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OPTIMAL GROWTH IN CONTINUOUS-TIME WITH CREDIT RISK

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We consider asset allocation strategies for the case where an investor can allocate his wealth dynamically between a risky stock, whose price evolves according to a geometric Brownian motion, and a risky bond, whose price is subject to negative jumps due to its credit risk and therefore has discontinuous sample paths. We derive optimal policies for a number of objectives related to growth. In particular, we obtain the policy that minimizes the expected time to reach a given target value of wealth in an exact explicit form. We also show that this policy is exactly equivalent to the policy that is optimal for maximizing logarithmic utility of wealth and, hence, the expected average rate at which wealth grows, as well as to the policy that maximizes the actual asymptotic rate at which wealth grows. Our results generalize and unify results obtained previously for cases where the bond was risk-free in both continuous- and discrete-time.

1. INTRODUCTION

In this paper we consider the optimal investment behavior for an investor who can diversify his wealth between two assets: a risky stock and a risky bond. The objectives we consider here relate solely to growth, in a sense soon to be defined. The price of the risky stock is assumed to follow a geometric Brownian motion, which is a standard model in financial economics (see, e.g., Merton [20]). A complication in our model not present in the usual one studied in the optimal investment literature (see, e.g., Browne [5,6], Hakansson [13], Merton [19,20]) is the fact that here the bond is not completely risk free, but is in fact stochastic, since it is subject to credit

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risk. The assumption we make about how credit risk affects the bond price is that, at certain random times, an event occurs which reduces the price of the bond by a random amount. This event is usually termed a "corporate reorganization" (see, e.g., Jarrow and Turnbull [15]) and refers to the fact that at a corporate reorganization epoch, there is a partial default on the bond. This is an extension of a model first considered in Merton [19]. Similar models are used in [15] and others, with the objective of pricing options on the corporate bonds (see also Merton [20, Ch. 9]). Here, we are interested in determining investment strategies that achieve optimal growth. There are, of course, a few different ways to define growth; for example, some define growth as the (average) rate at which wealth compounds and an optimal growth policy would therefore maximize this rate, or the expected value thereof. This is the approach taken in the discrete-time papers of Kelly [17], Hakansson [13], and Markowitz [18], among others. Breiman [3] considered an optimal growth strategy to be a strategy that minimizes the expected time to reach a given target level of wealth. He showed that in discrete-time, without credit risk, as the target level of wealth gets "large," the policy that maximizes the expected rate at which wealth compounds is also asymptotically optimal for the latter problem, namely of minimizing the expected time to the target level. Interesting analyses of optimal growth policies in discrete-time can be found in the papers cited above as well as in Thorp [22], Ethier and Tavare [9], Algoet and Cover [1], and others. In particular, Algoet and Cover [1] proved that the policy that maximizes the expected rate at which wealth compounds also maximizes the actual rate (as the time horizon goes to infinity) at which it compounds.

Optimal growth strategies have been studied in continuous-time as well, though mostly for models with a completely risk-free bond, hence without credit risk (see [20, Ch. 6] for a survey and discussion of the fundamental role optimal growth policies play in modern finance and [6] for extended optimality properties of these policies). Merton [19] solved for the policy that maximizes the expected logarithm and, hence, the expected growth rate. Heath, Orey, Pestien, and Sudderth [14] were the first to solve for the policy that minimizes the expected time to a goal in continuous-time. Unlike the discrete-time asymptotic results of Breiman [3], they found that the policy that maximizes the expected rate at which wealth compounds is in fact exactly optimal for the problem of minimizing the expected time to a goal, for any fixed goal. Merton [20, Ch. 6] recovered these results independently via a different route, and Browne [5] considered this and other goal-related objectives in a more general model that incorporates liabilities. The results of Algoet and Cover [1] have been extended to the continuous-time case in [16]. Central to the analysis in those papers was the fact that the (controlled) wealth process of the investor followed a diffusion process, with continuous sample paths. The introduction of the credit risk problem introduces some new difficulties into the model, since now the sample path of the wealth process is no longer continuous.

Here we consider the extension of optimal growth policies in continuous-time to the case where the bond is no longer risk-free, and is subject to credit risk. Credit risk here means that the organization paying the coupon on the bond may fail to meet

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the coupon. When a coupon payment is missed, the value of the bond goes down. There are many ways to model the credit risk problem, here we will use the model first introduced in Merton [19], which was extended and analyzed further in Jarrow and Turnbull [15], among others. In the next section, we introduce the model of the primitives, namely the stock price and the bond price processes. In Section 3, we give our main result, which shows that the same policy, which is given explicitly, is optimal for the following three problems: (1) minimizing the expected time to reach a given value of wealth; (2) maximizing the expected average rate at which wealth compounds over a fixed horizon; (3) maximizing the actual rate at which wealth compounds over the infinite horizon.

The explicit form of our policy given in Theorem 1 below is seen to be not only a generalization of the continuous-time results of Heath et al. [14], Merton [20], and others, but our results also unify the discrete-time optimal growth policies studied in Breiman [3], Ethier and Tavare [9], Finkelstein and Whitely [10], Hakansson [13], Kelly [17], and Thorp [22], with the aforementioned continuous-time results. In Section 4, we provide the proof of the theorem. The proofs for the three different parts are of varying difficulty: maximizing the expected average rate at which wealth compounds over a fixed horizon turns out to be a relatively simple problem and is solved using some basic facts from the general theory of stochastic integration. Minimizing the expected time to reach a given value of wealth, which is, in fact, the main contribution of this paper, is a more delicate problem and requires a new and more technical proof. We first show that the Hamilton-Jacobi-Bellman (HJB) equations of stochastic control are satisfied and then provide a rigorous proof of optimality by a martingale argument. Finally, using results from the general theory of stochastic integration, we show that the ratio of the wealth from any other strategy to the wealth from the growth-optimal strategy is a nonnegative supermartingale. This result then allows the martingale argument of [1] (see also [16]) to be applied here to verify optimality for maximizing the actual rate at which wealth compounds over the infinite horizon. We then conclude, in Section 5, with some suggestions for extensions.

2. THE MODEL

Without significant loss of generality, we assume that there is only one risky stock available for investment (e.g., a mutual fund), whose price at time t will be denoted by S_t. Following [19], we will assume that the price process of the risky stock follows a geometric Brownian motion; i.e., S_t satisfies the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t \tag{1}$$

where μ and σ are constants, and $\{W_t : t \geq 0\}$ is a standard Brownian motion. Thus,

$$S_t = S_0 \exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\}$$

and so for $\mu > \sigma^2/2$, the stock is viable over an infinite horizon, since we have $(1/t)\ln S_t \to (\mu - \frac{1}{2}\sigma^2) > 0$.

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The other investment opportunity is a (risky) bond (or money market account) whose price satisfies the following stochastic differential equation:

$$dB_t = rB_{t-}\,dt - B_{t-}\,dQ_t, \tag{2}$$

where Q_t is a *compound Poisson process*; that is,

$$Q_t = \sum_{i=1}^{N_t} Y_i, \tag{3}$$

where $\{N_t; t \geq 0\}$ is a simple homogeneous Poisson process with intensity λ, and $\{Y_i; i \geq 1\}$ is an i.i.d. sequence of random variables. The compound Poisson process Q and the Brownian motion W are assumed to be mutually independent. The interpretation of Y_i is the random percentage change in the bond price associated with the ith jump in N. In the finance literature, Y_i is called the "percentage writedown" associated with the ith corporate reorganization. As bond prices cannot become negative, we will assume that

$$P(0 \leq Y_1 \leq 1) = 1. \tag{4}$$

This model of Poisson bankruptcies has been used extensively in finance literature. Merton [19] introduced this model and studied the problem of how an investor should invest so as to maximize expected utility of terminal wealth for a special case with $Y_1 \equiv 1$, whereby the bond defaults completely at the first corporate reorganization. However, as he himself notes there, a complete default is rather rare, thus the model used here might be somewhat more realistic. Jarrow and Turnbull [15] use a simple Poisson bankruptcy model to price options on credit-risky securities. The empirical distributions of the writedowns for different classes of corporate bonds have been studied, for example, in Franks and Torous [12]. Here our interests lie in determining optimal portfolio strategies rather than pricing.

Elementary results on stochastic integration show that the solution to Eq. (2) is

$$B_t = B_0 e^{rt} \prod_{i=1}^{N_t} (1 - Y_i), \tag{5}$$

with the understanding that an empty product is 1. Thus, the bond price is always nonnegative for every $t < \infty$. Whether the bond is viable in the long run depends, of course, on the relationship among the parameters. For example, a simple conditioning argument shows that $EB_t = B_0 \exp\{t(r - \lambda E(Y_1))\}$, from which we may conclude that it is sufficient to assume $r/\lambda > EY_1$ to ensure that $(1/t)\ln EB_t \to r - \lambda EY_1 > 0$. However, it is also clear that, since $\ln B_t = \ln B_0 + rt + \sum_{i=1}^{N_t} \ln(1 - Y_i)$, the law of large numbers implies that

$$(1/t)\ln B_t \to (r + \lambda E \ln(1 - Y_1)),$$

which is positive only if $r/\lambda > -E \ln(1 - Y_1)$, which is, in fact, a stricter assumption than the former. (This of course follows from Jensen's inequality, and the fact that $r/\lambda > 1 - e^{-r/\lambda}$.) For the remainder of the paper, we do not particularly care about the

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long-term viability of the bond, and so we make no assumptions regarding the parameters or distribution of the writedowns.

We will let $\mathcal{F}$ denote the underlying filtration of interest, that is, we are given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ which supports the standard Brownian motion W and the compound Poisson process Q, where $\mathcal{F}_t$ is the P-augmentation of the natural filtration $\mathcal{F}_t^{W,Q} := \sigma\{(W_s, Q_s),\ 0 \le s \le t\}$.

Let f_t denote the fraction of wealth invested in the risky stock at time t under an investment policy f, where we assume that $\{f_t\}$ is a suitable, $\mathcal{F}_{t-}$ admissible adapted control process, that is, f_t is a nonanticipative function that satisfies $\int_0^T f_t^2\, dt < \infty$, a.s., for every $T < \infty$. We will constrain the control $\{f_t\}$ by requiring $0 \le f_t \le 1$ for all t. This rules out short sales as well as borrowing from some other external source.

Let X_t^f denote the wealth of the investor at time t, if it follows policy f, with $X_0 = x$. Since any amount not invested in the risky stock is held in the bond (which has a greater return than cash), this process then evolves as

$$\begin{aligned} dX_t^f &= f_{t-} X_{t-}^f \frac{dS_t}{S_t} + X_{t-}^f (1 - f_{t-}) \frac{dB_t}{B_t} \\ &= X_{t-}^f [r + f_{t-}(\mu - r)]\, dt + f_{t-} X_{t-}^f \sigma\, dW_t - (1 - f_{t-}) X_{t-}^f\, dQ_t \end{aligned} \tag{6}$$

upon substituting from Eqs. (1) and (2).

Thus, for Markov control processes f, where $f_{t-} = f(t-, X_{t-})$, and functions $g \in \mathcal{C}^2$, the generator of the wealth process is

$$\mathcal{A}^f g(x) = [\, f(\mu - r) + r]xg_x + \tfrac{1}{2} f^2 \sigma^2 x^2 g_{xx} + \lambda[Eg([1 - (1 - f)Y_1]x) - g(x)], \tag{7}$$

where the expectation operator in the last term is on the random percentage change, Y_1.

Recognizing that, if a jump in N occurs at time s, then the jump in the wealth process is

$$\Delta X_s^f := X_s^f - X_{s-}^f \equiv -X_{s-}^f (1 - f_{s-}) Y_{N(s-)+1}$$

allows us to use standard results in the theory of stochastic integration (e.g., Protter [21, Thm. II.36]) to solve Eq. (6) as

$$\begin{aligned} X_t^f = X_0 \exp\left\{ \int_0^t \left(r + f_u(\mu - r) - \frac{1}{2} f_u^2 \sigma^2 \right) du + \int_0^t \sigma f_u\, dW_u \right\} \\ \times \prod_{i=1}^{N_t} (1 - (1 - f_{\tau_i -}) Y_i), \end{aligned} \tag{8}$$

where $\{\tau_i, i \ge 1\}$ are the points of the Poisson process, $\{N_t, t \ge 0\}$.

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3. OPTIMAL GROWTH POLICIES

Our interest, in this paper, lies in determining optimal growth policies for the investor. As noted earlier though, growth can be interpreted in different ways. For example, we can take growth to mean reaching a given target value of wealth as quickly as possible. This investment objective was treated in [14] (see also [5,20]) for the case without credit risk (i.e., $N_t = 0$ for all $t \geq 0$, hence a deterministic bond). In the absence of credit risk ($N_t = 0$), it is clear from Eq. (8) that it is possible to find an investment policy that causes wealth to grow exponentially (e.g., simply take f_u to be an appropriate constant, possibly 0). Thus, it is natural to consider $t^{-1} \ln(X_t^f/X_0)$ as the average growth rate of wealth, and, therefore, one may call a policy that maximizes either the expected average growth rate (over a fixed time interval) or the limit of the average growth rate (called the actual asymptotic growth rate) a growth optimal policy as well. Maximizing the expected average growth rate over a fixed finite horizon, say $(0,T]$, is seen to be equivalent to maximizing $E \ln(X_T^f)$, over all admissible policies. It turns out that the optimal policy for the latter two objectives is equivalent to the one that is optimal for the problem of reaching a target goal in minimal expected time; thus, for that model (i.e., without credit risk), we can call it *the* optimal growth policy [6].

As we will show directly, a similar result holds for the model with credit risk, although the policy is somewhat different and, in fact, generalizes the previous results.

To formalize this, consider the following three problems associated with the controlled wealth process given in Eq. (6):

Problem 1. Choose an admissible investment policy to *minimize the expected time to reach a given level, say b, of wealth*, that is, for $X_0 = x < b$, let

$$\Psi^{(1)}(x) := \inf_{0 \leq f \leq 1} E_x(\tau_b^f), \tag{9}$$

where

$$\tau_b^f := \inf\{t \geq 0 : X_t^f = b\}. \tag{10}$$

Let $f^{(1)} = \{f_t^{(1)}; t \geq 0\}$ denote the optimal policy for this problem, that is,

$$f^{(1)} = \arg \inf_{0 \leq f \leq 1} E_x(\tau_b^f). \tag{11}$$

Problem 2. Choose an admissible investment policy to *maximize the logarithm of terminal wealth at the fixed terminal time T,* that is, for $X_0 = x$, let

$$\Psi^{(2)}(x) = \sup_{0 \leq f \leq 1} E_x[\ln(X_T^f)], \tag{12}$$

with the corresponding optimal policy $f^{(2)} = \{f_t^{(2)} : 0 \leq t \leq T\}$,

$$f^{(2)} := \arg \sup_{0 \leq f \leq 1} E_x[\ln(X_T^f)]. \tag{13}$$

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Problem 3. Choose an admissible investment policy to *maximize the actual growth rate of wealth*, that is, for $X_0 = x$, let

$$\Psi^{(3)}(x) = \sup_{0 \le f \le 1} \left\{ \liminf_{T \to \infty} \frac{1}{T} \ln\left(X_T^f\right) \right\}, \tag{14}$$

with the corresponding optimal policy $f^{(3)} = \{ f_t^{(3)} : t \ge 0 \}$,

$$f^{(3)} := \arg \sup_{0 \le f \le 1} \left\{ \liminf_{T \to \infty} \frac{1}{T} \ln\left(X_T^f\right) \right\}. \tag{15}$$

In the theorem below we show that the same policy is optimal for all three problems and give the policy explicitly. It is interesting to note that while it was to be expected a priori that Problem 2 should have a solution similar to that of Problem 3, there does not seem to be that direct of a connection between Problem 1 (for any arbitrary wealth level b) and the other two. Similar to the case without credit risk (see, e.g., [6]) this optimal growth policy invests a constant proportion of wealth in the stock with the remainder in the bond, independent of the size of wealth, although, of course, in our case the constant is different and our results (especially for Problem 1) could not have been obtained from the previous results.

Remark 1: It is clear from Problem 2 that $f^{(2)}$ is also the optimal policy for maximizing the expected growth rate, $\underline{\lim}_{T \to \infty} (1/T) E[\ln X_T^f]$. This is the problem first introduced in [17] for simple discrete-time problems. However, the criteria in Problem 3 is the actual growth rate, which is harder to establish.

Our main result is now given in the following theorem.

THEOREM 1: *Let $C(f)$ denote the function*

$$C(f) = r + f(\mu - r) - \tfrac{1}{2} f^2 \sigma^2 + \lambda E(\ln(1 - (1 - f) Y_1)) \tag{16}$$

and let f^* denote its maximizer, that is, let f^* denote the constant determined by

$$f^* = \arg \sup_{0 \le f \le 1} \{ r + f(\mu - r) - \tfrac{1}{2} f^2 \sigma^2 + \lambda E(\ln(1 - (1 - f) Y_1)) \}. \tag{17}$$

Then for every $t \ge 0$,

$$f_t^{(1)} = f_t^{(2)} = f_t^{(3)} \equiv f^* \tag{18}$$

with corresponding optimal value functions

$$\Psi^{(1)}(x) = \frac{1}{C(f^*)} \ln\left(\frac{b}{x}\right) \tag{19}$$

$$\Psi^{(2)}(x) = \ln x + C(f^*) T \tag{20}$$

$$\Psi^{(3)}(x) = C(f^*). \tag{21}$$

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Remark 2: Observe that the investor holds more stock in the (downward) Poisson jump model treated here than in the corresponding riskless case, since it can be shown that the optimal policy of Eq. (17) is increasing in λ.

Note that when $\lambda = 0$, Eq. (17) shows that f^* reduces to

$$f^*|_{\lambda=0} = \max\left\{1, \frac{\mu - r}{\sigma^2}\right\}. \tag{22}$$

This is indeed the optimal policy obtained in Heath et al. [14] for Problem 1 (see also [20,5]), as well as the optimal policy for Problems 2 and 3 obtained in Merton [19] and Karatzas [16]. When $r = \mu = \sigma = 0$, and when we take $Z_i = -Y_i$, whereby the only investment opportunity is a "bond" that appreciates at the points of a Poisson process, and set $1 - f = \alpha$, then f^* reduces to

$$\arg\sup_{\alpha} E \ln(1 + \alpha Z_1), \tag{23}$$

which is equivalent to the discrete-time optimal growth, or Kelly strategies, studied in Kelly [17], Breiman [3], Thorp [22], Finkelstein and Whitely [10], Ethier and Tavare [9], Bell and Cover [2], and others. A Bayesian version of the optimal growth policy in both discrete and continuous time is studied in Browne and Whitt [7]. Thus, in some sense, the optimal policy obtained here is a hybrid of the continuous-time and discrete-time optimal growth policies. Note, however, that in discrete-time it is only an asymptotic version of Problem 1 that is related to Problems 2 and 3, as analyzed in Breiman [3].

Remark 3: The computation of $C(f)$ and then the maximization thereof is quite straightforward, although simple closed-form solutions seem rare. For example, if the writedowns are assumed to be uniformly distributed on (0,1), then it is easy to show that

$$C(f) = r + f(\mu - r) - \frac{\sigma^2}{2} f^2 - \lambda - \lambda \frac{f \ln(f)}{1 - f}.$$

If the writedowns are taken to be a constant, say α, where $0 < \alpha < 1$, then f^* is seen to be the root to a simple quadratic equation.

4. PROOFS

We will prove the theorem in three steps: We will first solve for the optimal policy of Problem 2. A standard argument can be modified to treat this case so that no new difficulties arise. Problem 1, which is the main contribution of this paper, is the tricky one, and we will treat it next. We will first show that the optimal wealth process generated by the policy f^* does indeed solve the appropriate Dirichlet problem arising from the Hamilton–Jacobi–Bellman equations. We will then use a martingale argument to verify rigorously that it is indeed optimal. We will then treat Problem 3. The proof that the same policy is optimal for this case relies on the rather interesting fact, which we state as a theorem below, that the ratio of the wealth

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process obtained from any arbitrary admissible policy to the wealth process obtained from the optimal growth policy, f^*, is in fact a supermartingale. Once we establish this, we can use arguments similar to those in [1] (see also [16]) to complete the proof.

PROOF OF PROBLEM 2: We can modify a fairly standard argument to obtain this. To proceed, recall that the solution to Eq. (6) is

$$X_T^f = x \exp\left\{ \int_0^T \left(r + f_u(\mu - r) - \frac{1}{2} f_u^2 \sigma^2 \right) du + \sigma \int_0^T f_u \, dW_u \right\} \times \prod_{i=1}^{N_T} (1 - (1 - f_{\tau_i -}) Y_i), \tag{24}$$

where $\{\tau_i : i \geq 1\}$ are the points of the Poisson process $\{N_t : t \geq 0\}$, and $X_0 = x$. Taking logarithms on Eq. (24) therefore gives

$$\ln X_T^f = \ln x + \int_0^T \left(r + f_u(\mu - r) - \frac{1}{2} f_u^2 \sigma^2 \right) du + \sigma \int_0^T f_u \, dW_u + \sum_{i=1}^{N_T} \ln(1 - (1 - f_{\tau_i -}) Y_i). \tag{25}$$

Since we will be taking expectations on Eq. (25), it is convenient to recall here the following basic fact, which follows directly from the definition of stochastic intensity in Bremaud [4, Sect. II.3]. It is also a simple version of the Campbell theorem for compound Poisson processes with i.i.d. jumps (see, e.g., Daley and Vere-Jones [8, Sect. 12.1]).

LEMMA 1: *Let $\{N_t, t \geq 0\}$ denote a simple Poisson process with points $\{\tau_i, i \geq 1\}$ and intensity λ, and let $Q_t = \sum_{i=1}^{N_t} Y_i$, where $\{Y_i, i \geq 1\}$ are i.i.d., and independent of the Poisson process N. Then for any measurable function $g(x, y)$,*

$$E \sum_{i=1}^{N_t} g(\tau_i, Y_i) = \lambda \int_0^t E[g(u, Y_1)] \, du \tag{26}$$

where the expectation in the right-hand side of Eq. (26) *is with respect to the random variable Y_1.*

It follows from Eq. (26) that

$$E \sum_{i=1}^{N_T} \ln(1 - (1 - f_{\tau_i -}) Y_i) = \lambda \int_0^T E[\ln(1 - (1 - f_u) Y_1)] \, du. \tag{27}$$

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Taking expectations now on Eq. (25) using (27) gives

$$E \ln X_T^f = \ln x + E \int_0^T \left(r + f_u(\mu - r) - \frac{1}{2} f_u^2 \sigma^2 + \lambda E[\ln(1 - (1 - f_u) Y_1)] \right) du$$

$$\equiv \ln x + E \int_0^T C(f_u)\, du, \tag{28}$$

where $C(\cdot)$ was defined earlier in Eq. (16). Since there is no explicit dependence on the wealth process X_u^f in the integrand on the right-hand side of Eq. (28), it suffices to maximize the integrand, which yields the (constant) policy f^* of Eq. (17). Thus all assertions relating to Problem 2 in the theorem have been established. ■

Remark 4: Note that when we place the (constant) optimal policy, f^*, back into Eq. (6), we obtain an optimal wealth process, say X^*, which by Eq. (24) is

$$X_t^* = x \exp\left\{ \left(r + f^*(\mu - r) - \frac{1}{2} (f^*)^2 \sigma^2 \right) t + \sigma f^* W_t \right\} \prod_{i=1}^{N_t} (1 - (1 - f^*) Y_i). \tag{29}$$

Since the Brownian motion $\{W_t, t \geq 0\}$, the Poisson process $\{N_t, t \geq 0\}$, and the i.i.d. writedowns, $\{Y_i, i \geq 1\}$, are all assumed to be mutually independent, it is straightforward to take expectations on Eq. (29) to find that

$$E(X_T^*) = x \exp\{[r + f^*(\mu - r) - \lambda(1 - f^*) E(Y_1)] T\}.$$

We now move on to consider the substantially harder problem of minimizing the expected time to get to a target level of wealth.

Proof of Problem 1: Define now $\tau^* := \inf\{t \geq 0 : X_t^* = b\}$, where X^* is the wealth process obtained from using the strategy f^*. To prove that f^* is indeed optimal for this problem, we must show that

$$\inf_{0 \leq f \leq 1} E_x(\tau_b^f) = \Psi^{(1)}(x) = E_x(\tau^*), \tag{30}$$

where $\Psi^{(1)}(x)$ is given in Eq. (19).

Observe first that for any policy, the goal b is reached by diffusion and not by jumps. Hence, a necessary condition for the first equality in (30) to hold is that the value function $\Psi^{(1)}(x)$ solves the HJB optimality equation (see, e.g., Fleming and Soner [11])

$$\inf_{0 \leq f \leq 1} \mathcal{A}^f \Psi^{(1)}(x) + 1 = 0,$$

with the boundary condition $\Psi^{(1)}(b) = 0$, where $\mathcal{A}^f$ is the generator of Eq. (7).

Let $\mathcal{A}^*$ denote the generator of the (claimed optimal) wealth process X_t^* (i.e., when $f_t = f^*$). Then the fact that f^* solves the HJB optimality equation, and the

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second equality in (30), can be verified directly by observing that the function $\Psi^{(1)}(x)$ solves the Dirichlet problem

$$\mathcal{A}^*\Psi^{(1)}(x) + 1 = 0, \quad \text{for } 0 \le x < b \quad \text{and} \quad \Psi^{(1)}(b) = 0. \tag{31}$$

This can be seen by substituting f^* and $\Psi^{(1)}(x)$ of Eq. (19) into the generator (7), which gives

$$\begin{aligned}\mathcal{A}^*\Psi^{(1)}(x) + 1 &= \mathcal{A}^*\left(\frac{1}{C(f^*)}\ln\left(\frac{b}{x}\right)\right) + 1 \\ &= -\frac{1}{C(f^*)}\left[r + f^*(\mu - r) - \frac{1}{2}(f^*)^2\sigma^2\right. \\ &\qquad \left. + \lambda E\ln(1 - (1 - f^*)Y_1)\right] + 1 \equiv 0,\end{aligned} \tag{32}$$

where the last equality follows from the definition of $C(f^*)$ and f^*.

Since we have shown that $\Psi^{(1)}(x)$ solves the appropriate HJB equations, it remains only to prove sufficiency, that is, to verify that f^* is in fact optimal for the problem of minimizing the expected time to the goal. To do this, we will make use of the *martingale optimality principle*. For this case, this means finding an appropriate functional which is a uniformly integrable martingale under the policy f^*, but is a supermartingale under any other admissible policy.

To proceed, observe that by Eq. (29) we have for the process X^*,

$$\begin{aligned}\ln(X_t^*) = \ln x + &\left\{\left(r + f^*(\mu - r) - \frac{1}{2}(f^*)^2\sigma^2\right)t + \sigma f^* W_t\right\} \\ &+ \sum_{i=1}^{N_t}\ln(1 - (1 - f^*)Y_i)\end{aligned} \tag{33}$$

and, hence, using Eq. (27), we find that

$$E\ln(X_t^*|\mathcal{F}_s) = \ln(X_s^*) + C(f^*)(t - s).$$

From this it follows that the process $\{\ln(X_t^*) - tC(f^*), t \ge 0\}$ is a martingale with respect to the filtration $\mathcal{F} = \{\mathcal{F}_t : t \ge 0\}$.

Define now the function

$$M(t,x) := \Psi^{(1)}(x) + t \equiv C(f^*)^{-1}\ln\left(\frac{b}{x}\right) - t, \quad \text{for } 0 < x \le b. \tag{34}$$

Consider now any other admissible policy, say $h = \{h_t, t \ge 0\}$, which determines a corresponding wealth process $\{X_t^h, t \ge 0\}$. Let τ_b^h denote the first hitting time to the wealth level b for this process, that is, let

$$\tau_b^h := \inf\{t \ge 0 : X_t^h = b\}.$$

We are interested only in control policies h for which $E\tau_b^h < \infty$.

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Optimality of the policy f^* for the problem of minimizing the expected time to a goal will now follow directly from the following:

LEMMA 2: *Let h denote an admissible policy with $E\tau_b^h < \infty$. Then for any such h,*

$$E(M(t, X_t^h)|\mathcal{F}_s) \geq M(s, X_s^h), \quad \text{for } 0 \leq s \leq t \leq \tau_b^h \tag{35}$$

with equality holding if and only if $h = f^*$, where f^* is the policy given by Eq. (17).

PROOF: For $t < \tau_b^h$, Eq. (24) shows that

$$X_t^h = x \exp\left\{ \int_0^t \left(r + h_u(\mu - r) - \frac{1}{2} h_u^2 \sigma^2 \right) du + \sigma \int_0^t h_u \, dW_u \right\} \times \prod_{i=1}^{N_t} (1 - (1 - h_{\tau_i -})Y_i), \tag{36}$$

which allows us to write, for $t < \tau_b^h$

$$M(t, X_t^h) = M(s, X_s^h) + (t - s) - \frac{1}{C(f^*)} \times \left[\int_s^t \left(r + h_u(\mu - r) - \frac{1}{2} h_u^2 \sigma^2 \right) du + \int_s^t \sigma h_u \, dW_u + \sum_{i=N_s}^{N_t} \ln(1 - (1 - h_{\tau_i -})Y_i) \right]. \tag{37}$$

Since $\{N_u\}$ is a simple Poisson process, it follows that

$$E\left(\sum_{i=N_s}^{N_t} \ln(1 - (1 - h_{\tau_i -})Y_i) | \mathcal{F}_s \right) = \lambda E\left(\int_s^t E \ln(1 + h_u Y_1) \, du | \mathcal{F}_s \right),$$

and by the admissibility of h, it follows that $E\int_s^t h_u \, dW_u = 0$.

Hence, after some manipulation, we can write, again for $t < \tau_b^h$,

$$E(M(t, X_t^h)|\mathcal{F}_s) = M(s, X_s^h) + (t - s) - \frac{1}{C(f^*)} \times E\left(\int_s^t \left[r + h_u(\mu - r) - \frac{1}{2} h_u^2 \sigma^2 + \lambda E \ln(1 - (1 - h_u)Y_1) \right] du | \mathcal{F}_s \right), \tag{38}$$

which it is convenient to rewrite, using Eq. (16), as

$$E(M(t, X_t^h)|\mathcal{F}_s) = M(s, X_s^h) + \frac{1}{C(f^*)} E\left(\int_s^t (C(f^*) - C(h_u)) \, du | \mathcal{F}_s \right), \quad \text{for } t \leq \tau_b^h. \tag{39}$$

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But it follows from the definition of $C(f^*)$ that $C(f^*) \geq C(h)$ for any h, hence the integrand above is always nonnegative and is equal to 0 only for $h_u = f^*$ for all $u \geq 0$. Hence, the lemma is proved, and so we may conclude that, in fact, f^* is optimal for Problem 1. ■

Remark 5: This proof holds also for the case $\lambda = 0$, thus offering an alternative proof for the case without credit risk treated earlier in [14,20,5].

We next move on to consider the maximization of the *actual* growth rate of wealth.

Problem 3. To prove that $f^* = f_t^{(3)}$, we will first show that for any other wealth process, say X_t^h, the ratio $Z_t := X_t^h / X_t^*$ is a (nonnegative) local martingale. If this is the case, then since a nonnegative local martingale is also a nonnegative supermartingale, we can adapt the arguments of Algoet and Cover [1] (see also [16, Sect. 9.6]) to this case to show the comparison

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \ln X_T^h \leq \overline{\lim_{T\to\infty}} \frac{1}{T} \ln X_T^* \equiv C(f^*) \tag{40}$$

holds almost surely for every admissible portfolio policy h and its associated wealth process X^h. Thus, the assertions in Theorem 1 regarding Problem 3 will follow if we prove:

LEMMA 3: *Let f^* denote the constant policy of Eq.* (17), *and let $h = \{h_t\}$ denote any other admissible policy. Then the ratio of the corresponding wealth processes, $\{Z_t\}$ where $Z_t = X_t^h / X_t^*$, is a* (*nonnegative*) *supermartingale, that is, for $s \leq t$,*

$$E\left(\frac{X_t^h}{X_t^*} \middle| \mathcal{F}_s \right) \leq \frac{X_s^h}{X_s^*}. \tag{41}$$

PROOF: Let $Z_t := X_t^h / X_t^*$. Recognizing that

$$\Delta Z_t := Z_t - Z_{t-} \equiv -\left(1 - \frac{(1-h_{t-})Y_{N(t-)+1}}{(1-f^*)Y_{N(t-)+1}} \right) Z_{t-}$$

allows us to apply the multidimensional general version of Ito's formula [21, Thm. II.33] to show, after much simplification, that Z_s satisfies

$$\begin{aligned} Z_t = Z_s &+ \int_s^t Z_{u-}((\mu - r)(h_u - f^*) + \sigma^2[(f^*)^2 - f^* h_u])\, du \\ &+ \int_s^t Z_{u-}(h_u - f^*)\, dW_u \\ &- \sum_{i=N_s}^{N_t} Z_{\tau_i -} \left(1 - \frac{1 - (1-h_{\tau_i -})Y_i}{1-(1-f^*)Y_i} \right). \end{aligned} \tag{42}$$

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Since Z is a nonnegative process, the stochastic integral term in Eq. (42) is a nonnegative local martingale, hence a nonnegative supermartingale. As such, taking expectations on Eq. (42) gives

$$E(Z_t|\mathcal{F}_s) \le Z_s + E\left(\int_s^t Z_{u-}((\mu - r)(h_u - f^*) + \sigma^2[(f^*)^2 - f^* h_u])\,du|\mathcal{F}_s\right) - E\left(\sum_{i=N_s}^{N_t} Z_{\tau_i-}\left(1 - \frac{1-(1-h_{\tau_i-})Y_i}{1-(1-f^*)Y_i}\right)\middle|\mathcal{F}_s\right). \quad \textbf{(43)}$$

An application of Eq. (26) allows us to evaluate the last term in Eq. (43) as

$$E\left(\sum_{i=N_s}^{N_t} Z_{\tau_i-}\left(1 - \frac{1-(1-h_{\tau_i-})Y_i}{1-(1-f^*)Y_i}\right)\middle|\mathcal{F}_s\right) = \lambda E\left(\int_s^t Z_{u-} E\left(1 - \frac{1-(1-h_u)Y_1}{1-(1-f^*)Y_1}\right)du\middle|\mathcal{F}_s\right)$$

which in turn shows that Eq. (42) can be written as

$$E(Z_t|\mathcal{F}_s) \le Z_s + E\left(\int_s^t Z_{u-} G(h_u)\,du|\mathcal{F}_s\right), \quad \textbf{(44)}$$

where the function G is defined by

$$G(y) := (\mu - r)(y - f^*) + \sigma^2[(f^*)^2 - f^* y] - \lambda E\left(1 - \frac{1-(1-y)Y_1}{1-(1-f^*)Y_1}\right), \quad \textbf{(45)}$$

where f^* is given by Eq. (17) and where the expectation in the last term is on the random variable Y_1.

Equation (44) shows that (41) will be established if $G(y) \le 0$, which we now show is indeed the case.

PROPOSITION 1: *Let $G(y)$ denote the function defined in Eq.* (45). *Then $G(y) \le 0$ for $0 \le y \le 1$.*

PROOF: Some manipulations show that we can write $G(y)$ as

$$G(y) = (y - f^*)C'(f^*), \quad \textbf{(46)}$$

where $C'(\cdot)$ is the derivative of the function given by Eq. (16), that is,

$$C'(g) = (\mu - r) - \sigma^2 g + \lambda E\left(\frac{Y_1}{1-(1-g)Y_1}\right).$$

Consider then the nonlinear programming problem of

$$\max C(f) \quad \text{subject to} \quad f \le 1, f \ge 0.$$

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Setting up the Lagrangian $L(f,\theta) := C(f) + \theta(1-f)$, where θ is the Lagrangian multiplier, the Kuhn–Tucker optimality conditions are: (i) $C'(f^*) \le \theta^*$; (ii) $f^*(C'(f^*) - \theta^*) = 0$; (iii) $f^* \le 1$; (iv) $\theta^*(1-f^*) = 0$; (v) $\theta^* \ge 0$.

Hence, it is immediate that

$$G(y) \equiv (y - f^*)C'(f^*) \le (y - f^*)\theta^*$$

with $\theta^* = 0$ if $f^* < 1$, and $\theta^* > 0$ if $f^* = 1$. If the former case holds, then we have $G(y) = 0$, while if the latter case holds we have $G(y) \le 0$, since $y \le 1$. ■

Therefore, it follows now from Eq. (44) that $E(Z_t|\mathcal{F}_s) \le Z_s$, which establishes Eq. (41). ■

To continue now, we have shown that Z_t is a nonnegative supermartingale, with $Z_0 = 1$, hence it converges, that is, $Z_t \to Z_\infty < \infty$. Therefore, we may now simply repeat the argument in [1] or [16] to complete the proof, that is, by Kolmogorov's inequality

$$P\left(\sup_{n\le t<\infty} \frac{X_t^h}{X_t^*} > e^{\delta n}\right) \le e^{-\delta n}$$

for every integer $n \ge 1$ and $\delta > 0$. It follows, therefore, that

$$\sum_{n=1}^{\infty} P\left(\sup_{n\le t<\infty} \ln\left(\frac{X_t^h}{X_t^*}\right) > \delta n\right) \le \sum_{n=1}^{\infty} e^{-\delta n} < \infty,$$

and so, by the Borel–Cantelli lemma, there exists an integer valued random variable N_δ such that

$$\ln\left(\frac{X_t^h}{X_t^*}\right) \le \delta n \le \delta t, \quad \text{for all } n \ge N_\delta \quad \text{and } t \ge n.$$

It follows that $\sup_{t\ge n}(1/t)\ln(X_t^h/X_t^*) \le \delta$ holds for every $n \ge N_\delta$, and therefore we also have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln\left(\frac{X_t^h}{X_t^*}\right) \le \delta \quad \text{a.e.}$$

The inequality (40) now follows from the fact that δ is arbitrary. ■

5. CONCLUSIONS

We have shown that for the case of a single stock whose price process follows a geometric Brownian motion, and a risky bond whose price may fall by random amounts at the jumps of a homogeneous Poisson process, the same investment policy is optimal for the three objectives of (i) minimizing the expected time to reach a given level of wealth, (ii) maximizing the expected rate of growth over a finite horizon (which is equivalent to maximizing terminal logarithmic utility of wealth), and (iii) maximizing the actual rate of growth over an infinite horizon.

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It is a relatively simple matter to extend our results to the case of multiple stocks and bonds. It is also a simple matter to extend the model of the default rate to the case of an arbitrary point process, as well as to stock prices that follow stochastic differential equation with more general coefficients than the constant ones treated here. However, while we could then still show that the solutions to Problems 2 and 3 are equivalent, Problem 1, minimizing the expected time to the goal, which is the main point of this paper, cannot in general be solved in closed form for processes more general than those considered here. The reason for this, of course, is that any time dependence in the underlying parameters will also make the optimal policy for Problems 2 and 3 (they will in general be the same) time-dependent as well, which would in turn make the expected value of the logarithm of the resulting wealth process a nonlinear function of time. However, it is precisely the linearity of the expected logarithm of wealth that allowed the explicit analysis for Problem 1, so we cannot as yet prove that the optimal policy for the minimum time to the goal problem is indeed the same as that for the others for more general models. We conjecture at this point that indeed such is the case at least asymptotically (as $b \uparrow \infty$) by the results of Breiman [3].

It would, of course, be very interesting to know what the optimal policy is for Problem 1 for more general processes than those treated here, and we leave this problem open for future research. It is interesting to note that even if the constant coefficient model holds, together with a simple Poisson jump process but with negative as well as positive jumps allowed, then our results for Problem 1 are no longer valid. The reason for this is that, in that case, the wealth process may in fact reach the goal (and beyond) by taking a jump. In that case, the HJB equations need to be modified for this possibility, and there does not appear to be a closed form solution to the problem. We leave this problem, too, for future research.

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