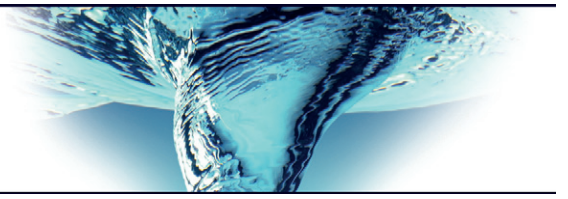
CrossMark

J. Fluid Mech. (2015), *vol.* 776, R1, doi:10.1017/jfm.2015.302



JFM RAPIDS
journals.cambridge.org/rapids

Simple and efficient representations for the fundamental solutions of Stokes flow in a half-space

Z. Gimbutas[1], **L. Greengard**[2,3] **and S. Veerapaneni**[4,†]

[1]Information Technology Laboratory, National Institute of Standards and Technology, 325 Broadway, Mail Stop 891.01, Boulder, CO 80305-3328, USA

[2]Simons Center for Data Analysis, Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA

[3]Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012-1110, USA

[4]Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA

(Received 3 April 2015; revised 20 May 2015; accepted 24 May 2015; first published online 2 July 2015)

We derive new formulae for the fundamental solutions of slow viscous flow, governed by the Stokes equations, in a half-space. They are simpler than the classical representations obtained by Blake and collaborators, and can be efficiently implemented using existing fast solver libraries. We show, for example, that the velocity field induced by a Stokeslet can be annihilated on the boundary (to establish a zero-slip condition) using a single reflected Stokeslet combined with a single Papkovich–Neuber potential that involves only a scalar harmonic function. The new representation has a physically intuitive interpretation.

Key words: boundary integral methods, low-Reynolds-number flows, Stokesian dynamics

1. Introduction

Viscous flow of passive and active suspensions in the presence of an infinite planar boundary is an important physical model in many areas of science and engineering. It serves as a useful paradigm for understanding the effect of confined geometries on the macroscopic flow behaviour of particulate flows, for example those of bacterial propulsion, cellular blood flow and colloidal suspensions (Kim & Karrila 1991; Cichocki & Jones 1998; Bhattacharya & Bławzdziewicz 2002; Veerapaneni *et al.* 2011; Spagnolie & Lauga 2012). In problems where the Reynolds number is low, the ambient fluid is governed by the Stokes equations:

$$\mu\Delta\boldsymbol{u}(\boldsymbol{x}) = \nabla p(\boldsymbol{x}), \quad \nabla\cdot\boldsymbol{u}(\boldsymbol{x}) = 0, \tag{1.1a,b}$$

† Email address for correspondence: shravan@umich.edu

© Cambridge University Press 2015

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

where μ is the fluid viscosity, $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), u_3(\boldsymbol{x}))$ is the velocity of the fluid and $p(\boldsymbol{x})$ is the pressure. Assuming that the plane wall is located at $x_3 = 0$ and that the flow velocity decays in the far field,

$$\boldsymbol{u}(\boldsymbol{x}) \to 0 \quad \text{as } |\boldsymbol{x}| \to \infty, \tag{1.2}$$

the no-slip boundary condition is

$$\boldsymbol{u}|_{x_3=0} = 0. \tag{1.3}$$

Boundary integral methods are particularly well suited for problems of this kind since they discretize the domain boundary alone (resulting in many fewer degrees of freedom) and impose the decay and the no-slip conditions exactly. Moreover, they avoid the need for artificial truncation of the computational domain and can be solved rapidly and with great accuracy using fast algorithms such as the fast-multipole method (FMM) and high-order accurate quadrature rules.

In order to reformulate the Stokes equations as a boundary integral equation, however, one needs to have access to the Green's function for the half-space (Lorentz 1896, 1907; Lamb 1932; Happel & Brenner 1983). A now classical approach to constructing this Green's function is due to Blake and others (Blake 1971, 1974; Blake & Chwang 1974; Aderogba & Blake 1978*a*,*b*; Pozrikidis 1992; Yu 2003). Unfortunately, the resulting formulae are rather complicated, making them somewhat difficult to implement.

Here, we show that a much simpler alternative to the Blake solution can be obtained by combining a free-space image, which annihilates the tangential components of velocity, with a Papkovich–Neuber potential (Papkovich 1932; Neuber 1934), which annihilates the normal component.

The present paper is organized as follows. We discuss Papkovich–Neuber potentials and the standard fundamental solutions for the Stokes equations in § 2 (see, for example, Lamb 1932; Happel & Brenner 1983; Kim & Karrila 1991; Pozrikidis 1992). We also review Blake's formula for the Stokeslet in a half-space. In § 3, we derive the image structures for Stokeslets, stresslets, rotlets and Stokes doublets.

In the appendices, we provide the analogous formulae for two-dimensional half-space Stokes kernels and for some problems of linear elasticity.

2. Fundamental solutions, the Papkovich–Neuber representation and Blake's formulae

Before discussing the various standard fundamental solutions for the Stokes equations in free space, we introduce the Papkovich–Neuber representation, originally developed in Papkovich (1932) and Neuber (1934) for problems of linear elasticity. Without loss of generality, we assume that the fluid viscosity $\mu = 1$ in the rest of the paper.

DEFINITION 2.1. Let $\phi(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$, be a harmonic function. Then, the induced Papkovich–Neuber representation is defined to be the paired vector field $\boldsymbol{u}$ and scalar field p given by

$$\boldsymbol{u}(\boldsymbol{x}) = x_3 \nabla \phi(\boldsymbol{x}) - \begin{bmatrix} 0 \\ 0 \\ \phi(\boldsymbol{x}) \end{bmatrix}, \quad p(\boldsymbol{x}) = 2\frac{\partial \phi(\boldsymbol{x})}{\partial x_3}. \tag{2.1a,b}$$

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

It is straightforward to verify that $(\boldsymbol{u}, p)$ satisfy the Stokes equations (1.1).

Suppose now that a force vector $\boldsymbol{f} = (f_1, f_2, f_3)$ is applied to a viscous fluid at a point $\boldsymbol{y}$. Then, it is well known that the induced velocity and pressure can be computed using the Stokeslet (the single-layer kernel):

$$S_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{8\pi}\left[\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right], \quad i, j = 1, 2, 3, \tag{2.2}$$

$$P_j(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi}\frac{x_j - y_j}{|\boldsymbol{x}-\boldsymbol{y}|^3}, \tag{2.3}$$

where δ_{ij} is the Kronecker delta. More precisely, the velocity vector $\boldsymbol{u}(\boldsymbol{x})$ and pressure $p(\boldsymbol{x})$ are given by

$$u_i(\boldsymbol{x}) = \sum_{j=1}^{3} S_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j, \quad p(\boldsymbol{x}) = \sum_{j=1}^{3} P_j(\boldsymbol{x}, \boldsymbol{y}) f_j. \tag{2.4a,b}$$

The stresslet (or double-layer kernel) for the Stokes equations describes the velocity induced by an infinitesimal displacement $\boldsymbol{g} = (g_1, g_2, g_3)$ (sometimes called the double-force source strength) at a point $\boldsymbol{y}$ with orientation vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$. It is given by

$$T_{ijk}(\boldsymbol{x}, \boldsymbol{y}) = \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}, \quad i, j, k = 1, 2, 3, \tag{2.5}$$

$$\Pi_{jk}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\pi}\left[-\frac{\delta_{jk}}{|\boldsymbol{x}-\boldsymbol{y}|^3} + \frac{3(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right], \tag{2.6}$$

and the corresponding formulae for the velocity and pressure at an arbitrary point $\boldsymbol{x}$ are

$$u_i(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x}, \boldsymbol{y}) \nu_k g_j, \quad p(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} \Pi_{jk}(\boldsymbol{x}, \boldsymbol{y}) \nu_k g_j. \tag{2.7a,b}$$

2.1. *Stokes flow in a half-space, Blake's formula*

Suppose now that a force vector $\boldsymbol{f}$ is applied to a viscous fluid in the upper half-space ($x_3 > 0$). Then, the corresponding Stokeslet-induced velocity field fails to satisfy the no-slip condition (1.3). In order to annihilate the velocity field while satisfying the homogeneous Stokes equations in the upper half-space, Blake (1971) proposed the following image structure:

$$S^W_{ij}(\boldsymbol{x}, \boldsymbol{y}) = S_{ij}(\boldsymbol{x}, \boldsymbol{y}) - S_{ij}(\boldsymbol{x}, \boldsymbol{y}^I) + 2y_3^2 S^D_{ij}(\boldsymbol{x}, \boldsymbol{y}^I) - 2y_3 S^{SD}_{ij}(\boldsymbol{x}, \boldsymbol{y}^I), \tag{2.8}$$

where $\boldsymbol{y}^I = (y_1, y_2, -y_3)$ is the reflected image location. Here, S^D is a modified source doublet, given by

$$S^D_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{8\pi}(1 - 2\delta_{j3})\frac{\partial}{\partial x_j}\frac{x_i - y_i}{|\boldsymbol{x}-\boldsymbol{y}|^3}, \tag{2.9}$$

and S^{SD} is a modified Stokes doublet, given by

$$S^{SD}_{ij}(\boldsymbol{x}, \boldsymbol{y}) = (1 - 2\delta_{j3})\frac{\partial S_{i3}(\boldsymbol{x}, \boldsymbol{y})}{\partial x_j}. \tag{2.10}$$

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

Similar, but more involved, decompositions for the Stokes doublet and stresslet in a half-space are given in Blake & Chwang (1974) and Pozrikidis (1992). It should be noted that the computation of the modified source doublet S_{ij}^D requires the evaluation of three distinct harmonic dipole fields.

3. A new image formula

In this section, we derive a simpler image structure, using the Papkovich–Neuber representation which involves only a single harmonic function. Here, $\boldsymbol{y}^I$, as above, will denote the image location $(y_1, y_2, -y_3)$. Now, however, we define reflected single-force, double-force and double-force-orientation vectors by negating their third components:

$$\boldsymbol{f}^I = (f_1, f_2, -f_3), \quad \boldsymbol{g}^I = (g_1, g_2, -g_3), \quad \boldsymbol{\nu}^I = (\nu_1, \nu_2, -\nu_3). \tag{3.1a–c}$$

We also make use of the harmonic potential due to a unit strength charge,

$$G^S(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}, \tag{3.2}$$

the harmonic potential due to a unit strength dipole with orientation vector $\boldsymbol{\nu}$,

$$G^D[\boldsymbol{\nu}](\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} \nu_i \frac{\partial}{\partial y_i} \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}, \tag{3.3}$$

and the harmonic potential due to a unit strength quadrupole with orientation vectors $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$,

$$G^Q[\boldsymbol{\nu}, \boldsymbol{\kappa}](\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \nu_i \kappa_j \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}. \tag{3.4}$$

Finally, we recall that, through the Papkovich–Neuber representation (2.1), the harmonic function $\phi(\boldsymbol{x})$ induces velocity and pressure fields that can be written in component form as

$$u_i(\boldsymbol{x}) = x_3 \frac{\partial}{\partial x_i} \phi(\boldsymbol{x}) - \delta_{i3} \phi(\boldsymbol{x}), \quad p(\boldsymbol{x}) = 2 \frac{\partial}{\partial x_3} \phi(\boldsymbol{x}). \tag{3.5a,b}$$

Of particular note is the fact that at $x_3 = 0$, the only non-zero velocity component is

$$u_3(\boldsymbol{x}) = -\phi(\boldsymbol{x}). \tag{3.6}$$

This suggests a simple two-step strategy. First, annihilate the tangential components of the velocity field induced by a Stokeslet (single-layer) or stresslet (double-layer) kernel. It is easy to see that this can be accomplished by subtracting the influence of a reflected single force $\boldsymbol{f}^I = (f_1, f_2, -f_3)$ or double force $\boldsymbol{g}^I = (g_1, g_2, -g_3)$, $\boldsymbol{\nu}^I = (\nu_1, \nu_2, -\nu_3)$ located at the image point $\boldsymbol{y}^I$ respectively. It remains only to match the remaining non-zero normal component u_3, which we will do by a judicious choice of the Papkovich–Neuber potential $\phi(\boldsymbol{x})$.

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

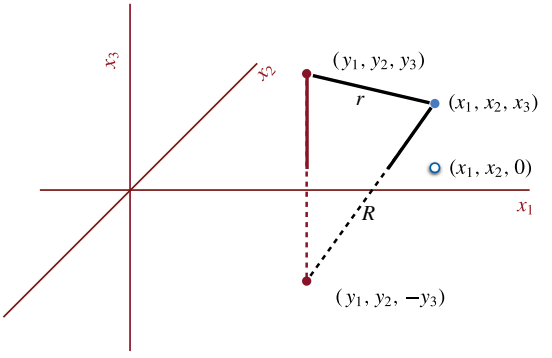


FIGURE 1. The original source location is in the upper half-space at (y_1, y_2, y_3) and the reflected image source is at $(y_1, y_2, -y_3)$. For target points $(x_1, x_2, 0)$ that lie on the half-space boundary, the distances r and R from the source and image are the same, simplifying the computation of $\boldsymbol{v}$ in (3.8).

3.1. *The Stokeslet correction*

Following the discussion above, let us write out in more detail the velocity field $\boldsymbol{v}$ remaining after subtracting the image Stokeslet at $\boldsymbol{y}^I$ from the original Stokeslet at $\boldsymbol{y}$ (figure 1):

$$v_i(\boldsymbol{x}) = \sum_{j=1}^{3} (S_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j - S_{ij}(\boldsymbol{x}, \boldsymbol{y}^I) f_j^I). \tag{3.7}$$

At the interface $x_3 = 0$, a straightforward computation yields

$$v_1(\boldsymbol{x}) = 0, \quad v_2(\boldsymbol{x}) = 0, \quad v_3(\boldsymbol{x}) = -\frac{f_3^I}{4\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}^I|} - \frac{y_3}{4\pi} \sum_{j=1}^{3} \frac{x_j - y_j^I}{|\boldsymbol{x} - \boldsymbol{y}^I|^3} f_j^I. \tag{3.8a–c}$$

By inspection of (3.8), it is clear that the harmonic potential ϕ required to cancel the non-zero normal component v_3 at the interface is that induced by a charge of strength f_3^I and a dipole with orientation vector $\boldsymbol{f}^I$ of strength y_3 located at $\boldsymbol{y}^I$. Thus, the velocity field $\boldsymbol{u}^W$ satisfying the desired no-slip boundary condition (1.3) can be expressed as

$$u_i^W(\boldsymbol{x}) = u_i^A(\boldsymbol{x}) - u_i^B(\boldsymbol{x}) - u_i^C(\boldsymbol{x}), \tag{3.9}$$

where $\boldsymbol{u}^A$ and $\boldsymbol{u}^B$ are the velocity fields induced by the original free-space Stokeslet and the reflected image force vector respectively:

$$u_i^A(\boldsymbol{x}) = \sum_{j=1}^{3} S_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j, \quad u_i^B(\boldsymbol{x}) = \sum_{j=1}^{3} S_{ij}(\boldsymbol{x}, \boldsymbol{y}^I) f_j^I. \tag{3.10a,b}$$

Here, u_i^C is the Papkovich–Neuber correction,

$$u_i^C(\boldsymbol{x}) = x_3 \frac{\partial}{\partial x_i} \phi(\boldsymbol{x}) - \delta_{i3} \phi(\boldsymbol{x}), \tag{3.11}$$

where the harmonic potential ϕ is that due to a simple charge and dipole, both located at $\boldsymbol{y}^I$:

$$\phi(\boldsymbol{x}) = f_3^I G^S(\boldsymbol{x}, \boldsymbol{y}^I) + y_3 G^D[\boldsymbol{f}^I](\boldsymbol{x}, \boldsymbol{y}^I). \tag{3.12}$$

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

3.2. *The stresslet correction*

We state the image structure for the stresslet (the double-layer kernel) in the form of a theorem.

THEOREM 1. *Let a double force* $\boldsymbol{g}$ *with orientation vector* $\boldsymbol{v}$ *be located at* $\boldsymbol{y}$*, resulting in the free-space velocity field*

$$u_i^T(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x},\boldsymbol{y})\nu_k g_j. \tag{3.13}$$

Then, the corresponding velocity satisfying the no-slip boundary condition (1.3) is given by

$$u_i^{T,W}(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x},\boldsymbol{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x},\boldsymbol{y}^I)\nu_k^I g_j^I - \left[x_3\frac{\partial}{\partial x_i}\phi^T(\boldsymbol{x}) - \delta_{i3}\phi^T(\boldsymbol{x})\right], \tag{3.14}$$

where

$$\phi^T(\boldsymbol{x}) = 2(\boldsymbol{v}^I\cdot\boldsymbol{g}^I)G^D[\boldsymbol{h}](\boldsymbol{x},\boldsymbol{y}^I) + 2y_3 G^Q[\boldsymbol{v}^I,\boldsymbol{g}^I](\boldsymbol{x},\boldsymbol{y}^I), \tag{3.15}$$

with $\boldsymbol{h} = (0, 0, 1)$.

Proof. It should be noted first that the symmetric part of the Stokes doublet is given by

$$T_{ijk}^S(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\left[-\frac{(x_i-y_i)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{jk} + \frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right]. \tag{3.16}$$

Furthermore, the tangential components of the velocity field

$$v_i(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}^S(\boldsymbol{x},\boldsymbol{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}^S(\boldsymbol{x},\boldsymbol{y}^I)\nu_k^I g_j^I \tag{3.17}$$

are zero when $x_3 = 0$. Thus, the harmonic potential ϕ^S annihilating v_3 at the interface $x_3 = 0$ is that due to a single quadrupole source located at y^I with orientation vectors $\boldsymbol{v}^I, \boldsymbol{g}^I$ and strength $2y_3$:

$$\phi^S(\boldsymbol{x}) = 2y_3 G^Q[\boldsymbol{v}^I,\boldsymbol{g}^I](\boldsymbol{x},\boldsymbol{y}^I). \tag{3.18}$$

It is straightforward to check that the stresslet is simply related to the symmetric part of the Stokes doublet,

$$T_{ijk}(\boldsymbol{x},\boldsymbol{y}) = \frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5} = \frac{1}{4\pi}\frac{x_i-y_i}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{jk} + T_{ijk}^S(\boldsymbol{x},\boldsymbol{y}). \tag{3.19}$$

Thus, it remains only to annihilate the velocity field induced by the first term on the right-hand side of (3.19) and its reflected image, given by

$$v_i(\boldsymbol{x}) = \frac{1}{4\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{x_i-y_i}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{jk}\nu_k g_j - \frac{1}{4\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{x_i-y_i^I}{|\boldsymbol{x}-\boldsymbol{y}^I|^3}\delta_{jk}\nu_k^I g_j^I. \tag{3.20}$$

At $x_3 = 0$, the tangential components vanish and the normal component is easily computed to be $2(\boldsymbol{v}^I\cdot\boldsymbol{g}^I)G^D[h](\boldsymbol{x},\boldsymbol{y}^I)$, where $\boldsymbol{h} = (0, 0, 1)$. The desired result follows. □

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

3.3. *The rotlet correction*

Similar representations are easily derived for other fundamental solutions, such as the rotlet – the antisymmetric part of the Stokes doublet:

$$T^R_{ijk}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\left[-\frac{(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{ik} + \frac{(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{ij}\right]. \tag{3.21}$$

The corresponding velocity satisfying the no-slip boundary condition (1.3) is given by

$$u^{R,W}_i(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T^R_{ijk}(\boldsymbol{x},\boldsymbol{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T^R_{ijk}(\boldsymbol{x},\boldsymbol{y}^I)\nu^I_k g^I_j - \left[x_3\frac{\partial}{\partial x_i}\phi^R(\boldsymbol{x}) - \delta_{i3}\phi^R(\boldsymbol{x})\right], \tag{3.22}$$

where the Papkovich–Neuber correction ϕ^R is due to two dipoles,

$$\phi^R(\boldsymbol{x}) = -2\nu^I_3 G^D[\boldsymbol{g}^I](\boldsymbol{x},\boldsymbol{y}^I) + 2g^I_3 G^D[\boldsymbol{\nu}^I](\boldsymbol{x},\boldsymbol{y}^I). \tag{3.23}$$

3.4. *The Stokes doublet correction*

Finally, the Stokes doublet is the sum of its symmetric and antisymmetric parts,

$$\begin{aligned} T^D_{ijk}(\boldsymbol{x},\boldsymbol{y}) \;=\; & \frac{1}{4\pi}\left[-\frac{(x_i-y_i)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{jk} - \frac{(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{ik} + \frac{(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{ij}\right. \\ & \left. +\; \frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right] = T^S_{ijk}(\boldsymbol{x},\boldsymbol{y}) + T^R_{ijk}(\boldsymbol{x},\boldsymbol{y}). \end{aligned} \tag{3.24}$$

By combining (3.18) and (3.23), we obtain the image formula for the Stokes doublet,

$$u^{D,W}_i(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T^D_{ijk}(\boldsymbol{x},\boldsymbol{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T^D_{ijk}(\boldsymbol{x},\boldsymbol{y}^I)\nu^I_k g^I_j - \left[x_3\frac{\partial}{\partial x_i}\phi^D(\boldsymbol{x}) - \delta_{i3}\phi^D(\boldsymbol{x})\right], \tag{3.25}$$

where the Papkovich–Neuber potential is given by

$$\phi^D(\boldsymbol{x}) = -2\nu^I_3 G^D[\boldsymbol{g}^I](\boldsymbol{x},\boldsymbol{y}^I) + 2g^I_3 G^D[\boldsymbol{\nu}^I](\boldsymbol{x},\boldsymbol{y}^I) + 2y_3 G^Q[\boldsymbol{\nu}^I,\boldsymbol{g}^I](\boldsymbol{x},\boldsymbol{y}^I). \tag{3.26}$$

4. Conclusions

We have derived very simple image formulae for Stokes flow in a half-space induced by any of the standard fundamental solutions – the Stokeslet, stresslet, rotlet and Stokes doublet. In each case, all that is required is a reflected fundamental solution and a Papkovich–Neuber correction based on a single harmonic potential.

The velocity (and pressure) due to the 'direct' and reflected fundamental solutions can be computed together with any software that handles Stokeslets, stresslets, etc. in free space. Furthermore, the Papkovich–Neuber potential requires only the evaluation of a single additional harmonic function – itself requiring only software for free-space harmonic sources, dipoles and quadrupoles. Many efficient schemes exist for these various steps, such as those described in Fu *et al.* (1998), Fu & Rodin (2000), Yoshida, Nishimura & Kobayashi (2001), Ying, Biros & Zorin (2004), Frangi (2005), Wang & Yao (2005), Gumerov & Duraiswami (2006), Wang *et al.* (2006, 2007), Tornberg & Greengard (2008), Fong & Darve (2009), Gimbutas & Greengard (2012).

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

Using the formulae derived here, we extended the scope of our existing FMM libraries to half-space Stokes flow problems. They can be downloaded from the Courant Mathematics and Computing Laboratory webpage (Gimbutas & Greengard 2012).

Acknowledgements

The work of Z.G. was supported in part by the Office of the Assistant Secretary of Defense for Research and Engineering and AFOSR under NSSEFF Program Award FA9550-10-1-0180, and in part by the National Science Foundation under grant DMS-0934733. Contributions by staff of NIST, an agency of the US Government, are not subject to copyright within the United States. The work of L.G. was supported in part by the Office of the Assistant Secretary of Defense for Research and Engineering and AFOSR under NSSEFF Program Award FA9550-10-1-0180, by the National Science Foundation under grant DMS-0934733 and by the Applied Mathematical Sciences Program of the US Department of Energy under Contract DEFGO288ER25053. The work of S.V. was supported by the NSF under grants DMS-1418964 and DMS-1224656.

Appendix A. Extension to the two-dimensional problems

The two-dimensional Stokes flow representations in a half-space can be derived similarly. They lead to identical Papkovich–Neuber corrections with corresponding charge, dipole and quadrupole potentials replaced by their two-dimensional equivalents. Similarly to the three-dimensional case, for any $\boldsymbol{x} \in \mathbb{R}^2$, the half-plane Green's function is composed of three terms:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^A(\boldsymbol{x}) - \boldsymbol{u}^B(\boldsymbol{x}) - \boldsymbol{u}^C(\boldsymbol{x}), \tag{A 1}$$

where the first term is the free-space Green's function, the second term is the reflected image about the plane wall annihilating the tangential component of the velocity field and the third term is a Papkovich–Neuber correction term in the following form:

$$\boldsymbol{u}^C(\boldsymbol{x}) = x_2 \nabla \phi(\boldsymbol{x}) - \begin{bmatrix} 0 \\ \phi(\boldsymbol{x}) \end{bmatrix}, \quad p(\boldsymbol{x}) = 2\frac{\partial \phi(\boldsymbol{x})}{\partial x_2}. \tag{A 2a,b}$$

The correction potentials for various fundamental solutions are

$$\phi(\boldsymbol{x}) = f_2^I G^S(\boldsymbol{x}, \boldsymbol{y}^I) + y_2 G^D[\boldsymbol{f}^I](\boldsymbol{x}, \boldsymbol{y}^I) \tag{A 3}$$

for the two-dimensional Stokeslet and

$$\phi^T(\boldsymbol{x}) = 2(\boldsymbol{v}^I \cdot \boldsymbol{g}^I) G^D[h](\boldsymbol{x}, \boldsymbol{y}^I) + 2y_2 G^Q[\boldsymbol{v}^I, \boldsymbol{g}^I](\boldsymbol{x}, \boldsymbol{y}^I), \tag{A 4}$$

$$\phi^R(\boldsymbol{x}) = -2v_2^I G^D[\boldsymbol{g}^I](\boldsymbol{x}, \boldsymbol{y}^I) + 2g_2^I G^D[\boldsymbol{v}^I](\boldsymbol{x}, \boldsymbol{y}^I), \tag{A 5}$$

$$\phi^D(\boldsymbol{x}) = -2v_2^I G^D[\boldsymbol{g}^I](\boldsymbol{x}, \boldsymbol{y}^I) + 2g_2^I G^D[\boldsymbol{v}^I](\boldsymbol{x}, \boldsymbol{y}^I) + 2y_2 G^Q[\boldsymbol{v}^I, \boldsymbol{g}^I](\boldsymbol{x}, \boldsymbol{y}^I) \tag{A 6}$$

for the two-dimensional stresslet, rotlet and Stokes doublet respectively, where the images of the source, single-force, double-force and double-force-orientation vectors are

$$\boldsymbol{y}^I = (y_1, -y_2), \quad \boldsymbol{f}^I = (f_1, -f_2), \quad \boldsymbol{g}^I = (g_1, -g_2), \quad \boldsymbol{v}^I = (v_1, -v_2), \tag{A 7a–d}$$

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

with the orientation vector $\boldsymbol{h} = (0, 1)$, and the free-space Laplace Green's functions in two dimensions are given by

$$G^S(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{y}|, \tag{A 8}$$

$$G^D[\nu](\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{2} \nu_i \frac{\partial}{\partial y_i} G^S(\boldsymbol{x}, \boldsymbol{y}) \quad \text{and}$$

$$G^Q[\boldsymbol{\nu}, \boldsymbol{\kappa}](\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \kappa_j \frac{\partial^2}{\partial y_i \partial y_j} G^S(\boldsymbol{x}, \boldsymbol{y}). \tag{A 9a,b}$$

Appendix B. Extension to linear elasticity kernels

The single-layer kernel for linear isotropic elasticity in R^3 is given by Kelvin's solution (Mindlin 1936),

$$U_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{8\pi\mu} \left[(2 - \alpha) \frac{\delta_{ij}}{|\boldsymbol{x} - \boldsymbol{y}|} + \alpha \frac{(x_i - y_i)(x_j - y_j)}{|\boldsymbol{x} - \boldsymbol{y}|^3} \right], \quad i, j = 1, 2, 3, \tag{B 1}$$

where λ, μ are Lame's parameters and $\alpha = (\lambda + \mu)/(\lambda + 2\mu)$. It is easy to see that the displacement field

$$u_i(\boldsymbol{x}) = \sum_{j=1}^{3} S_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j - \sum_{j=1}^{3} S_{ij}(\boldsymbol{x}, \boldsymbol{y}^I) f_j^I - u^C(\boldsymbol{x}) \tag{B 2}$$

satisfies the no-displacement boundary condition $\boldsymbol{u}(\boldsymbol{x}) = 0$ at $x_3 = 0$, where the Papkovich–Neuber correction is

$$u_i^C(\boldsymbol{x}) = \alpha x_3 \frac{\partial}{\partial x_i} \phi(\boldsymbol{x}) - (2 - \alpha) \delta_{i3} \phi(\boldsymbol{x}), \tag{B 3}$$

$$\phi(\boldsymbol{x}) = \frac{f_3^I}{\mu} G^S(\boldsymbol{x}, \boldsymbol{y}^I) + \frac{y_3}{\mu} G^D[\boldsymbol{f}^I](\boldsymbol{x}, \boldsymbol{y}^I). \tag{B 4}$$

References

ADEROGBA, K. & BLAKE, J. R. 1978*a* Action of a force near the planar surface between semi-infinite immiscible liquids at very low Reynolds numbers: addendum. *Bull. Austral. Math. Soc.* **19** (2), 309–318.

ADEROGBA, K. & BLAKE, J. R. 1978*b* Action of a force near the planar surface between two semi-infinite immiscible liquids at very low Reynolds numbers. *Bull. Austral. Math. Soc.* **18** (3), 345–356.

BHATTACHARYA, S. & BŁAWZDZIEWICZ, J. 2002 Image system for Stokes-flow singularity between two parallel planar walls. *J. Math. Phys.* **43** (11), 5720–5731.

BLAKE, J. R. 1971 A note on the image system for a Stokeslet in a no-slip boundary. *Math. Proc. Camb. Phil. Soc.* **70** (2), 303–310.

BLAKE, J. R. 1974 Singularities of viscous flow. *J. Engng Maths* **8** (2), 113–124.

BLAKE, J. R. & CHWANG, A. T. 1974 Fundamental singularities of viscous flow. *J. Engng Maths* **8** (1), 23–29.

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press

CICHOCKI, B. & JONES, R. B. 1998 Image representation of a spherical particle near a hard wall. *Physica* A **258** (3), 273–302.

FONG, W. & DARVE, E. 2009 The black-box fast multipole method. *J. Comput. Phys.* **228** (23), 8712–8725.

FRANGI, A. 2005 A fast multipole implementation of the qualocation mixed-velocity–traction approach for exterior Stokes flows. *Engng Anal. Bound. Elem.* **29** (11), 1039–1046.

FU, Y., KLIMKOWSKI, K. J., RODIN, G. J., BERGER, E., BROWNE, J. C., SINGER, J. K., VAN DE GEIJN, R. A. & VEMAGANTI, S. K. 1998 A fast solution method for three-dimensional many-particle problems of linear elasticity. *Intl J. Numer. Meth. Engng* **42** (7), 1215–1229.

FU, Y. & RODIN, G. J. 2000 Fast solution method for three-dimensional Stokesian many-particle problems. *Commun. Numer. Meth. Engng* **16** (2), 145–149.

GIMBUTAS, Z. & GREENGARD, L. 2012, STFMMLIB3 – fast multipole method (FMM) library for the evaluation of potential fields governed by the Stokes equations in R^3, http://www.cims.nyu.edu/cmcl/fmm3dlib/fmm3dlib.html.

GUMEROV, N. A. & DURAISWAMI, R. 2006 Fast multipole method for the biharmonic equation in three dimensions. *J. Comput. Phys.* **215** (1), 363–383.

HAPPEL, J. & BRENNER, H. 1983 *Low Reynolds Number Hydrodynamics: with Special Applications to Particulate Media*. Springer.

KIM, S. & KARRILA, S. J. 1991 *Microhydrodynamics: Principles and Selected Applications*. Butterworth-Heinemann.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LORENTZ, H. A. 1896 Eene algemeene stelling omtrent de beweging eener vloeistof met wrijving en eenige daaruit afgeleide gevolgen. *Versl. K. Akad. W. Amsterdam* **5**, 168–175.

LORENTZ, H. A. 1907 Ein allgemeiner Satz, die Bewegung einer reibenden Flüssigkeit betreffend, nebst einigen Anwendungen desselben. *Abhand. Theor. Phys.* **1**, 23–42.

MINDLIN, R. D. 1936 Force at a point in the interior of a semi-infinite solid. *J. Appl. Phys.* **7** (5), 195–202.

NEUBER, H. 1934 Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie. Der Hohlkegel unter Einzellast als Beispiel. *Z. Angew. Math. Mech.* **14** (4), 203–212.

PAPKOVICH, P. F. 1932 Solution générale des équations differentielles fondamentales d'élasticité exprimée par trois fonctions harmoniques. *C. R. Acad. Sci. Paris* **195**, 513–515.

POZRIKIDIS, C. 1992 *Boundary Integral and Singularity Methods for Linearized Viscous Flow*. Cambridge University Press.

SPAGNOLIE, S. E. & LAUGA, E. 2012 Hydrodynamics of self-propulsion near a boundary: predictions and accuracy of far-field approximations. *J. Fluid Mech.* **700**, 105–147.

TORNBERG, A. K. & GREENGARD, L. 2008 A fast multipole method for the three-dimensional Stokes equations. *J. Comput. Phys.* **227** (3), 1613–1619.

VEERAPANENI, S. K., RAHIMIAN, A., BIROS, G. & ZORIN, D. 2011 A fast algorithm for simulating vesicle flows in three dimensions. *J. Comput. Phys.* **230** (14), 5610–5634.

WANG, H., LEI, T., LI, J., HUANG, J. & YAO, Z. 2007 A parallel fast multipole accelerated integral equation scheme for 3D Stokes equations. *Intl J. Numer. Meth. Engng* **70** (7), 812–839.

WANG, H. T. & YAO, Z. H. 2005 A new fast multipole boundary element method for large scale analysis of mechanical properties in 3d particle-reinforced composites. *Comput. Model. Engng Sci.* **7** (1), 85–95.

WANG, X., KANAPKA, J., YE, W., ALURU, N. R. & WHITE, J. 2006 Algorithms in FastStokes and its application to micromachined device simulation. *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.* **25** (2), 248–257.

YING, L., BIROS, G. & ZORIN, D. 2004 A kernel-independent adaptive fast multipole algorithm in two and three dimensions. *J. Comput. Phys.* **196** (2), 591–626.

YOSHIDA, K., NISHIMURA, N. & KOBAYASHI, S. 2001 Application of fast multipole Galerkin boundary integral equation method to elastostatic crack problems in 3d. *Intl J. Numer. Meth. Engng* **50** (3), 525–547.

YU, H. Y. 2003 Fundamental singularities in a two-fluid Stokes flow with a plane interface. *J. Mech.* **19** (1), 263–270.

https://doi.org/10.1017/jfm.2015.302 Published online by Cambridge University Press