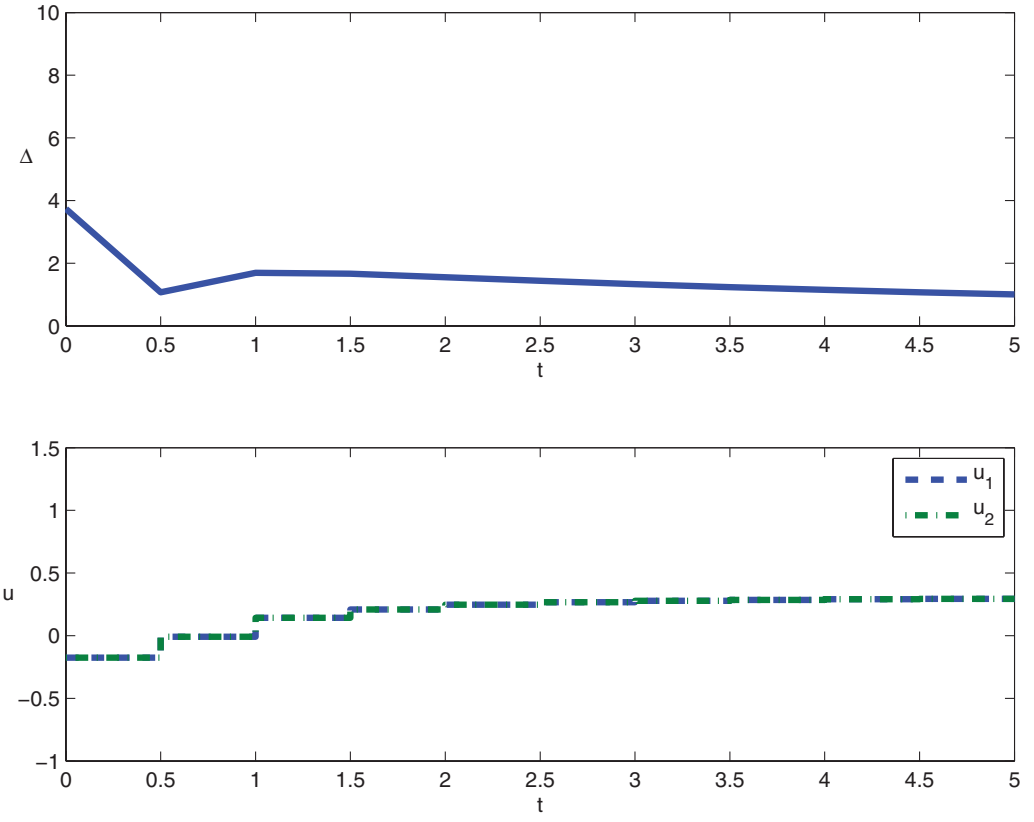


FIGURE 3. (Colour online) Results obtained with the MPC scheme applied to the PDP. Top: the norm difference between the desired target PDFs and the controlled PDFs. Bottom: the evolution of the control functions.

We apply our MPC control strategy (Algorithm 2) for this linear-filter PDP stochastic model to obtain optimal controls $u = (u_1, u_2)$ at each time window that drive the system to a desired PDF.

In the first experiment, the tracking functions are specified choosing the following $m_1(t) = 0.7(1 - e^{-t})$, $m_2(t) = -m_1(t)$, $\sigma^2(t) = 10^{-2}(1 + t)$. We consider a time horizon with $T = 5$ and time windows of size $\Delta t = 0.5$, so that $t_k = k\Delta t, k = 1, \ldots 10$.

For the discretization of the forward and adjoint FP systems, we choose $h = 0.01$ and δt is given by the Courant–Friedrichs–Lewy condition (5.5). The weight of the cost of the control in the cost functional is $v = 0.1$. Results of numerical experiments for this setting are reported in Figure 3, where we depict the max-norm of the difference between the desired target and the calculated PDFs, $\Delta(t_k) = \|f(\cdot, t_k) - f^T(\cdot, t_k)\|_\infty + \|g(\cdot, t_k) - g^T(\cdot, t_k)\|_\infty$. We see that this tracking error decreases along the time evolution. In the same figure, we show the evolution of the control functions that are indistinguishable since the two target PDFs have a symmetric dynamics.

In the next experiment, we consider a pair of Gaussian densities travelling with different velocities for the distribution mean as follows, $m_1(t) = 1 - e^{-t}$ and $m_2(t) = -m_1(t)/3$, so that the pair is asymmetric with respect to zero. The results of our control strategy for

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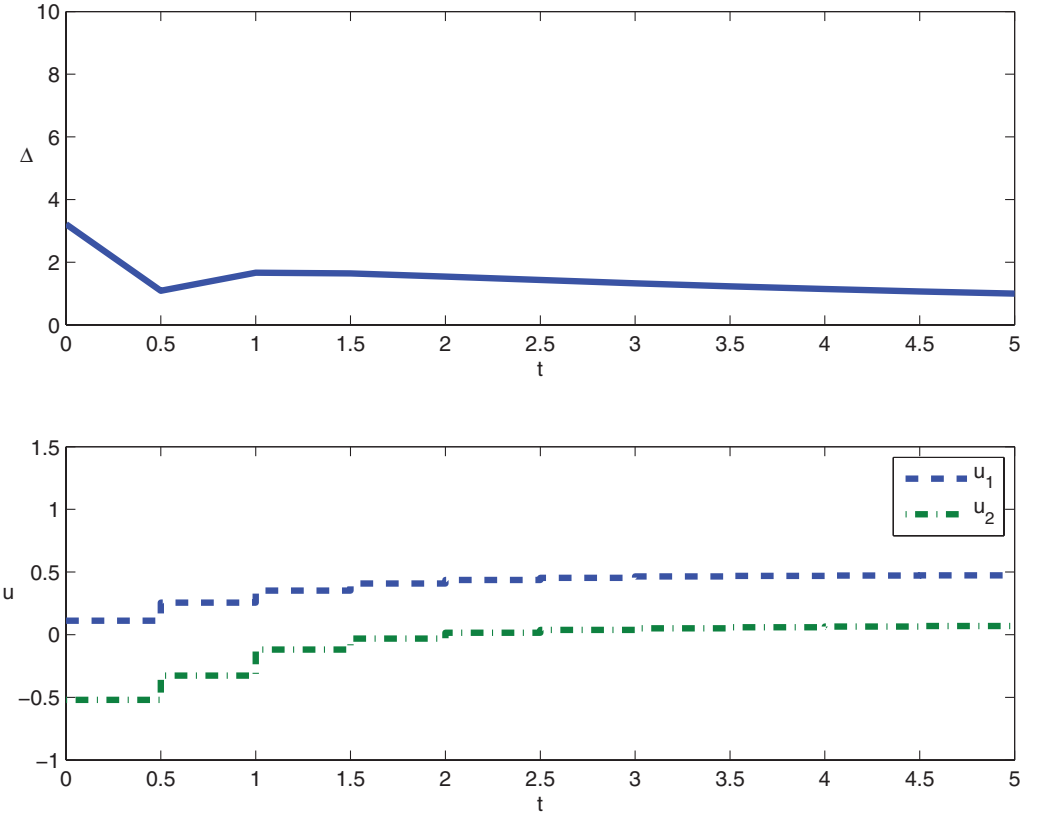


FIGURE 4. (Colour online) Results obtained with the MPC scheme applied to the PDP. Top: the norm difference between the desired target PDFs and the controlled PDFs. Bottom: the evolution of the control functions.

this case are reported in Figure 4. We see that the control functions are distinct according to the need to track the asymmetric pair of Gaussian PDFs. We show in Figure 5 the desired and obtained PDFs at the final time. The ability of the control to drive the process close to the desired target is demonstrated by a reduction of the tracking error: $\Delta = 1.00$ with control and $\Delta = 1.19$ without control. Figure 5 shows that both marginal densities are driven towards the respective targets.

In another series of experiments, we take the transition rate $\mu = 0.8$. A value of $\mu < 1$ results in a PDF evolution that tends to produce two peaks at the borders of the invariant set, one for each state of the system. In Figure 6, results for this case are shown for the uncontrolled and controlled case. In this figure, we can see that the control task effectively drives the PDFs to the desired target configuration. Figure 7 shows the resulting uncontrolled and controlled PDFs at final time.

Similar results are obtained by choosing a value of the weight of the cost $\nu = 10^{-4}$. The results corresponding to this choice, with $\mu = 2$ and asymmetric targets, are shown in Figures 8 and 9.

It is important to validate our control strategy directly with the PDP stochastic model. For this reason, once we compute the optimal controls, we insert them in the PDP model

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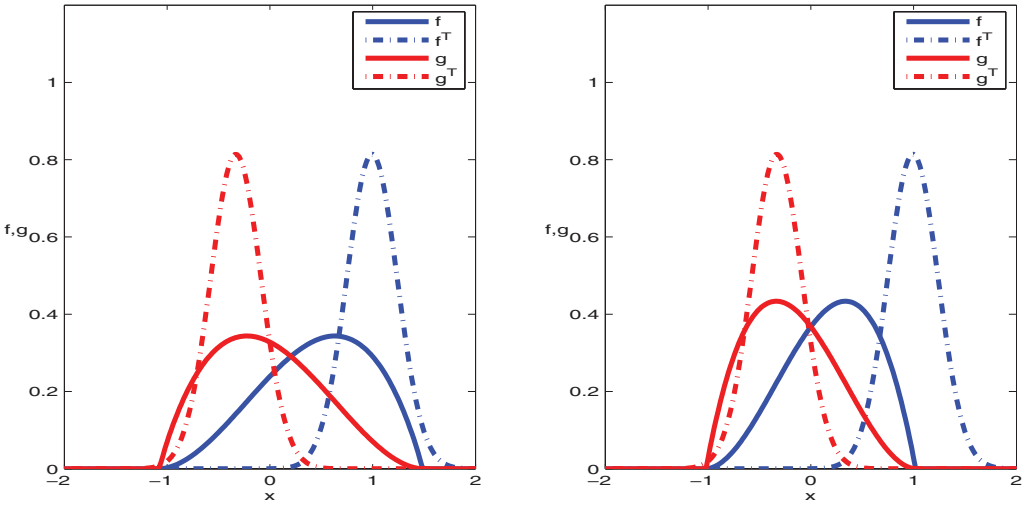


FIGURE 5. (Colour online) Results obtained with the MPC scheme applied to the PDP, $\mu = 2$ and $\nu = 10^{-1}$. Dotted lines denote the desired PDFs target. Solid lines represent the PDFs resulting from the FP evolution. Left: the controlled process; right: the uncontrolled process.

and perform Monte Carlo simulations with the PDP equations using the given optimal controls. Note that the same control functions are used for all instances of the PDP. In Figure 10, we show the trajectories of the uncontrolled and controlled PDP with $\mu = 2$ and $\nu = 10^{-4}$. We obtain that at $t = T$, the controlled trajectories tend to distribute consistently with the desired PDFs; compare with Figure 9. In the uncontrolled case, they distribute approximating the equilibrium PDF.

In the second part of this section, we discuss the optimal control of a non-linear filtering PDP with dichotomic noise [45]. In this case, we have $A_1(x, u_1) = -(\beta + u_1)(\sin(x) - 1)$ and $A_2(x, u_2) = -(\beta + u_2)(\sin(x) + 1)$. The FP equation corresponding to this model is given by

$$\partial_t f(x,t) - (\beta + u_1)\partial_x\left((\sin(x) - 1)\, f(x,t)\right) = -\mu\, f(x,t) + \mu\, g(x,t), \tag{7.2}$$

$$\partial_t g(x,t) - (\beta + u_2)\partial_x\left((\sin(x) + 1)\, g(x,t)\right) = +\mu\, f(x,t) - \mu\, g(x,t). \tag{7.3}$$

With $u_1 = u_2 = 0$, we have the following sum of equilibrium PDFs of the two states [45]

$$(f_{eq} + g_{eq})(x) = \frac{\sec(x)^2}{2K_1(2\alpha)} e^{-2\alpha \sec(x)}, \quad |x| \leqslant \frac{\pi}{2}, \quad \alpha = \frac{\mu}{\beta}, \tag{7.4}$$

where K_1 is a modified Bessel's function of the second kind. The equilibrium state of the PDP is the purpose of our control. However, only the sum of the equilibrium PDFs is known while we need the single PDFs.

We remark that the controls do not affect the invariant set of this process, that is the bounded set $\Omega = (-\pi/2, \pi/2)$. In this experiment, we consider initial distributions given by two Gaussian densities, both with mean values $m_{1,2} = -0.4$ and variance $\sigma^2 = 10^{-2}$, and we set $\mu = 3$ and $\beta = 1$. We determine the single-target PDFs by solving the forward

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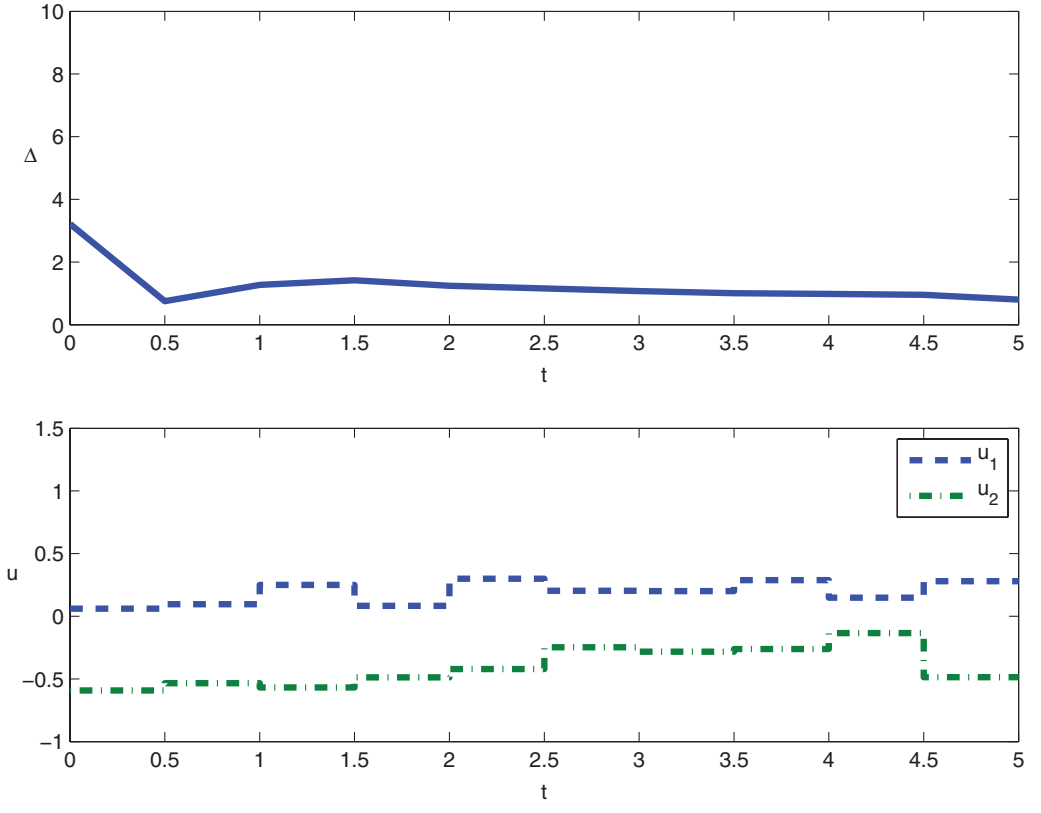


FIGURE 6. (Colour online) Results obtained with the MPC scheme applied to the PDP, $\mu = 0.8$. Top: the norm difference between the desired tracking functions and the controlled PDFs. Bottom: the evolution of the control functions.

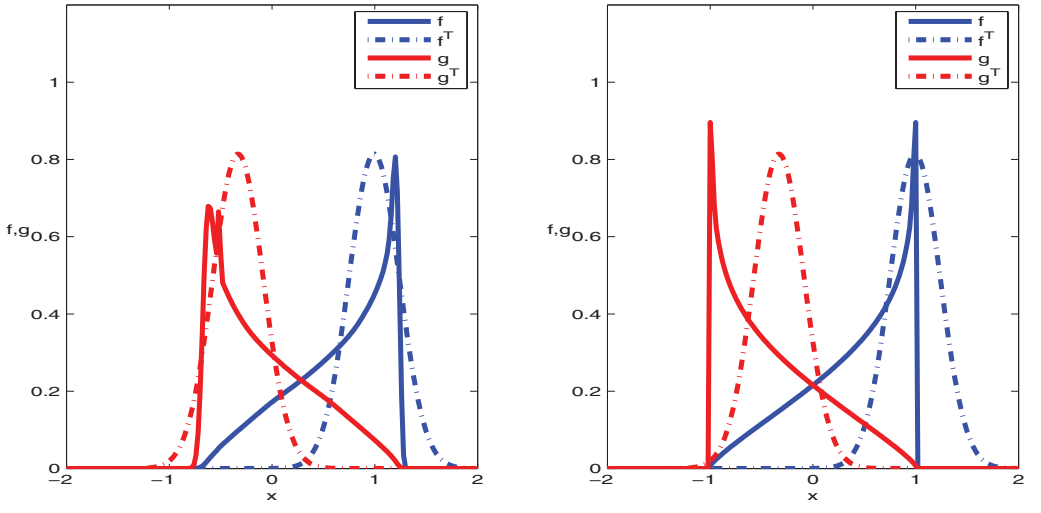


FIGURE 7. (Colour online) Results obtained with the MPC scheme applied to the PDP with transition rate $\mu = 0.8$. Dotted lines denote the desired PDF target. Solid lines represent the PDFs resulting from the FP evolution. Left is for the controlled process; right is the uncontrolled process.

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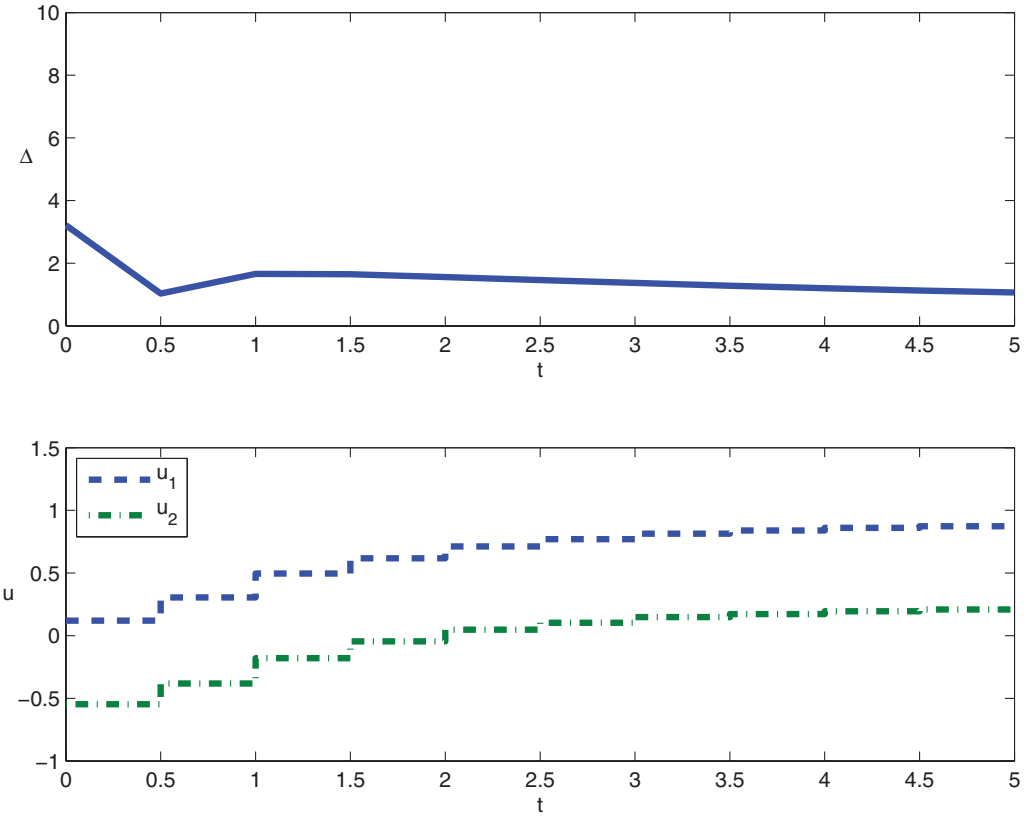


FIGURE 8. (Colour online) Results obtained with the MPC scheme applied to the PDP with transition rate $\mu = 2$ and cost weight $v = 10^{-4}$. Top: the norm difference between the desired tracking functions and the controlled PDFs. Bottom: the evolution of the control.

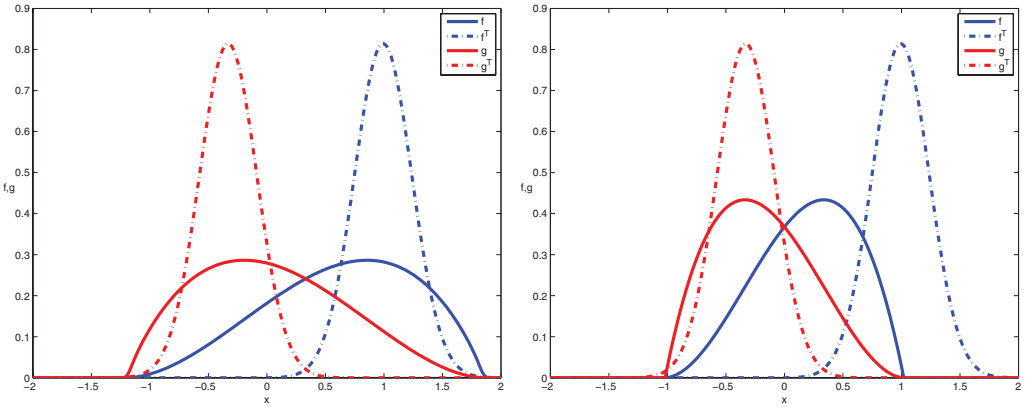


FIGURE 9. (Colour online) Left: results with the MPC scheme applied to the PDP with transition rate $\mu = 2$ and cost weight $v = 10^{-4}$; dotted lines represent the desired marginal densities. Right: uncontrolled solution.

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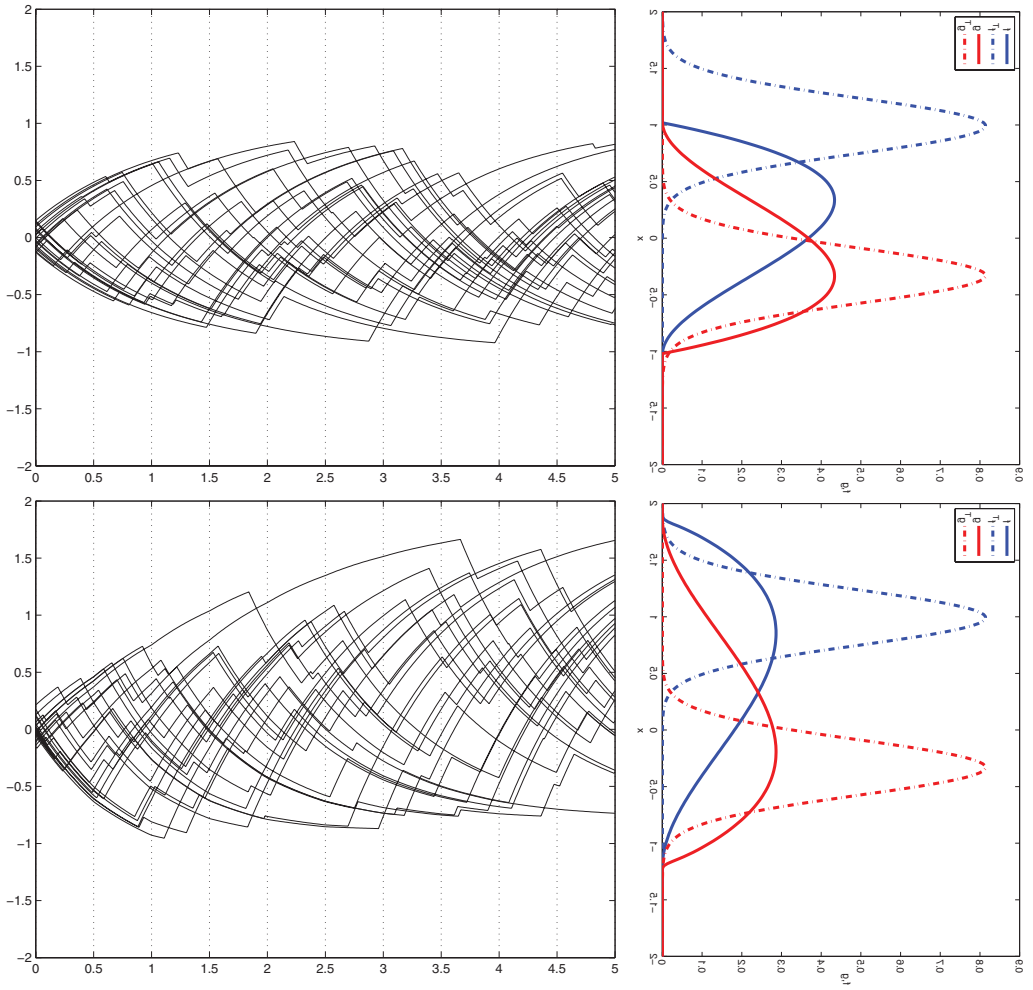


FIGURE 10. (Colour online) Left: trajectories of the PDP obtained with Monte Carlo simulation. Right: dotted lines represent desired PDFs, solid lines represent the resulting PDFs. Top: uncontrolled process. Bottom: controlled trajectories.

FP system with $T = 15$, where we assume that the equilibrium state is reached with a good approximation, in the sense that $\|f(\cdot, T) + g(\cdot, T) - (f_{eq}(\cdot) + g_{eq}(\cdot))\|_\infty \leqslant \varepsilon_{eq}$.

Once the equilibrium configurations for f and g are computed (we choose $\varepsilon_{eq} = 10^{-2}$), we start our optimal control procedure of Algorithm (2) using these numerical PDFs in the tracking functional. In Table 1, we report results of numerical experiments with $T = 3$ and $\nu = 1$. For each MPC time window, we show in one column the max-norm of the difference between the uncontrolled total PDF and the equilibrium PDF given by (7.4). Also in this table, we report the values of the control functions. In Table 1, we report results of a similar experiment and $\nu = 10^{-3}$. In this case, the action of the controls results in faster convergence to the equilibrium state. Moreover, consistently with the optimal

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Table 1. *Results for the non-linear filtering model with equilibrium target. Space mesh* $N_x = 600$, $\varepsilon_{eq} = 9.43\,10^{-3}$

	Time	$\|f^*(t_k) - f_{eq}\|_\infty$	$\|f(t_k) - f_{eq}\|_\infty$	u_1	u_2
$v = 1$	0	3.34E+00	3.34E+00	0	0
	1	3.45E−01	3.00E−01	7.07E−02	−3.13E−02
	2	8.26E−02	5.38E−02	1.56E−02	−1.27E−02
	3	3.50E−02	2.14E−02	4.77E−03	−4.44E−03
$v = 10^{-3}$	0	3.34E+00	3.34E+00	0	0
	1	3.45E−01	1.76E−01	3.08E−01	−1.34E−01
	2	8.26E−02	1.58E−02	1.17E−01	−8.72E−03
	3	3.50E−02	9.28E−03	2.28E−03	−1.19E−03

control formulation we see that a more effective control action is obtained when choosing a smaller weight of the cost of the control.

8 Conclusion

A new framework for the optimal control of PDP was presented. This framework is based on a hyperbolic FP system that governs the time evolution of the PDFs of PDP and on deterministic tracking objectives with terminal target PDFs. To obtain a closed-loop control scheme, the solution of a sequence of open-loop FP optimality systems in a MPC framework was considered.

The resulting first-order hyperbolic optimality systems were discretized by first-order schemes that guarantee positivity and conservativeness of the numerical PDFs. The effectiveness of the proposed computational control framework to control the PDF of PDP was successfully validated considering PDP with dichotomic noise.

We remark that our deterministic control aims at controlling the entire PDF. This is in contrast to existing approaches where the mean value of functionals are used. In fact, the present methodology is more general since it can accommodate any meaningful objective and control strategy. In particular, the controls can have continuous dependence on time and on the state variables.

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