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Substreetutions and more on trees

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Abstract. We define a notion of substitution on colored binary trees that we call substreetution. We show that a point fixed by a substreetution may (or not) be almost periodic, and thus the closure of the orbit under the $\mathbb{F}_2^+$-action may (or not) be minimal. We study one special example: we show that it belongs to the minimal case and that the number of preimages in the minimal set increases just exponentially fast, whereas it could be expected a super-exponential growth. We also give examples of periodic trees without invariant measures on their orbit. We use our construction to get quasi-periodic colored tilings of the hyperbolic disk.

Key words: tree shift, entropy, group action, minimal systems
2020 Mathematics Subject Classification: 05C05 (Primary); 37B05 (Secondary)

1. *Introduction*

1.1. *Background and main motivations.*

In the present paper, we define and study a notion of substitution on colored trees. In the following, they will be called substreetutions to make clear they are substitutions on trees. The term substitution will be used for 'classical ones' acting on the line. These substreetutions are new objects and generate many natural questions, within the dynamical viewpoint.

This work is a first step in view to export thermodynamic formalism to colored trees and $\mathbb{F}_2^+$-actions. Our first main theorem (see Theorem A) states the existence of a substreetution H which admits a fixed point $\mathfrak{J} = H(\mathfrak{J})$, such that the closure of the $\mathbb{F}_2^+$-orbit of $\mathfrak{J}$ $X := \overline{\{T_\omega(\mathfrak{J}),\ \omega \in \mathbb{F}_2^+\}}$ is a minimal invariant set. This construction is an extension to the case of colored trees and $\mathbb{F}_2^+$ actions of well-known objects for the case $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}$-actions, e.g. the Fibonacci or the Thue–Morse substitutions. However, we point out that minimality for $\mathbb{F}_2^+$-actions is different from minimality for $\mathbb{N}$-actions, and does not only mean that any

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point has a dense orbit. We refer the reader to [10, 11]. In particular, we exhibit an example (see Theorem 1.4) of a periodic orbit which is non-minimal. In future work, we expect to study thermodynamic formalism for $(X, \mathbb{F}_2^+)$.

The first motivation comes from thermodynamic formalism in ergodic theory. One of our final goals is to exhibit and study examples of freezing phase transitions (hopefully) with ground states supported in quasi-crystal for several semi-group actions. A series of works [1, 6, 7] investigates this question for $\mathbb{Z}$-actions and makes links with substitutions. The question of higher-dimensional actions naturally appears, but conflicts with the fact that 1D thermodynamic formalism deeply uses the transfer operator, in particular to detect freezing phase transitions. Indeed, there is no equivalent of this operator for $\mathbb{Z}^d$-actions with $d > 1$. Actually, this operator needs (to be well defined) a kind of canonical order in the action, which does not hold for $\mathbb{Z}^d$, $d > 1$. As $\mathbb{F}_2^+$ has a canonical order, we have thus been led to study substitutions associated to $\mathbb{F}_2^+$-actions.

The second motivation is the existence of works on Sturmian trees (see [5, 15]). We were motivated to define substitutions on colored trees and to study the closure of the fixed point, as it is done for the Fibonacci sequence. We remind that the Fibonacci sequence is both Sturmian and substitutive.

Coming from the ergodic theory world, the question of invariant measures is of prime interest. It seems that little is known on invariant measures for trees. We remind that $\mathbb{F}_2^+$ is not amenable, which means the standard construction of invariant measures fails. Literature mentions examples for simple periodic trees with high symmetries, Bernoulli measures on the vertices, or Markov inspired measures (see [4, 24]). In [13], some stationary measures are studied but it is not clear that they are $\mathbb{F}_2^+$-invariant measures. We also mention the work of Rozikov *et al* (see [23]) in the statistical mechanics context.

We remind that thermodynamic formalism is inspired from statistical mechanics in physics. In thermodynamic formalism, invariant measures are usually obtained as *equilibrium states*, and more precisely as *conformal* measures coming from a transfer operator. It turns out that the minimal set we get in Theorem A seems to be sufficiently (dynamically) rich to contain many invariant measures. Contrary to what could be expected (see the first motivation), we finally have hope that the machinery with the transfer operator could work to produce invariant measures inside X and not only external measures with freezing phase transition in X.

If one wants to mimic or adapt the classical proofs, one of the first key points is to control the spectral radius of the transfer operator. Still for the classical case, the logarithm of the spectral radius is the topological entropy of the system. Entropy is a way to quantify the complexity of the system, and this is the core of the study in dynamical systems. Usually, entropy is given by the exponential growth rate of the number of configurations the system sees with respect to their length. In the case of transitive shifts of finite type, for example, defined by means of an adjacency matrix A that shows the allowed transitions among elements of the alphabet, entropy is precisely the logarithm of the Perron–Frobenius eigenvalue of A. For labeled trees, it is natural to expect a super-exponential growth rate (see [19]).

It is interesting to remember that for classical substitutions, complexity is in $O(n)$, $O(n \log\log n)$, $O(n \log n)$, or $O(n^2)$. This means that chaos is not observed at

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exponential scale but at a lower scale. Substitutions are chaotic but zero-entropy systems. Here, we show in our second main result, Theorem B, that the growth rate for the number of preimages of any tree in X is at most exponential, with an upper bound on the speed. As the growth rate for preimages is usually expected to coincide with the growth rate for configurations [21], this indicates that it is likely that our minimal system X behaves as a classical expanding dynamical system, despite the acting semi-group being $\mathbb{F}_2^+$ and not $\mathbb{N}$.

Copying the vocabulary for classical substitution, we *mostly* consider constant length-2 substitutions on regular binary trees in the present paper. More general substreetutions can easily be defined; however, some computations show that complexity may become large extremely fast.

Several examples of constant length-2 substreetutions are described and studied in the paper. In one example, the fixed point is the usual Thue–Morse sequence lifted in a tree (each line has a constant color). The closure of the orbit of this tree is obviously minimal but does not really exploit the tree structure.

Furthermore, in Theorem 1.5, we give an example of a periodic tree without an $\mathbb{F}_2^+$-invariant measure.

The referee has kindly pointed out to us the paper [2], which continues a general program and aims to expand the theory of aperiodic order to non-abelian locally compact groups and to study the resulting dynamical systems. We hope that our present article participates in this general program.

Many results we use on substitutions are well known and can be found in the general book on substitutions [12].

1.2. *Settings and results*

1.2.1. *Binary trees and colored trees.* We consider the free monoid with two generators $\mathbb{F}_2^+$ whose elements are the identity and all the finite words, of any given length, that can be written with the alphabet $\{a, b\}$. The letter e stands for the empty word; $|\cdot|$ is the length of a given word, meaning the number of letters it has; and we also set $|e| = 0$. The product of two words is just the word obtained by the concatenation. The lexicographic order on $\mathbb{F}_2^+$ is the one given by $a < b$ and $\omega < \omega'$ if $|\omega| < |\omega'|$.

Fix a set $\mathcal{A}$, called the alphabet; a *colored tree (also called configuration)* is a map $\mathfrak{B}\colon \mathbb{F}_2^+ \to \mathcal{A}$, say $\mathfrak{B} \in \mathcal{A}^{\mathbb{F}_2^+}$. In the following, one shall set $\mathfrak{B}_x$ for the value $\mathfrak{B}(x)$ and, for simplicity, we also fix $\mathcal{A}$ as being the set $\{0, 1\}$. We will only consider colored trees, and thus simply use the word tree.

If $\mathfrak{A}$ is a tree and ω belongs to $\mathbb{F}_2^+$, $\mathfrak{A}_\omega$ is called the digit at position ω. Hence, if the tree $\mathfrak{A}$ is fixed, elements of $\mathbb{F}_2^+$ will also be called *sites*. The *generation* of a site ω is $|\omega|$. Hence, the *root*, which corresponds to the site e, is generation 0, the two first descendants form the set of sites at generation 1, and so on.

We shall always represent trees vertically, going down, with the root at the top. With this representation, the digits of $\mathfrak{A}$ corresponding to a fixed generation $|\omega|$ endowed with the lexicographic order of its sites are called a *line*; line 0 corresponds to $\mathfrak{A}_e$ (the root), line 1 corresponds to $(\mathfrak{A}_a, \mathfrak{A}_b)$, line 2 to $(\mathfrak{A}_{aa}, \mathfrak{A}_{ab}, \mathfrak{A}_{ba}, \mathfrak{A}_{bb})$, and so on.

If ω is a site, $\omega.a$ is called its a-follower. Equivalently, we shall say that the digit $\mathfrak{A}_{\omega a}$ is the a-follower of digit $\mathfrak{A}_\omega$ and the digit $\mathfrak{A}_{\omega b}$ is the b-follower of digit $\mathfrak{A}_\omega$. The sites $\omega.a$

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and $\omega.b$ are said to be brothers. Followers will also be called children. An n-descendant for ω is any digit at site $\omega\omega'$ with $|\omega'| = n$. Children are 1-descendants.

The site ω is the *1-ascendant* of sites $\omega.a$ and $\omega.b$. We shall also call it the *father*. The father of the father is called the *grandfather*. It is also the 2-ancestor. By induction, one may define the n-ancestor of any site ω with $|\omega| > n$. Two sites are called *cousins* if they have the same grandfather but they are not brothers.

A backward path in the tree is a path going from one site to one of its ancestor and containing all intermediate ancestors. A forward (finite) path is a backward path read in the opposite direction.

The distance between two trees $\mathfrak{A}$ and $\mathfrak{B}$ is $2^{-N(\mathfrak{A},\mathfrak{B})}$, where $N(\mathfrak{A}, \mathfrak{B})$ is the minimal integer n such that $\mathfrak{A}_\omega \neq \mathfrak{B}_\omega$ and $|\omega| = n$.

In other words, $d(\mathfrak{A}, \mathfrak{B}) = 2^{-n}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ have different roots if $n = 0$, and if $n \geq 1$, then $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$ coincide for every ω with $|\omega| \leq n - 1$ and for at least one ω with $|\omega| = n$, one of the followers of $\mathfrak{A}_\omega$ is different to the same follower for $\mathfrak{B}_\omega$.

Note that the space of trees $\mathcal{A}^{\mathbb{F}_2^+}$ is compact (for the metric we introduced) as a product of compact spaces. The subset of trees with root equal to 0 (or 1) is also compact as a closed set included into a compact set.

1.2.2. *Canonical dynamics on trees.* There is a natural $\mathbb{F}_2^+$-action on trees defined as follows.

For any given $x \in \mathbb{F}_2^+$ and $\mathfrak{A} \in \mathbb{F}_2^+ \to \{0, 1\}$, take

$$(T_a\mathfrak{A})_x := \mathfrak{A}_{ax} \quad \text{and} \quad (T_b\mathfrak{A})_x := \mathfrak{A}_{bx}.$$

For the definition of distance, one can easily see that we have

$$d(T_\omega(\mathfrak{A}), T_\omega(\mathfrak{B})) \leq 2d(\mathfrak{A}, \mathfrak{B}) \quad \text{for } \omega = a, b,$$

and thus the maps T_a and T_b are Lipschitz continuous.

As usual, given $\omega \in \mathbb{F}_2^+$, $\omega = \omega_1\omega_2 \cdots \omega_k$, we just write $T_\omega = T_{\omega_1\omega_2\cdots\omega_k} = T_{\omega_1} \circ T_{\omega_2} \circ \cdots \circ T_{\omega_k}$. It is important to observe that

$$(T_{\omega_1}T_{\omega_2} \cdots T_{\omega_k}\mathfrak{A})_x = (T_{\omega_2} \cdots T_{\omega_k}\mathfrak{A})_{\omega_1 x} = (T_{\omega_3\cdots T_{\omega_k}}\mathfrak{A})_{\omega_2\omega_1 x} = \cdots = \mathfrak{A}_{\omega_k\cdots\omega_2\omega_1 x}.$$

This can be simply written as $(T_\omega\mathfrak{A})_x = \mathfrak{A}_{\tilde{\omega}x}$ where the symbol $\tilde{}$ reverses the order of the word ω, say, $\tilde{\omega} := \omega_k\omega_{k-1} \cdots \omega_2\omega_1$.

Definition 1.1. Let $\mathfrak{A}$ be a tree. A patch (of size $n \geq 1$) is any ball $B(T_\omega\mathfrak{A}, 2^{-n})$ where $\omega \in \mathbb{F}_2^+$.

Definition 1.2. A tree $\mathfrak{A}$ is called pre-periodic if there exists a finite set of trees $\mathcal{O} := \{\mathfrak{A}(1), \mathfrak{A}(2), \ldots, \mathfrak{A}(n)\}$ satisfying:

(1) $\mathcal{O}$ contains $\mathfrak{A}$;
(2) $\mathcal{O}$ is T_a- and T_b-invariant.

The tree $\mathfrak{A}$ is said to be periodic if $T_a(\mathcal{O}) \cup T_b(\mathcal{O}) = \mathcal{O}$.

Equivalently, $\mathfrak{A}$ is periodic and $\mathcal{O} := \{\mathfrak{A}(1), \mathfrak{A}(2), \ldots, \mathfrak{A}(n)\}$ is its orbit means that if we consider the oriented graph whose states are the elements of $\mathcal{O}$ with edges representing

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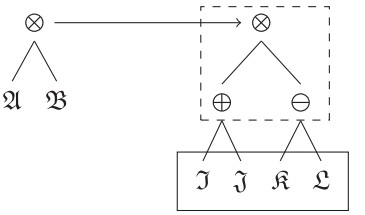


FIGURE 1. A marked substreetution.

images by T_a or T_b, then each vertex is the initial point for two edges and the end point for at least one edge.

Notation 1. For simplicity, we shall set $T^{-1}(\mathfrak{A})$ for $T_a^{-1}(\mathfrak{A}) \cup T_b^{-1}(\mathfrak{A})$.

1.2.3. *Substreetution*

Definition 1.3. A *substreetution* (actually, this is a *constant length-2* substitution) on trees is a map H on the set of configurations defined by concatenation as follows.

(1) H maps each site to a triple (actually a root with two followers), the value depending only on the value of the digit at the site. See box with a dashline in Figure 1.
(2) H connects images of subtrees (followers) as indicated in Figure 1, with $\mathfrak{I}, \mathfrak{J}, \mathfrak{K}, \mathfrak{L} \in \{H(\mathfrak{A}), H(\mathfrak{B})\}$.

The order word $\mathfrak{IJKL}$ is called the *grammar* of the substreetution and is explained below.

The substreetution is said to be *marked* if $H(0) = \overset{i}{/\ \backslash}$ and $H(1) = \overset{1-i}{/\ \backslash}$, $i = 0, 1$.

Identifying the trees and their name, and re-employing notation from above, the word $\mathfrak{IJKL}$ is a word in $\{H(\mathfrak{A}), H(\mathfrak{B})\}^4$ and can canonically be identified with a word in $\{\mathfrak{A}, \mathfrak{B}\}^4$. This defines the *grammar* of the substreetution H and indicates the H-image of which sub-tree as to be connected on which slot at generation 2.

Example. The grammar $ABBA$ means $\mathfrak{I} = \mathfrak{L} = H(\mathfrak{A})$ and $\mathfrak{J} = \mathfrak{K} = H(\mathfrak{B})$. In this case, it is easy to see that the function H satisfies the following relations: $HT_a = T_{aa}H = T_{bb}H$ and $HT_b = T_{ab}H = T_{ba}H$.

More generally, we let the reader check that

$$\{T_{aa}H, T_{ab}H, T_{ba}H, T_{bb}H\} = \{HT_a, HT_b\}, \tag{1}$$

which is rewritten under the form $T^2 \circ H = H \circ T$. We call this relation the *renormalization equation*.

We emphasize that equation (1) is a natural extension to the $\mathbb{F}_2^+$-action case of the property $\sigma^2 \circ H = H \circ \sigma$, which holds for the shift considered as an $\mathbb{N}$-action for constant length-2 substitutions.

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It has been pointed out to us that our procedure could have some similarities with the one introduced in [9]. This procedure deals with theoretical computer science. The vocabulary and probably the questions are quite different from those in Dynamical Systems (at least as used or studied by the authors). This makes the understanding of similarities or differences harder. At least, we see a difference as the Damm procedure seems to study finite trees whereas we are interested in infinite trees. We emphasize that our procedure has been defined from the need that the renormalization equality in equation (1) holds.

1.2.4. *Results.* The first two results describe a property of one specific substreetution.



THEOREM A. *Let H denote the substreetution given by $0 \mapsto 1\ \ 0$ and $1 \mapsto 1\ \ 0$, equipped with the grammar BBAB. Then there exists a unique fixed point $\mathfrak{J}$ with root 0. The closure of its orbit is a minimal dynamical system X and is not periodic.*

There also exists a unique fixed point $\mathfrak{J}'$ with root 1. It coincides with $\mathfrak{J}$ except at the root.

As T is continuous and X is compact, $T(X) \subset X$ and minimality yield that $T(X) = X$. Hence, T (restricted to X) is onto, which is *a priori* not obvious. This makes sense to study the backward complexity in X. Furthermore, it is not clear that $\mathfrak{J}'$ belongs to X. However, we show this actually holds, see Corollary 5.17.

THEOREM B. *With the same H, let $\mathfrak{A}$ be in X. Set $p(n, \mathfrak{A}) := \#\{\mathfrak{B} \in X, T_\omega(\mathfrak{B}) = \mathfrak{A}, |\omega| = n\}$. Then for every n,*

$$p(n, \mathfrak{A}) \leq 3^n.$$

The bound in Theorem B is not sharp at all. What is remarkable is that it exists. We emphasize that in the whole space of trees $\mathcal{A}^{\mathbb{F}_2^+}$, for any tree $\mathfrak{A}$, $T^{-1}(\mathfrak{A})$ has cardinality of the continuum. Furthermore, we conjecture a result similar to what holds for the tree-pressure in [20]: for *most* of the elements $\mathfrak{A}$ in X,

$$\lim_{n\to+\infty} \frac{1}{n} \log p(n, \mathfrak{A}) = \sup_{\mathfrak{B}\in X} \lim_{n\to+\infty} \frac{1}{n} \log p(n, \mathfrak{B}),$$

and both quantities exist.

Re-employing vocabulary from 1D ergodic theory, balls of size 2^{-n} form a partition of X. We denote by R_n its cardinality. Here, R_n is the number of patches of size n that we see in $\mathfrak{J}$. We conjecture that

$$\lim_{n\to+\infty} \frac{1}{n} \log R_n = \sup_{\mathfrak{B}\in X} \lim_{n\to+\infty} \frac{1}{n} \log p(n, \mathfrak{B})$$

holds.

The next results show that substreetutions may not generate a minimal set. The final result shows that the existence of invariant measures on a periodic set of trees does not always hold.

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THEOREM 1.4. *The substreetution given by* $0 \mapsto \begin{smallmatrix} 0 \\ / \ \backslash \\ 0 \quad 1 \end{smallmatrix}$ *and* $1 \mapsto \begin{smallmatrix} 1 \\ / \ \backslash \\ 1 \quad 0 \end{smallmatrix}$, *equipped with the grammar ABBA is periodic but non-minimal.*

THEOREM 1.5. *There exist periodic trees with no invariant measure on their orbit.*

1.3. *More general substreetutions.* Once we agree that the way to define a substitution on trees is to consider that equality in the renormalization equation (1) holds for sets, then it is very to define more general substreetutions.

Actually, H can be defined by:

- fixing the image of each 'color' as a finite tree;
- fixing the grammar for each *slot* at the end of each $H(\otimes)$.

This may be defined for colored trees involving a greater alphabet (more than two colors) and even not necessarily binary (or even regular) trees. Of course, the grammar has to consider possible irregularities in the number of followers.

Example. We can consider a substreetution on binary 2-colored trees defined by

$$H\left(\begin{smallmatrix} 0 \\ / \ \backslash \\ A \quad B \end{smallmatrix}\right) = \begin{smallmatrix} 0 \\ / \ \backslash \\ 0 \quad H(B) \\ / \ \backslash \\ H(A) \quad H(B) \end{smallmatrix} \quad \text{and} \quad H\left(\begin{smallmatrix} 1 \\ / \ \backslash \\ A \quad B \end{smallmatrix}\right) = \begin{smallmatrix} 1 \\ / \ \backslash \\ 0 \qquad 0 \\ / \ \backslash \quad / \ \backslash \\ H(A) \quad H(B) \quad H(B) \quad H(A) \end{smallmatrix}.$$

In this paper, we will mostly consider constant length-2 substreetutions, because these objects are already complicated and there are many situations that we have not yet studied. For constant length-2 substitutions on alphabets with two letters, there are $2^3 = 8$ possibilities. For constant length-2 substreetutions on regular binary trees, there are 128 possibilities. Even if these numbers may be diminished due to symmetries, the number of possibilities for trees is much higher than the number of possibilities for the line.

Furthermore, we will deeply use the constant-length structure to define tools (as the *source and the* χ*-procedure*). These tools could be defined for general substreetutions but would be much more complicated and harder to work with.

1.4. *Plan of the paper.* In §2, we first give some general results on substreetution, as the existence of fixed points. Then, we prove some points of Theorem A: existence of two fixed points that differ only from the root. This property corresponds, according to [**17**, Definition 30], to say that the constant length substitution considered in Theorem A is *not* strongly injective.

In §3, we give a precise description of each line of the fixed point $\mathfrak{J}$. For that, we use the notion of source which corresponds to unsubstitution in the classical.

Section 4 is devoted to the proof of the remaining points in the Main Theorem A: that X is minimal and that $\mathfrak{J}$ is not periodic.

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Minimality follows from [10]. We show that $\mathfrak{J}$ is almost periodic. The proof that $\mathfrak{J}$ is not periodic is done by contradiction. Assuming it is periodic, we use the Perron–Frobenius theorem on positive matrices to show that the proportion of 1s along the lines of $\mathfrak{J}$ should have finitely many accumulation points that are all positive. Contradiction comes from the fact that the proportion of 1s goes to 0 along special subsequences of lines.

Section 5 explores several symmetries and rigidities of the configuration $\mathfrak{J}$ and its images. This allows us to determine the types (namely odd and even) of trees in X.

We also show that knowing one subtree in $\mathfrak{A} \in X$ allows one to recover its brother and even its cousins.

Section 6 is devoted to the proof of Main Theorem B. For any type of tree defined above, we compute the number of possible preimages in X and give a precise description of them.

Finally, in §7, we show Theorems 1.4 and 1.5. We also give an interesting example of substreetution generating the well-known Thue–Morse sequence along the paths.

We give a picture of $\mathfrak{J}$ and show how to get a colored quasi-periodic tiling of $\mathbb{D}^2$ using elements of X.

2. *Some more general facts on (constant length-2) substreetutions*

2.1. *Source and unsubstreetuted tree.*

We consider a *constant length-2* substreetution as defined in Definition 1.3.

Recall the renormalization equation $T^2 \circ H = H \circ T$. From this we get the following lemma.

LEMMA 2.1. *Let H be a substreetution. For every site ω with even length, there exists a word $s(\omega)$ of length $|\omega|/2$ such that*

$$T_\omega \circ H = H \circ T_{s(\omega)}.$$

The site $s(\omega)$ is called the source of the site ω. Furthermore, given two words of even length ω and ψ, then $s(\omega\psi) = s(\omega)s(\psi)$.

Proof. The first part follows as a direct application of the renormalization equation by induction. For the second, just notice that

$$T_{\omega\psi} \circ H = T_\omega \circ T_\psi \circ H = T_\omega \circ H \circ T_{s(\psi)} = H \circ T_{s(\omega)} \circ T_{s(\psi)} = H \circ T_{s(\omega)s(\psi)} = H \circ T_{s(\omega\psi)}. \quad \square$$

We emphasize that a kind of converse is true. Given any word ω, there exists ω' with $|\omega'| = 2|\omega|$ such that $T_{\omega'} \circ H = H \circ T_\omega$. Nevertheless, such an ω' is not necessarily unique. It is entirely determined by the grammar.

2.2. *Fixed points for marked substitution.*

We start with a basic observation.

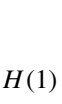
LEMMA 2.2. *A marked substreetution is one-to-one.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two trees coinciding up to generation $n-1$. Let ω, with $|\omega| = n$, be such that $\mathfrak{A}_\omega \neq \mathfrak{B}_\omega$. Let ω' be such that $s(\omega') = \omega$. By definition, $T_\omega(\mathfrak{A})$ and $T_\omega(\mathfrak{B})$ have different roots. Because H is marked, $H \circ T_\omega(\mathfrak{A}) = T_{\omega'}(H(\mathfrak{A}))$ and $H \circ T_\omega(\mathfrak{B}) = T_{\omega'}(H(\mathfrak{B}))$ have different roots. Therefore, $H(\mathfrak{A}) \neq H(\mathfrak{B})$. $\square$

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Assume that $\mathfrak{A}$ is a fixed-point for a substreetution H. Let ω be a site with even length. Set $\omega' = s(\omega)$. Then, $T_\omega(\mathfrak{A}) = H(T_{\omega'}(\mathfrak{A}))$. We say that $T_{\omega'}(\mathfrak{A})$ is the *unsubstreetuted tree* from $T_\omega(\mathfrak{A})$. By Lemmas 2.1 and 2.2, it is well defined and uniquely determined. However, because the source operation is not one-to-one, it may be that several sites have the same unsubstreetuted tree. This holds if and only if the subtrees with roots on these sites are equal.

Now, we show that a well-known result on classical substitutions holds for substreetutions.

PROPOSITION 2.3. *If* $H(0) =$  *(respectively* $H(1) =$  *) then, independently of the chosen grammar, there is a unique fixed point for* H *starting with* 0 *(respectively* 1*). They will respectively be denoted by* ${}_0\mathfrak{A}$ *and* ${}_1\mathfrak{A}$.

Proof. Let us assume that $H(0) = (0, ., .)$ holds, the other case being totally symmetric.

By definition of H, if $\mathfrak{A}$ and $\mathfrak{B}$ are two trees coinciding up to generation n, with $n \geq 1$, then $H(\mathfrak{A})$ and $H(\mathfrak{B})$ do coincide up to generation $2n$ because each site produces two generations of new sites. Therefore,

$$d(H(\mathfrak{A}), H(\mathfrak{B})) \leq d^2(\mathfrak{A}, \mathfrak{B}) \leq \tfrac{1}{2} d(\mathfrak{A}, \mathfrak{B}).$$

Hence, H acts as a contraction on the compact set of trees whose root is 0. It admits a unique fixed point on this set. □

Because the set of trees is compact, and thus is a Hausdorff space, the fixed point ${}_0\mathfrak{A}$ is obtained as the limit of $H^n(\mathfrak{A})$ as n goes to $+\infty$, where $\mathfrak{A}$ is any tree with root 0. Furthermore, ${}_0\mathfrak{A}$ can also be obtained as $\lim_{n\to+\infty} H^n(0)$.

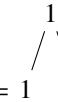
In Theorem A, we consider the substitution $0 \mapsto$  and $1 \mapsto$ , equipped with the grammar BBAB.

It fulfills conditions to ensure that there is a fixed point with root 0 and a fixed point with root 1. We denote by $\mathfrak{J}$ the fixed tree with root 0 and by $\mathfrak{J}'$ the fixed point with root 1. It is easy to check (by induction) that each odd line is a string of 10s.

We give here a first description of these two fixed points.

PROPOSITION 2.4. *The two trees* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *only differ on the root.*

Proof. We recall that $\mathfrak{J}$ and $\mathfrak{J}'$ are obtained as $\lim_{n\to+\infty} H^n(0)$ and $\lim_{n\to+\infty} H^n(1)$. Hence, the proof is done by induction on the number of iterations for H. We show that $H^n(0)$ and $H^n(1)$ only differ in the root.

Because $H(0) = 1$  and $H(1) = 1$ , then level 1 for $\mathfrak{J}$ is equal to level 1 for $\mathfrak{J}'$.

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For $n \geq 2$, $H^n(0) = H^{n-1}\begin{pmatrix} & 0 & \\ / & & \backslash \\ 1 & & 0 \end{pmatrix}$. This finite tree is the tree $\mathfrak{A}^{n-1} := H^{n-1}(0)$, where we glue the trees $H^{n-1}(0)$ or $H^{n-1}(1)$ at the extremities. The gluing only depends on the grammar rules.

Similarly, the finite tree $H^n(1) = H^{n-1}\begin{pmatrix} & 1 & \\ / & & \backslash \\ 1 & & 0 \end{pmatrix}$ is equal to the tree $\mathfrak{B}^{n-1} := H^{n-1}(0)$, where we glue the trees $H^{n-1}(0)$ or $H^{n-1}(1)$ at the extremities. The gluing for that tree respects the same rules as for $H^n(0)$.

By assumption, $\mathfrak{A}$ and $\mathfrak{B}$ only differ in the root. This finishes the proof by induction. □

In the following, we write $\mathcal{O}(\mathfrak{J})$ for $\{T_\omega(\mathfrak{J}),\ \omega \in \mathbb{F}_2^+\}$. We remind that $X = \overline{\mathcal{O}(\mathfrak{J})}$.

3. *Description of the Jacaranda tree* $\mathfrak{J}$

In all the following, except if it is especially mentioned, we shall only consider the substitution from Theorem A: $0 \mapsto \begin{smallmatrix} & 0 & \\ / & & \backslash \\ 1 & & 0 \end{smallmatrix}$ and $1 \mapsto \begin{smallmatrix} & 1 & \\ / & & \backslash \\ 1 & & 0 \end{smallmatrix}$, equipped with the grammar BBAB. In this case, the renormalization equations take the form

$$T_aT_aH = T_bT_aH = T_bT_bH = T_b, \quad T_aT_bH = HT_a, \tag{2}$$

where $\mathfrak{J}$ and $\mathfrak{J}'$ are the two fixed points (for H) respectively with root 0 and 1. Additionally, $\mathfrak{J}$ is referred to as the *Jacaranda tree*.

As mentioned above, each odd line in $\mathfrak{J}$ is a concatenation of 10s. The goal of this section is first to describe each even line (see Proposition 3.11) and to get an estimation for the number of 1s on even lines. This later result will be used to show that $\mathfrak{J}$ is not periodic (nor pre-periodic).

3.1. *Inverse of the source function for* $\mathfrak{J}$.

As defined previously in Lemma 2.1, the source function s is defined by

$$T_\omega H = HT_{s(\omega)}.$$

From equation (2), it is easy to see that

$$\begin{cases} s(aa) = s(ba) = s(bb) = b, \\ s(ab) = a. \end{cases}$$

Obtaining s for a general even length word ω is now easy with the use of the concatenation property described in Lemma 2.1.

Here we want to adapt the source function s to understand how to describe the tree $\mathfrak{J}$; more precisely, given an even length word ω, we want to find a function S such that $\mathfrak{J}_\omega = \mathfrak{J}_{S(\omega)}$.

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We remind the reader that for a word $\psi = \psi_1\psi_2 \dots \psi_k$, we set $\widetilde{\psi} = \psi_k\psi_{k-1} \dots \psi_1$; we easily get $\widetilde{\psi\eta} = \widetilde{\eta}\widetilde{\psi}$. By extension, we also set $\widetilde{s}(\omega) := \widetilde{s(\omega)}$.

We claim that $S(\omega) = \widetilde{s}(\widetilde{\omega})$ holds. To prove the claim, we first recall that the map H has the property that for any given tree $\mathfrak{C}$, we get $(H(\mathfrak{C}))_e = \mathfrak{C}_e$. Then,

$$\mathfrak{J}_\omega = (T_{\widetilde{\omega}}\mathfrak{J})_e = (T_{\widetilde{\omega}} \circ H(\mathfrak{J}))_e = (H \circ T_{s(\widetilde{\omega})}\mathfrak{J})_e = (T_{s(\widetilde{\omega})}\mathfrak{J})_e = \mathfrak{J}_{\widetilde{s}(\widetilde{\omega})} = \mathfrak{J}_{S(\omega)}.$$

Hence, a is the image of ba under S. In a similar way, we can obtain that the images of S on the points aa, ab, bb, and ab are:

$$\begin{cases} S(aa) = S(ab) = S(bb) = b, \\ S(ba) = a. \end{cases}$$

More generally, we get the following lemma.

LEMMA 3.1. *Given two words P and Q with even length, then $S(PQ) = S(P)S(Q)$.*

Proof. The proof uses the properties of the source function s presented in Lemma 2.1: we have

$$S(PQ) = \widetilde{s}(\widetilde{PQ}) = \widetilde{s}(\widetilde{Q}\widetilde{P}) = s(\widetilde{\widetilde{Q}})s(\widetilde{P}) = \widetilde{s}(\widetilde{P})\widetilde{s}(\widetilde{Q}) = S(P)S(Q). \qquad \square$$

Now we want to see the inverse image of S, which we denote by θ, as a multivalued function that maps words to sets of words. It is defined in such a way that $S(\theta(p)) = \{p\}$ for any word p and $\theta(S(P)) \ni P$ for any even length word P. In the definition of θ, we also set $\theta(\emptyset) = \emptyset$ and $\theta(e) = \{e\}$.

The main property of the function θ is the following lemma.

LEMMA 3.2. *Given the words p and q, then $\theta(pq) = \theta(p)\theta(q)$ (where the product above is the set of words obtained by concatenating any word in $\theta(p)$ with any another word in $\theta(q)$).*

Proof. ($\theta(p)\theta(q) \subseteq \theta(pq)$): Let $R \in \theta(p)\theta(q)$; then $R = PQ$, where $P \in \theta(p)$ and $Q \in \theta(q)$. Hence, $S(P) = p$ and $S(Q) = q$, and Lemma 3.1 yields $S(R) = S(PQ) = S(P)S(Q) = pq$, showing that pq is the source for R. Therefore, $R \in \theta(pq)$ holds.

($\theta(pq) \subseteq \theta(p)\theta(q)$): Let $R \in \theta(pq)$; hence, $S(R) = pq$. Since R is a word with even length (because it is in the image of θ), we can write $R = AB$ with $|A| = 2|p|$ and $|B| = 2|q|$. Then $S(R) = S(AB) = S(A)S(B)$. This shows that $S(A)$ is the prefix of $S(R) = pq$ with length $|p|$. Similarly, $S(B)$ is the suffix of pq with length $|q|$. Hence, $S(A) = p$ and $S(B) = q$. This implies that $A \in \theta(p)$ and $B \in \theta(q)$, showing that $R = AB \in \theta(p)\theta(q)$ as claimed, concluding the proof. $\square$

A simple recurrence yields

$$\theta(p_1p_2 \dots p_k) = \theta(p_1)\theta(p_2) \dots \theta(p_k). \tag{3}$$

It is also easy to see from the definition that $\theta(a) = ba$ and $\theta(b) = \{aa, ab, bb\}$; hence, using the property above, we can obtain θ for any given word.

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3.2. *The χ-procedure for even lines.* Odd lines for $\mathfrak{J}$ and $\mathfrak{J}'$ are just strings of 10s. We want to get a description for even lines. We start with the last even line for finite trees.

3.2.1. *Word in the last even line for image of finite colored trees.* Here, we want to describe how the word in the last line for a colored tree generates the word in the penultimate line for its image by H. This procedure is recursive.

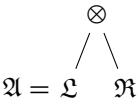
Definition 3.3. A family-block in a colored tree is the collection of all n-descendants of a site ω.

Note that two brothers and four cousins form a family block. A family block has cardinality 2^n, where n is such that the first common ancestor is their n-ancestor.

Definition 3.4. Let $\mathfrak{A}$ be a finite colored tree whose last line is a full family block. Then, the word in the last line is called the bottom-word of $\mathfrak{A}$. It is denoted by $\mathrm{bw}(\mathfrak{A})$. The word in the penultimate line is called the vice-bottom-word. It is denoted by $\mathrm{vbw}(\mathfrak{A})$.

The next lemma is a direct consequence of the definition of H, depending strongly on the chosen grammar.

LEMMA 3.5. *Let $\mathfrak{A}$ be a finite colored tree whose last line is a full family block. Set*



Then,

$$vbw(H(\mathfrak{A})) = vbw(H(\mathfrak{R}))vbw(H(\mathfrak{R}))vbw(H(\mathfrak{L}))vbw(H(\mathfrak{R})).$$

Note that the equality $\mathrm{bw}(H(\mathfrak{A})) = \mathrm{bw}(H(\mathfrak{R}))\mathrm{bw}(H(\mathfrak{R}))\mathrm{bw}(H(\mathfrak{L}))\mathrm{bw}(H(\mathfrak{R}))$ also holds but is not interesting as the bottom-word is only composed by a chain of 10s.

Now, to introduce χ, we notice that we can represent any configuration $\mathfrak{C} \in \{0,1\}^{\mathbb{F}_2^+}$ by means of the sets of addresses of the 1s in each of its lines (it is clear that this can also be done using zeros, our choice of '1' here is completely arbitrary). Let us define the set $W_n = \{a,b\}^n$ of words of length $n \geq 1$ over the alphabet $\{a,b\}$, where W_0 is the singleton $\{e\}$ that represents the root.

Given a configuration $\mathfrak{C}$, we have, on the line $l \geq 0$, a word of length 2^l on the alphabet $\{0,1\}$ whose addresses are words on the alphabet $\{a,b\}$, with length l; hence, the addresses corresponding to the line l are in W_l. Then we define $1_l(\mathfrak{C})$ as the set of addresses $\omega \in W_l$ where $\mathfrak{C}_\omega = 1$.

For example, $\mathbf{1}_2(0010) = \{ba\}$ and $\mathbf{1}_3(01000001) = \{aab, bbb\}$; in the case of words with length one (corresponding to the root of the configuration), we write $\mathbf{1}_0(0) = \emptyset$ and $\mathbf{1}_0(1) = e$.

Since the alphabet is $\{0,1\}$, we have a one-to-one correspondence between a word $P \in \{0,1\}^{2^l}$ (with length 2^l) and the set $\mathbf{1}_l(P) \subset W_l$.

In particular, this allows us to define a function χ that maps words of $\{0,1\}^{2^l}$ to words of $\{0,1\}^{2^{2l}}$: given a word $P \in \{0,1\}^{2^l}$, we take the corresponding addresses of the symbols '1' (say, $\mathbf{1}_l(P)$), take their image under θ, which gives a set of addresses in $\{a,b\}$ with

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length $2l$, and consider them as the addresses of the symbols '1' of a new word $\chi(P)$ of length 2^{2l}. More formally, we have the following definition.

Definition 3.6. χ is the function such that

$$\mathbf{1}_{2l}(\chi(P)) = \theta(\mathbf{1}_l(P)),$$

where P is a word of length 2^l over the alphabet $\{0, 1\}$.

Remark 1. In the expression above, as usual, $\theta(A) = \{\theta(a) \text{ for all } a \in A\}$.

From this definition, we get that the function χ can be described recursively (but it is valid only for the substitution H defined in the beginning of this section).

LEMMA 3.7. *We have that* $\chi(0) = 0$, $\chi(1) = 1$ *and, for any given word* P_1P_2 *with length* 2^l*, we have*

$$\chi(P_1P_2) = \chi(P_2)\chi(P_2)\chi(P_1)\chi(P_2).$$

Proof. We have

$$\mathbf{1}_0(\chi(0)) = \theta(\mathbf{1}_0(0)) = \theta(\emptyset) = \emptyset.$$

Hence, $\chi(0) = 0$. Similarly, $\mathbf{1}_0(\chi(1)) = \theta(\mathbf{1}_0(1)) = \theta(e) = e$, and so $\chi(1) = 1$.

Now

$$\mathbf{1}_{2l}(\chi(P_1P_2)) = \theta(\mathbf{1}_l(P_1P_2)).$$

However, $\mathbf{1}_l(P_1P_2) = \{a\mathbf{1}_{l-1}(P_1), b\mathbf{1}_{l-1}(P_2)\}$ and then

$$\begin{aligned}\theta(\mathbf{1}_l(P_1P_2)) &= \theta(\{a\mathbf{1}_{l-1}(P_1), b\mathbf{1}_{l-1}(P_2)\})\\ &= \{ba\theta(\mathbf{1}_{l-1}(P_1)), aa\theta(\mathbf{1}_{l-1}(P_2)), ab\theta(\mathbf{1}_{l-1}(P_2)), bb\theta(\mathbf{1}_{l-1}(P_2))\}\\ &= \{aa\mathbf{1}_{2l-2}(\chi(P_2)), ab\mathbf{1}_{2l-2}(\chi(P_2)), ba\mathbf{1}_{2l-2}(\chi(P_1)), bb\mathbf{1}_{2l-2}(\chi(P_2))\}\\ &= \mathbf{1}_{2l}(\chi(P_1P_2)).\end{aligned}$$

Hence, $\chi(P_1P_2) = \chi(P_2)\chi(P_2)\chi(P_1)\chi(P_2)$. □

Remark 2. We emphasize (and let the reader check):

(1) $\chi(10) = 0010$;

(2) any $\chi^u(10) = \underbrace{\chi \circ \ldots \circ \chi}_{u \text{ times}}(10)$ with $u \geq 2$ is a concatenation of blocks 0010 and 0000;

(3) $\chi^u(10)$ is a word of length 2^{2^u} in 0s and 1s.

PROPOSITION 3.8. *For any finite colored tree with full last line, say* $\mathfrak{A}$*, then* $vbw(H(\mathfrak{A})) = \chi(bw(\mathfrak{A}))$ *holds.*

Proof. The proof is done by induction on the number of lines of the finite colored tree $\mathfrak{A}$ (with full last line). If $\mathfrak{A} = 0, 1$ (that is only line 0), then $H(\mathfrak{A}) = \begin{array}{ccc} & \otimes & \\ 1 & & 0 \end{array}$ with $\otimes = \mathfrak{A}$.

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Moreover, $\mathrm{bw}(\mathfrak{A}) = \otimes$. Then,

$$\mathrm{vbw}(H(\mathfrak{A})) = \otimes = \chi(\otimes) = \chi(\mathrm{bw}(\mathfrak{A})).$$

Assume that the result holds for any finite colored tree with full last line with l lines.

Let $\mathfrak{A}$ be a finite colored tree with full last line and with $l+1$ lines. Set $\mathfrak{A} = \begin{array}{ccc} & \otimes & \\ \swarrow & & \searrow \\ \mathfrak{L} & & \mathfrak{R} \end{array}$ and $\mathrm{bw}(\mathfrak{A}) = w$. Then $w = \mathrm{bw}(\mathfrak{L})\mathrm{bw}(\mathfrak{R})$ holds with $\mathfrak{L}$ and $\mathfrak{R}$ two colored trees with full last line and with l-lines.

Lemma 3.5 and the induction assumption yield

$$\begin{aligned}\mathrm{vbw}(H(\mathfrak{A})) &= \mathrm{vbw}(H(\mathfrak{R}))\mathrm{vbw}(H(\mathfrak{R}))\mathrm{vbw}(H(\mathfrak{L}))\mathrm{vbw}(H(\mathfrak{R})) \\ &= \chi(\mathrm{bw}(\mathfrak{R}))\chi(\mathrm{bw}(\mathfrak{R}))\chi(\mathrm{bw}(\mathfrak{L}))\chi(\mathrm{bw}(\mathfrak{R})) \\ &= \chi(\mathrm{bw}(\mathfrak{L})\mathrm{bw}(\mathfrak{R})) = \chi(\mathrm{bw}(\mathfrak{A})).\end{aligned}$$

□

3.2.2. *Some technical lemmas on χ.* We give some extra properties of χ that will be used later.

LEMMA 3.9. *If ω is a word with length 2^l and ω' is the concatenation of 2^k words ω, then $\chi(\omega')$ is a concatenation of $\chi(\omega)$.*

Proof. The proof is done by induction on k. The result is obvious if $k = 1$. Let us assume it holds for k, and let us prove it for $k+1$.

Set ω' to be the concatenation of 2^k words ω and $\omega'' = \omega'\omega'$. Then,

$$\chi(\omega'') = \chi(\omega'\omega') = \chi(\omega')\chi(\omega')\chi(\omega')\chi(\omega').$$

Hence, the property holds for $k+1$ just by using the induction hypothesis. □

LEMMA 3.10. *For any $u \geq 1$, the word $\chi^u(10)$ has at least one 1 in the second half. The same holds for the first half if $u \geq 2$.*

Proof. The proof is done by induction.

First, $\chi(10) = 0010$. The second half of the word is 10 which contains 1.

Let us assume that the result holds for $\chi^u(10)$ and let us prove it for $\chi^{u+1}(10)$. Set $\chi^u(10) = P_1P_2$ with $|P_1| = |P_2|$.

By definition, $\chi^{u+1}(10) = \chi(\chi^u(10)) = \chi(P_1P_2) = \chi(P_2)\chi(P_2)\chi(P_1)\chi(P_2)$. The second half of $\chi^{u+1}(10)$ is the word $\chi(P_1)\chi(P_2)$ as the length of the image of a word under χ only depends on the length of the word.

By the induction hypothesis, P_2 contains some 1s. By the definition of χ, as $\chi(1) = 1$, $\chi(P_2)$ also contains some 1s. This shows that both halves contain some 1s. □

3.2.3. *Description of words on even lines in $\mathfrak{J}$*

PROPOSITION 3.11. *For every integer of the form $m = 2^u(2n+1)$ with $u \geq 0$, the word at generation m in $\mathfrak{J}$ is a concatenation of blocks $\chi^u(10)$.*

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Proof. The proof is done by induction on u.

For $u = 0$, we look at an odd line that is only composed by blocks 10 by the definition of H.

We assume that this the property holds for any line of the form $2^u(2n+1)$ and we prove it for a line of the form $2^{u+1}(2n+1)$.

At line $2^u(2n+1)$, we consider all the consecutive and juxtaposed blocks of length 2^{2^u}. They form a sequence of family blocks with common 2^u-ancestors. More precisely, the line $2^u(2n+1)$ is the juxtaposition of the bottom-words of all the finite subtrees in $\mathfrak{J}$ with root at line $2^{u+1}n$ and with size 2^u.

The tree $\mathfrak{J}$ is invariant by H, and hence the line $2^{u+1}(2n+1)$ is obtained from the line $2^u(2n+1)$. From Proposition 3.8 and the induction hypothesis, Lemma 3.9 shows that the line $2^{u+1}(2n+1)$ is the concatenation of blocks $\chi(\chi^u(10)) = \chi^{u+1}(10)$. □

We emphasize:

(1) odd lines are only composed by concatenations of 10s;
(2) lines in $4\mathbb{N}+2$ are only composed by 0010;
(3) lines in $4\mathbb{N}$ are composed by blocks 0010 and blocks 0000.

3.3. *Proportion of 1s on even lines in* $\mathfrak{J}$. Now we want to estimate how many 1s we have in even lines of the Jacaranda tree $\mathfrak{J}$.

First we start with special even lines.

PROPOSITION 3.12. *The number of 1s on the line* 2^n *is given by* $2^{2^n}(1 + f^n(1))^{-1}$, *where* $f(x) = x + x^{-1} + 1$ *and* $n \geq 0$.

Proof. First, we recall the map θ is defined such that $\theta(a) = ba$, $\theta(b) = \{aa, ab, bb\}$ and $\theta(pq) = \theta(p)\theta(q)$ (meaning the concatenation of any word of the set $\theta(p)$ with any word of the set $\theta(q)$).

Now consider the line $1 = 2^0$; since it is an odd line, the element that corresponds to 1 is on the site a. We define P_0 as the number of sites with 1 and Q_0 as the number of sites with 0 on the line 2^0; hence, $P_0 = Q_0 = 1$. This line is the source of line $2 = 2^1$ and so, to find the sites that correspond to 1, we need to find all the sites whose source is a. However, they are given exactly by $\theta(a) = ba$.

Then on the line 2^n, we have that the sites with 1 are given by $\theta^n(a)$ (and the sites with 0 are given by $\theta^n(b)$). Denoting by P_n (respectively Q_n) the cardinality of $\theta^n(a)$ (respectively $\theta^n(b)$), we have that $P_n + Q_n = 2^{2^n}$, the length of the line 2^n.

To obtain the line 2^{n+1}, whose source is 2^n, we use the fact that the 1s are on the positions $\theta(\theta^n(a)) = \theta^n(\theta(a)) = \theta^n(ba) = \theta^n(b)\theta^n(a)$; hence the sites of the 1s are the concatenation of any site of $\theta^n(b)$ with any from $\theta^n(a)$. This implies that the number of 1s is given recursively by $P_{n+1} = P_n Q_n$. To obtain Q_{n+1}, we just recall that this number is the number of elements of the line 2^{n+1} minus the number of elements that are equal to 1: $Q_{n+1} = (P_n + Q_n)^2 - P_{n+1} = (P_n + Q_n)^2 - P_n Q_n$.

For any $n \geq 1$, we have then the recursion

$$P_{n+1} = P_n Q_n \quad \text{and} \quad Q_{n+1} = (P_n + Q_n)^2 - P_n Q_n.$$

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Writing $R_n = Q_n/P_n$, we obtain that

$$R_{n+1} = \frac{Q_{n+1}}{P_{n+1}} = \frac{(Q_n + P_n)^2 - P_n Q_n}{P_n Q_n}$$
$$= \frac{Q_n^2 + P_n^2 + P_n Q_n}{P_n Q_n} = \frac{Q_n}{P_n} + \frac{P_n}{Q_n} + 1 = R_n + \frac{1}{R_n} + 1 = f(R_n),$$

where $f(x) = x + 1/x + 1$ and $R_1 = 3$.

Hence,

$$2^{2^n} = P_n + Q_n = P_n + R_n P_n \Rightarrow P_n = \frac{2^{2^n}}{1 + f^{n-1}(3)} = \frac{2^{2^n}}{1 + f^n(1)}$$

as claimed. □

COROLLARY 3.13. *Using the same notation of the previous proposition, we have the following.*

(1) *The proportion of 1s in the line 2^n is given by $P_n/(2^{2^n}) = (1 + f^n(1))^{-1}$, which is a strictly decreasing function of n, and this proportion goes to zero as n goes to infinity.*
(2) *The proportion of 1s in the lines $m = 2^u(2n + 1)$ is given by $(1 + f^u(1))^{-1}$.*

Proof. (1) Notice that $f(x) = x + 1/x + 1 > x + 1$; hence, we have that $f^k(1) \geq 1 + k$, showing that $f^k(1) \to \infty$ as k goes to infinity, and so

$$\frac{P_n}{2^{2^n}} = \frac{1}{1 + f^n(1)} \to 0 \quad \text{as } n \to \infty.$$

It is also clear that $f^n(1) = f(f^{n-1}(1)) = f^{n-1}(1) + 1/(f^{n-1}(1)) + 1 > f^{n-1}(1)$ showing that $f^n(1)$ increases with n, and so the proportion of 1s decreases as n increases.

(2) As a consequence of Proposition 3.11, we have that the lines $m = 2^u(2n + 1)$ are a concatenation of blocks $\chi^u(10)$; since in each one of those blocks the proportion of 1s is given by $(1 + f^u(1))^{-1}$, the proportion of 1s at the line m is also given by $(1 + f^u(1))^{-1}$. □

A very useful consequence of the results above is the following corollary.

COROLLARY 3.14. *A concatenation of blocks $\chi^u(10)$ is the same as a concatenation of blocks $\chi^v(10)$ if and only if $u = v$.*

Proof. One implication is automatic, let us concentrate on the other. Suppose $v > u$ (the case $v < u$ being completely similar); now take a concatenation of blocks $\chi^u(10)$. The proportion of 1s in this word is the same as in $\chi^u(10)$, that is, larger than the proportion of 1s in $\chi^v(10)$. However, in the concatenation of $\chi^v(10)$, the proportion of 1s is the same as in $\chi^v(10)$, showing that both blocks are distinct when $v \neq u$. □

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4. *Proof that X is minimal*

4.1. *The fixed point is minimal.* The grammar BBAB yields $T_aT_aH = T_bT_aH = T_bT_bH = HT_b$ and $T_aT_bH = HT_a$. Thus, the source is given by

$$\begin{cases} s(aa) = s(ab) = s(bb) = b, \\ s(ba) = a, \end{cases}$$

and more generally, $s(p_1p_2 \dots p_{2n}) = s(p_1p_2) \dots s(p_{2n-1}p_{2n})$.

PROPOSITION 4.1. *There exists N such that for any position $\omega = \omega_1 \dots \omega_n$, for any word $\omega_{n+1} \dots \omega_{n+N}$, there exists $k \le N$ such that $n + k$ is even and $\mathfrak{J}_{\omega_1 \dots \omega_{n+k}} = 0$.*

Proof. The result is obvious if n is even and $\mathfrak{J}_{\omega_1 \dots \omega_n} = 0$. For the rest of the proof, we assume that $\mathfrak{J}_{\omega_1 \dots \omega_n} = 1$ holds.

Every odd line is followed by an even one. Hence, we only need to prove the proposition for n even, as the result for odd lines is an immediate consequence of the result for even n.

For the rest of proof, we consider that n is even and that the digit at position $\omega_1 \dots \omega_n$ is 1.

(1) The case $n \in 8\mathbb{N}$. Let us set $n = 8n_1$. Because H is marked, the source of the digit at position $\omega_1 \dots \omega_n$ is 1 and is an even line. Its source is at line $2n_1$ and is again 1, and the source of this last digit is at line n_1 and is 1. Hence, we have line n_1, 1, and the following configurations at lines $2n_1$, $4n_1$, and $8n_1$.

In Figure 2, site $\boxed{0}$ at level $4n_1 + 1$ is mapped by H on sites $\boxed{0}$ at level $8n_1 + 2$. The subtree in frame at level $4n_1 + 1$ is mapped on the subtree at level $8n_1 + 2$ in frame (+ children). At level $8n_1 + 2$, three grand-children are 0. At level $8n_1 + 4$, all the grand-children from the site with digit 1 at level $8n_1 + 2$ are 0.

Consequently, for any α with $|\alpha| \ge 4$, there exists a digit in $\mathfrak{J}$ at position $\omega_1 \dots \omega_n\alpha_1 \dots \alpha_k$ with $k \le 4$ whose value is 0 and such that $n + k$ is even.

(2) Other cases. Let us set $n = 8n_1 + j$ with $j = 2, 4, 6$. Let α be any word with length at least 10. Then, $\omega_1 \dots \omega_n\alpha_1 \dots \alpha_{8-j}$ has length in $8\mathbb{N}$. Either there is some 0 at even position from position $\omega_1 \dots \omega_n$ to position $\omega_1 \dots \omega_n\alpha_1 \dots \alpha_{8-j}$, or we are back to the previous case. □

In the following for $\omega \in \mathbb{F}_2^+ = \omega_0 \dots \omega_n$, $\widetilde{\omega}$ stands for $\omega_n \dots \omega_0$.

PROPOSITION 4.2. *There exists N_0 such that for any $\omega = \omega_1 \dots \omega_n \in \mathbb{F}_2^+$, there exists $k \le N_0$ and $\alpha \in \mathbb{F}_2^+$ with $|\alpha| \le k$ such that $T_{\widetilde{\omega}}(\mathfrak{J})$ belongs to $T_\alpha(B(\mathfrak{J}, 1))$.*

Proof. From Proposition 4.1, a path from position $\omega' := \omega_1 \dots \omega_{n-N}$ to $\omega_1 \dots \omega_n$ contains some 0 at an even level. Having a digit equal to 0 at an even level implies being in $B(\mathfrak{J}, 1)$. Thus, the result holds with N_0 equal to N from Proposition 4.1. □

PROPOSITION 4.3. *For every m, there exists N_m such that for any $\omega = \omega_1 \dots \omega_n \in \mathbb{F}_2^+$, there exists $k \le N_m$ and $\alpha \in \mathbb{F}_2^+$ with $|\alpha| \le k$ such that $T_{\widetilde{\omega}}(\mathfrak{J})$ belongs to $T_\alpha(B(\mathfrak{J}, 2^{-m}))$.*

Proof. The proof is done by induction on m. For $m = 0$, this follows from Proposition 4.2.

Let us assume that the proposition holds for m and let us prove it for $m + 1$.

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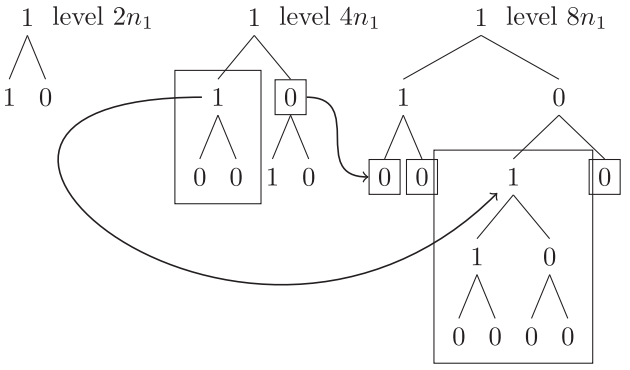


FIGURE 2. Different parts of the tree $\mathfrak{J}$.

Let $\omega = \omega_1 \dots \omega_n \in \mathbb{F}_2^+$. For simplicity, we assume that $|\omega|$ is bigger than $2N_m + N_0$. Again, we assume that n is even. From Proposition 4.2, we can find $k \leq N_0$ such that $n - k$ is even and

$$\mathfrak{J}_{\omega_1 \dots \omega_{n-k}} = 0.$$

Let us consider the source for this digit. It lives at level $n' = (n-k)/2$. Note that n' is greater than or equal to N_m.

Let $\beta = \beta_1, \dots \beta_{n'} \in \mathbb{F}_2^+$ be its position. Then, by the induction hypothesis, there exists $j \leq N_m$ such that $T_{\beta_{n'-j} \dots \beta_1}(\mathfrak{J})$ belongs to $B(\mathfrak{J}, 2^{-m})$.

By definition, $\beta_1 \dots \beta_{n'} = S(\omega_1\omega_2) \dots S(\omega_{n-k-1}\omega_{n-k})$, and then

$$\beta_1 \dots \beta_{n'-j} = S(\omega_1\omega_2) \dots S(\omega_{n-k-2j-1}\omega_{n-k-2j}) = s(\omega_2\omega_1) \dots s(\omega_{n-k-2j}\omega_{n-k-2j-1}).$$

Then we use the renormalization equality and apply $T_{\beta_{n'-j} \dots \beta_1}$. This yields

$$\begin{aligned} H \circ T_{\beta_{n'-j} \dots \beta_1}(\mathfrak{J}) &= H \circ T_{s(\omega_{n-k-2j}\omega_{n-k-2j-1}) \dots s(\omega_2\omega_1)}(\mathfrak{J}) \\ &= T_{\omega_{n-k-2j}\omega_{n-k-2j-1} \dots \omega_2\omega_1} H(\mathfrak{J}) \\ &= T_{\omega_{n-k-2j} \dots \omega_1}(\mathfrak{J}), \end{aligned}$$

where we use $H(\mathfrak{J}) = \mathfrak{J}$.

However, we remind that H acts as a contraction on trees. Hence, $T_{\beta_{n'-j} \dots \beta_1}(\mathfrak{J}) \in B(\mathfrak{J}, 2^{-m})$ yields

$$B(\mathfrak{J}, 2^{-m-1}) = B\big(H(\mathfrak{J}), \tfrac{1}{2}2^{-m}\big) \ni T_{\omega_{n-k-2j} \dots \omega_1}(\mathfrak{J}).$$

Note that $k + 2j$ is lower or equal to $2N_m + N_0$. If we set $\alpha = \omega_n \dots \omega_{n-k-2j+1}$, then

$$T_{\widetilde{\omega}}(\mathfrak{J}) \in T_\alpha B(\mathfrak{J}, 2^{-(m+1)}).$$

This proves that the property holds for $m + 1$, and thus it holds for every m. □

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Following notation from [10, Ch. IV(1.2)] for group actions, and [11, Proposition 5.21 and Definition 5.22], Proposition 4.3 shows that $\mathfrak{J}$ is almost periodic (this is also referred to as uniformly recurrent in [11]). Therefore, the closure of its orbit is a minimal invariant set.

4.2. *The fixed point $\mathfrak{J}$ is not periodic.* We are now in a position to prove the main result of this subsection.

PROPOSITION 4.4. *The fixed point $\mathfrak{J}$ is not periodic.*

Proof. Because odd lines are alternations of 1 and 0, we cannot have $T_\omega\mathfrak{J} = \mathfrak{J}$ with $|\omega|$ odd, because it would send even lines to odd lines, and in each odd line, the ratio between the 0s and the 1s is equal to 1.

We do the proof by contradiction. Let us assume that there are finitely many trees $\mathfrak{J}(1), \ldots, \mathfrak{J}(N)$ with $\mathfrak{J}(1) = \mathfrak{J}$ such that the orbit of $\mathfrak{J}$ is $\mathcal{O} = \{\mathfrak{J}(1), \mathfrak{J}(2), \ldots, \mathfrak{J}(N)\}$. This means that we can construct the graph whose vertices are the $\mathfrak{J}(i)$ terms and there is an arrow (with label a or b) going from $\mathfrak{J}(i)$ to $\mathfrak{J}(j)$ if $\mathfrak{J}(j) = T_c(\mathfrak{J}(i))$ with $c = a, b$. By the definition of being the orbit, each vertex has two outgoing arrows and at least one incoming arrow.

We denote by M the matrix associated to this graph. If we have at least one of the equalities $\mathfrak{J}(j) = T_a\mathfrak{J}(i)$ or $\mathfrak{J}(j) = T_b\mathfrak{J}(i)$, then we set $M_{i,j} = 1$; hence, $M_{ij} = 1$ means that we can go from $\mathfrak{J}(i)$ to $\mathfrak{J}(j)$ on at least one path (T_a or T_b, perhaps both).

By definition, each row and each column of M has at least one 1. By construction, $(M^n)_{i,j} > 0$ means that there exists $\omega \in \mathbb{F}_2^+$ with $|\omega| = n$ and $T_\omega.\mathfrak{J}(i) = \mathfrak{J}(j)$.

We let the reader check that the matrix M is exactly of the form of the matrices considered in [14, Ex. 1.9.4]. The spectral decomposition yields a partition of $\mathcal{O} = \{\mathfrak{J}(1), \ldots, \mathfrak{J}(N)\}$ in finitely many disjoint sets $\mathcal{O}_1, \ldots, \mathcal{O}_{N'}$, with

$$T_a\mathcal{O}_i \cup T_b\mathcal{O}_i = \mathcal{O}_{i+1},$$

with the convention $N' + 1 = 1$. Moreover, the dynamics for return in each component $\mathcal{O}_i$ is mixing in the following sense: there exists N'' such that for every $n \geq N''$, for every i, for every k_i and j_i such that $\mathfrak{J}(k_i)$ and $\mathfrak{J}(j_i)$ belong to $\mathcal{O}_i$, there exists $\omega \in \mathbb{F}_2^+$ with $|\omega| = nN'$ and $T_\omega\mathfrak{J}(k_i) = \mathfrak{J}(j_i)$.

In other words, the transition matrices for the return in each $\mathcal{O}_i$ are irreducible. We can apply the Perron–Frobenius theorem which yields for each $\mathcal{O}_i$ an eigenvector with positive entries (if we only consider the indexes k_i such that $\mathfrak{J}(k_i) \in \mathcal{O}_i$) associated to the eigenvalue $\lambda^{N'}$, where λ is the spectral radius (and thus the unique single dominating eigenvalue) for M. We denote these eigenvectors by $\vec{\gamma}_i$.

The $\vec{\gamma}_i$ terms yield an interesting property for the lines of $\mathfrak{J}$. We remind that $\mathfrak{J} = \mathfrak{J}(1)$ and thus belongs to one of the $\mathcal{O}_i$ terms, say $\mathcal{O}_{i_0}$. The dynamics of the transition matrix M represent the elements of the orbit of $\mathfrak{J}$ that can be reached by applying the $\mathbb{F}_2^+$-action. Hence, each line of value KN' in $\mathfrak{J}$ is only composed by roots of the elements of $\mathcal{O}_{i_0}$ (and they do not appear elsewhere, since the sets $\mathcal{O}_i$ are disjoint). More generally, each line $KN' + i$ in $\mathfrak{J}$ is only composed by roots of elements of $\mathcal{O}_{(i_0+i \bmod N')}$.

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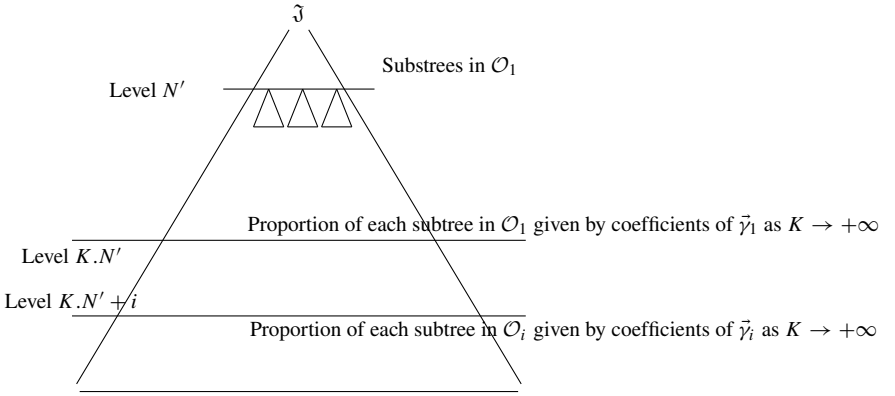


FIGURE 3. Proportion of 1s is bounded away from 0 as the generation increases.

Furthermore, the entries of the vector $(1, 0, \ldots, 0).M^{KN'+i}$ give exactly how many $T_{\omega_1 \ldots \omega_{KN'+i}}\mathfrak{J}$ belong to each $\mathfrak{J}(k_i) \in \mathcal{O}_{(i_0+i \bmod N')}$. In addition, as a consequence of the Perron–Frobenius theorem, their relative proportion is asymptotically given by the ratio between the entries of the eigenvectors $\vec{\gamma}_{i_0+i}$ as K goes to $+\infty$. We refer to Figure 3 for a graphic representation of this fact. Hence, the ratio between the number of 1s and the number of 0s must be uniformly (in i and K) bounded away from 0 and away from 1. This is in contradiction with Corollary 3.13. □

COROLLARY 4.5. *There is no $\omega \in \mathbb{F}_2^+$ such that $T_\omega(\mathfrak{J}) = \mathfrak{J}'$.*

Proof. The proof is done by contradiction. Let us assume that $T_\omega(\mathfrak{J}) = \mathfrak{J}'$. Note that $\omega \neq e$, as $\mathfrak{J}$ and $\mathfrak{J}'$ have different root. Then, $T_\omega(\mathfrak{J}') = \mathfrak{J}'$. Set $\omega = \omega_1 \ldots \omega_n$.

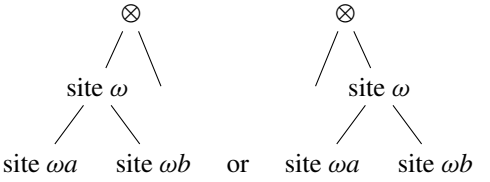
Pick m and consider a concatenation k times ω such that its length is bigger than N_m from Proposition 4.3. As $T_\omega(\mathfrak{J}') = \mathfrak{J}'$, we get that for some $1 \leq j \leq n$, $T_{\omega_1} \ldots \omega_j(\mathfrak{J}')$ belongs to $B(\mathfrak{J}, 2^{-m})$. Note that $j = j(m)$ but there are finitely many possibilities for j. Letting $m \to +\infty$ yields that for some j, $T_{\omega_1} \ldots \omega_j(\mathfrak{J}')$ belongs to infinitely many $B(\mathfrak{J}, 2^{-m})$, and hence $T_{\omega_1} \ldots \omega_j(\mathfrak{J}') = \mathfrak{J}$. Then $T_{\omega\omega_1 \ldots \omega_j}(\mathfrak{J}) = \mathfrak{J}$. This is a contradiction with Proposition 4.4. □

5. *Self-similarities, types, unsubstreetutions, and rigidity*

5.1. *Self-similarities in $\mathfrak{J}$*

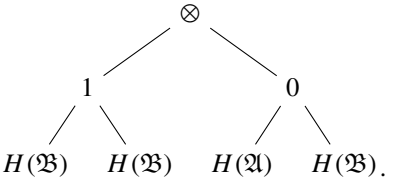
LEMMA 5.1. *Let ωa and ωb be two brothers with even length. Let ω' and ω'' respectively denote the sources of ωa and ωb. Then either $\omega' = \omega''$ or they are brothers.*

Proof. Let $\otimes$ be the color for the grandfather of ωa. We have the configuration



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In any case, $\otimes$ lays at an even level and we can consider its source. At the source, one sees  which yields at the level of the grandfather



If ω is the a-follower of $\otimes$, then $\omega' = \omega''$. If ω is the b-follower, then ω' and ω'' are brothers. □

We remind that odd lines of $\mathfrak{J}$ are composed by strings of 10. We remind equalities $\chi(1) = 1$, $\chi(0) = 0$, $\chi(10) = 0010$, and

$$\begin{aligned}\chi^2(10) &= \chi(0010)\\ &= \chi(10)\chi(10)\chi(00)\chi(10)\\ &= 0010\ 0010\ 0000\ 0010.\end{aligned}$$

Clearly, $\chi^n(0000)$ is a chain of 0s.

Hence, by Proposition 3.11, even lines in $\mathfrak{J}$ are composed by blocks 0000 or 0010.

More precisely, we have the following lemma.

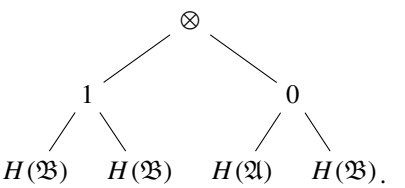
LEMMA 5.2. *Any line in* $\mathfrak{J}$ *at level in* $4\mathbb{N}+2$ *is only composed with blocks* 0010*. Any group of* 16 *digits in* $\mathfrak{J}$ *at level in* $4\mathbb{N}$ *with the same* 4*-ancestor is either composed only by* 0*s or is equal to*

$$0010\ 0010\ 0000\ 0010.$$

LEMMA 5.3. *Let* ω *be a site such that all its grandchildren in* $\mathfrak{J}$ *have root equal to* 0*. Then,*

$$T_{aa\widetilde{\omega}}(\mathfrak{J}) = T_{ba\widetilde{\omega}}(\mathfrak{J}) = T_{ab\widetilde{\omega}}(\mathfrak{J}) = T_{bb\widetilde{\omega}}(\mathfrak{J}).$$

Proof. Because all sites ωaa ωab, ωba, and ωbb have color 0, it means that they lay at an even level. Hence, $|\omega|$ is even and we can consider its source. At the source, we see , which yields at site ω



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We also know that the configurations $\mathfrak{A}$ and $\mathfrak{B}$ are both equal to 0 on their roots, which corresponds to the sources of the sites ωaa, ωab, ωba, and ωbb.

From the hypothesis, we know that $(T_{aa\tilde{\omega}}(\mathfrak{J}))_e = (T_{ba\tilde{\omega}}(\mathfrak{J}))_e = (T_{ab\tilde{\omega}}(\mathfrak{J}))_e = (T_{bb\tilde{\omega}}(\mathfrak{J}))_e$.

Now, let us set $|\omega ba| = 2^u(2n+1)$. We can consider the unsubstreetuted trees for $H(\mathfrak{B})$ and $H(\mathfrak{A})$ (respectively at site ωba and ωbb) u-times. These two resulting trees are at level $2n+1$. By Lemma 5.1, they are either equal or 'brother' subtrees. Furthermore, their roots are equal to 0. However, on an odd level, one only sees strings of 10. This yields that the subtrees are equal, and hence $H(\mathfrak{A}) = H(\mathfrak{B})$. □

5.2. *Types and unsubstreetution.* We remind $\mathcal{O}(\mathfrak{J}) = \{T_\omega(\mathfrak{J}),\ \omega \in \mathbb{F}_2^+\}$ and $X = \overline{\mathcal{O}(\mathfrak{J})}$.

As $H(\mathfrak{J}) = \mathfrak{J}$, equation (1) shows that the set X is H-invariant. A natural question is to inquire about in which $H^u(X)$ a given element belongs. If $\mathfrak{A} = H^u(\mathfrak{B})$, then we say that $\mathfrak{B}$ is the u-times unsubstreetuted tree of $\mathfrak{A}$. Unsubstreetution has been defined for elements in $\mathcal{O}(\mathfrak{J})$ in §2.1, via the source function (see Lemma 2.1). The goal of this section is to extend this notion to general elements in X and not only those in the orbit of $\mathfrak{J}$.

At this stage, it is not yet proved that $\mathfrak{J}'$ belongs to X. The proof is done in Corollary 5.17 in §5.3. Nevertheless, we point out that it is not an error to consider the case $\mathfrak{A} \in X$ and $\mathfrak{A} \neq \mathfrak{J}'$. Furthermore, Corollary 5.17 comes from Lemma 5.16. The lemma could be stated earlier but only for $\mathfrak{J}$. As we need it for any element of X, it appeared easier to postpone the proof of $\mathfrak{J}' \in X$.

5.2.1. *Type 2^u for elements in X*

Definition 5.4. We say that $\mathfrak{A} \in X$ is of type 2^u, with $u \in \mathbb{N}$, if there is a sequence ω_k with $|\omega_k| = 2^u(2n_k+1)$ such that

$$\mathfrak{A} = \lim_{k\to+\infty} T_{\omega_k}(\mathfrak{J}).$$

An element of type 1 (that is, 2^0) is also said to be of odd type. An element which is not of odd type is said to be of even type.

The set of odd (respectively even) trees will be denoted by X_{od} (respectively X_{ev}).

Note that any $T_\omega(\mathfrak{J})$ with $|\omega| = 2^u(2n+1)$ is of 2^u-type. It is *a priori* not clear that the type is well defined for any element in X. The purpose of this part is to prove that it holds.

LEMMA 5.5. *Let $\mathfrak{A}$ be in $X := \overline{\{T_\omega(\mathfrak{J}),\ \omega \in \mathbb{F}_2^+\}}$. Let (ω_k) be a sequence in $\mathbb{F}_2^+$ such that $|\omega_k| \to_{k\to\inf} +\infty$ and $\mathfrak{A} = \lim_{k\to\infty} T_{\omega_k}(\mathfrak{J})$. Then only the two following cases may happen:*

(1) *$(|\omega_k|)$ eventually is even;*
(2) *$(|\omega_k|)$ eventually is odd.*

Proof. We remind that any odd line for $\mathfrak{J}$ is composed by a string of 10, and every even line of $\mathfrak{J}$ is composed by blocks 0000 or 0010. This yields that if $|\omega|$ is even and $|\omega'|$ is odd,

$$d(T_\omega(\mathfrak{J}), T_{\omega'}(\mathfrak{J})) \geq 2^{-3}.$$

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A converging sequence is also a Cauchy sequence, and thus all lengths must have the same parity for sufficiently large index. □

A direct consequence of Lemma 5.5 is that any $\mathfrak{A} \in X$ is either of odd type or even type.

LEMMA 5.6. *Let $\mathfrak{A}$ be in X of even type. Let (ω_k) be a sequence such that $\lim_{k\to+\infty} T_{\omega_k}(\mathfrak{J}) = \mathfrak{A}$. Set $|\omega_k| = 2^{u_k}(2n_k + 1)$. If $u_k \to +\infty$, then $\mathfrak{A} = \mathfrak{J}$ or $\mathfrak{A} = \mathfrak{J}'$.*

Proof. Let us set $\mathfrak{A}^k := T_{\omega_k}(\mathfrak{J})$. The u_k-source for ω_k is a site ω'_k that is at level $2n_k + 1$. The digit is either 0 or 1 (for every sufficiently large k, only one option is possible because H is marked). Hence, $\mathfrak{A}^k$ starts with $H^{u_k}(0)$ or $H^{u_k}(1)$. To simplify the proof, we assume that the source is 0.

The assumption $u_k \to +\infty$ yields that for every integer m, and for every sufficiently large k, $\mathfrak{A}^k$ starts as $H^m(0)$. This implies that $\mathfrak{A}$ starts as $H^m(0)$, and this holds for every m. Hence, $\mathfrak{A} = \mathfrak{J}$.

The same argument holds if the source is 1. In that case, $\mathfrak{A} = \mathfrak{J}'$. □

PROPOSITION 5.7. *Let $\mathfrak{A} \in X$ be of even type. If $\mathfrak{A} \neq \mathfrak{J}, \mathfrak{J}'$, then it is of 2^u-type for some integer $u \geq 1$ and u is unique.*

Proof. Let u and k be positive integers. Let ω and ω' in $\mathbb{F}_2^+$ such that $|\omega| = 2^u(2n + 1)$ and $|\omega'| = 2^{u+k}(2m + 1)$. Set $\mathfrak{B} := T_\omega(\mathfrak{J})$ and $\mathfrak{B}' := T_{\omega'}(\mathfrak{J})$.

Line 2^u in $\mathfrak{B}$ is at level $2^{u+1}(n + 1) =: 2^{u+a}(2l + 1)$ in $\mathfrak{J}$, with $a \geq 1$. By Proposition 3.11, this line is a concatenation of blocks $\chi^{u+a}(10)$. Now, line 2^u in $\mathcal{B}'$ is at level $2^u(2^k(2m + 1) + 1)$. Hence, it is a concatenation of blocks $\chi^u(10)$. Therefore, Corollary 3.13 yields

$$d(\mathfrak{B}, \mathfrak{B}') \geq 2^{-u}. \tag{4}$$

Now, consider (ω_k) such that $\mathfrak{A} = \lim_{k\to\infty} T_{\omega_k}(\mathfrak{J})$ with $|\omega_k| = 2^{u_k}(2n_k + 1)$. Because we have assumed $\mathfrak{A} \neq \mathfrak{J}, \mathfrak{J}'$, Lemma 5.6 shows that (u_k) is bounded. Hence, it only takes finitely many values. We may consider the smallest one which is taken infinitely many times. We denote it by u.

The sequence $(T_{\omega_k}(\mathfrak{J}))$ is converging and thus it is a Cauchy sequence. By equation (4), we must have $u_k \leq u$ for sufficiently big k. This shows that the sequence (u_k) is eventually constant.

Furthermore, Corollary 3.13 yields that u is unique because trees in $\mathfrak{J}$ with root at lines $2^u(2n + 1)$ and trees with root at lines $2^{u+t}(2m + 1)$ are uniformly (with respect to n, m, and position of the sites) far away from each other. □

Notation 2. In the following, we shall say that $\mathfrak{J}$ and $\mathfrak{J}'$ are of 2^∞-type.

Remark 3. Any element of X is of a fixed type 2^u with $0 \leq u \leq +\infty$. This also holds for element in the orbit of $\mathfrak{J}$. However, for elements in $\mathcal{O}(\mathfrak{J})$, we also shall specify the type by the line: $T_\omega(\mathfrak{J})$ is of the type 2^u if $|\omega| = 2^u(2n + 1)$.

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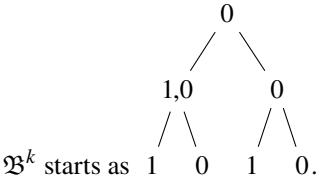
Now, we give two results about sequences. The next lemma is a kind of converse result of Lemma 5.6.

LEMMA 5.8. *Let* $\mathfrak{A}^k := T_{\omega_k}(\mathfrak{J})$ *be such that* $|\omega_k| = 2^{u_k}(2n_k+1)$ *and* $\mathfrak{J} = \lim_{k\to+\infty} \mathfrak{A}^k$. *Then* $\lim_{k\to+\infty} u_k = +\infty$.

The same holds if $\mathfrak{J}' = \lim_{k\to+\infty} \mathfrak{A}^k$.

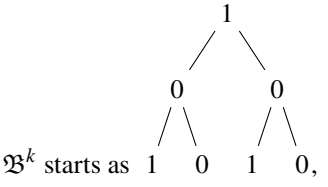
Proof. Assume, by contradiction, that this is not the case. Up to a sub-sequence, we can assume that $u_k = u \in \mathbb{N}$ for every k. We can also assume that for every k, $\mathfrak{A}^k_e = 0$ because $\lim_{k\to+\infty} \mathfrak{A}^k = \mathfrak{J}$.

Set $\mathfrak{A}^k =: H^u(\mathfrak{B}^k)$. Each $\mathfrak{B}^k$ is an odd-type tree. They all have root equal to 0 because H is marked and all $\mathfrak{A}^k$ have root equal to 0. Hence, taking the source, we see a configuration



Because H is marked, $H^u(\mathfrak{B}^k)$ differs with $\mathfrak{J}$, either at line 2^u+1 or at line $2^{u+1}+1$. This means that $\lim_{k\to+\infty} \mathfrak{A}^k \neq \mathfrak{J}$, which is a contradiction.

If now we assume $\mathfrak{J}' = \lim_{k\to+\infty} \mathfrak{A}^k$, we re-employ the same notation, and mimic the proof of first point, in that case, we have



because a 1 at an odd level must have its two followers equal to 0. Hence, $H^u(\mathfrak{B}^k)$ differs with $\mathfrak{J}'$ at line 2^u+1. □

PROPOSITION 5.9. *Let* $\mathfrak{A}$ *be in* X *and* $\mathfrak{A}^k$ *of type* 2^{u_k} *in* $\mathcal{O}(\mathfrak{J})$ *such that* $\mathfrak{A} = \lim_{k\to+\infty} \mathfrak{A}^k$. *Then either* $u_k \to +\infty$ *or* (u_k) *is eventually stationary.*

Proof. This is an immediate consequence of equation (4) and the fact that a converging sequence is also a Cauchy sequence. □

In view to be able to use Proposition 5.9, we explain how to detect the type 2^u for any even tree. The case $u = +\infty$ is extremely simple because we must see either $\mathfrak{J}$ or $\mathfrak{J}'$. Hence, only the finite case is relevant.

If k is an integer, we denote by $v_2(k)$ its dyadic valuation. This means

$$v_2(k) = n \iff k = 2^n(2m+1),\ m \in \mathbb{N}.$$

We let the reader check the following result.

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LEMMA 5.10. *Let k, k' be two positive integers.*

(1) *If $k' \geq k$, then for any m, $v_2(2^k(2m+1)+2^{k'+1}) = k$.*
(2) *If $k' = k-1$, then for any m, $v_2(2^k(2m+1)+2^{k'+1}) \geq k+1$.*
(3) *If $k' \leq k-2$, then for any m, $v_2(2^k(2m+1)+2^{k'+1}) = k'+1 < k$.*

LEMMA 5.11. *Let $\mathfrak{A} \in X$ of 2^u-type with $u < +\infty$. Then, u is the unique integer k such that any line $2^{k+1}p$, $p \geq 1$ in $\mathfrak{A}$ is composed by blocks $\chi^k(10)$.*

Proof. Let $\mathfrak{A}^k$ be a sequence of trees in $\mathfrak{J}$ which converges to $\mathfrak{A}$. We may assume that all the roots appear at levels $2^u(2n_k+1)$. Hence, any line numbered $2^u 2n$ in $\mathfrak{A}^k$ corresponds to a piece of line $2^u(2(n_k+n)+1)$ in $\mathfrak{J}$. By Proposition 3.11, such a line (in $\mathfrak{J}$) is a concatenation of $\chi^u(10)$. This cannot be detected at line 0 in $\mathfrak{A}^k$ but it is detectable from line 2^{u+1} where we already have a full block $\chi^u(10)$.

This yields that for any line in $\mathfrak{A}^k$ multiple of 2^{u+1}, the proportion of 1s with respect to the 0s is fixed and is equal to the proportion of 1s in $\chi^u(10)$. This passes to the limit, and thus this also holds in the tree $\mathfrak{A}$.

This shows that the set of integers l such that any line $2^{l+1}n$ ($n \in \mathbb{N}^*$) is a concatenation of blocks $\chi^l(10)$ is non-empty and contains u. We now prove that u is the unique integer with this property.

- Indeed, let us pick $v' < u$. Choose any v such that $v'+1+v > u$. Any line $2^{v'}2n$ in $\mathfrak{A}^k$ corresponds to a piece of line $2^u(2n_k+1)+2^{v'}2n = 2^{v'+1}(n+2^{u-v'-1}(2n_k+1))$ in $\mathfrak{J}$. Now, choose a positive n of the form $n = 2^v(2m+1) - 2^{u-v'-1}(2n_k+1)$ (with m sufficiently big). Then the line $2^{v'}2n$ in $\mathfrak{A}^k$ corresponds to a piece of line $2^{v'+1+v}(2m+1)$ in $\mathfrak{J}$. It is a concatenation of blocks $\chi^{v'+1+v}(10)$, but it may be truncated. As m can increase as wanted, we can fix it such that the line $2^{v'}2n$ in $\mathfrak{A}^k$ is a non-truncated concatenation of blocks $\chi^{v'+1+v}(10)$. Now, we remind that Corollary 3.14 states that a concatenation of blocks $\chi^{v'+1+v}(10)$ cannot be equal to a concatenation of blocks $\chi^u(10)$.

  Because v can be chosen arbitrarily, this shows that the proportion of 1s in lines $2^{v'}2n$ does not converge because it has several different accumulation points.
- Now for $v' > u$ and for any n, a line $2^{v'}2n$ in $\mathfrak{A}$ is also a line $2^u 2.2^{v'-u}n$, and hence is a concatenation of blocks $\chi^u(10)$.

This finishes to prove that u is unique integer such that all lines $2^l 2n$ in $\mathfrak{A}$ are concatenations of blocks $\chi^l(10)$. □

5.2.2. *Unsubstreetution*

PROPOSITION 5.12. *Let $u \geq 0$ be an integer. Let $\mathfrak{A} \in X$ be of type 2^u. Then there exists a unique $\mathfrak{B} \in X_{\text{od}}$ such that $\mathfrak{A} = H^u(\mathfrak{B})$. The tree $\mathfrak{B}$ is called the u-times unsubstreetuted tree of $\mathfrak{A}$, and the map $\mathfrak{A} \mapsto \mathfrak{B}$ is called the u-unsubstreetution.*

Proof. The result is trivial if $u = 0$ (that is, odd-type tree). We only do the proof for $u \geq 1$ (that is, even-type tree).

We consider a sequence $(\mathfrak{A}^k)$ in $\mathcal{O}(\mathfrak{J})$ converging to $\mathfrak{A}$. By Proposition 5.9, we can assume that all the $\mathfrak{A}^k$ are of the form

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$$\mathfrak{A}^k = H^u\left(\begin{array}{ccc} & \otimes & \\ \mathfrak{C}^k & & \mathfrak{D}^k \end{array}\right),$$

with $\otimes = 0, 1$. The tree $\begin{array}{ccc} & \otimes & \\ \mathfrak{C}^k & & \mathfrak{D}^k \end{array}$ is of odd-type. With this notation, we prove that $(\mathfrak{C}^k)$ and $(\mathfrak{D}^k)$ do converge in X.

The tree $H^u\left(\begin{array}{ccc} & \otimes & \\ \mathfrak{C}^k & & \mathfrak{D}^k \end{array}\right)$ is of the form $H^u(0)$, and then at each slot at line $2^u + 1$, we connect either the tree $H^u(\mathfrak{C}^k)$ or the tree $H^u(\mathfrak{D}^k)$. If we say that $\mathfrak{A}^k$ coincides with $\mathfrak{A}$ for the first n_k-lines, the convergence $\mathfrak{A}^k \to \mathfrak{A}$ yields $n_k \to +\infty$ as $k \to +\infty$. Hence, $H^u(\mathfrak{C}^k)$ and $H^u(\mathfrak{D}^k)$ coincide with the right $T_\omega(\mathfrak{A})$ with $|\omega| = 2^u$ for at least $n_k - 2^u$ lines.

Let us fix N. If we assume that for any $k \geq k_0$, $n_k > N + 2^u$, then for any k and k' greater than k_0, $\mathfrak{C}^k$ and $\mathfrak{C}^{k'}$ on the one hand, and $\mathfrak{D}^k$ and $\mathfrak{D}^{k'}$ on the other hand do coincide for at least $\lfloor N2^{-u} - 1 \rfloor$ digits. In other words, the sequences $(\mathfrak{C}^k)$ and $(\mathfrak{D}^k)$ are Cauchy sequences, and hence converge. □

We emphasize that the proof of Proposition 5.12 also yields the following lemma.

LEMMA 5.13. *If $\mathfrak{A}^k = H^u(\mathfrak{B}^k)$, then $(\mathfrak{B}^k)$ converges to $\mathfrak{B}$ as $k \to +\infty$ if and only if $(\mathfrak{A}^k)$ converges to $\mathfrak{A}$ and $\mathfrak{A} = H^u(\mathfrak{B})$.*

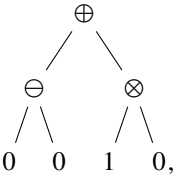
5.3. *Rigidity.* We remind that X stands for the closure of the orbit of $\mathfrak{J}$:

$$X := \overline{\{T_\omega(\mathfrak{J}),\ \omega \in \mathbb{F}_2^+\}}.$$

If $\mathfrak{A} = \begin{array}{ccc} & \oplus & \\ \mathfrak{A}' & & \mathfrak{A}'' \end{array}$ is an element in X, then there are actually some relations between the three objects $\oplus$, $\mathfrak{A}'$, and $\mathfrak{A}''$. This is what we name *rigidity*. Indeed, it turns out that knowing $\mathfrak{A}'$ or $\mathfrak{A}''$ actually yields knowledge of the other one. This also may lay down some specific value for $\oplus$.

5.3.1. *Blocks at even levels*

LEMMA 5.14. *If we see in X a tree starting as*

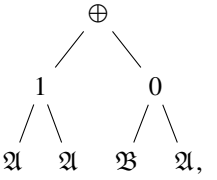


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then:

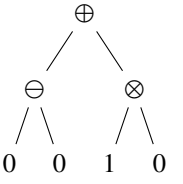
(1) *the tree is of even type;*

(2) *it is equal to*



with $\mathfrak{A}_e = 0$ *and* $\mathfrak{B}_e = 1$.

Proof. Due to Lemma 5.5 and Proposition 5.9, it suffices to prove the statement for configurations in $\mathfrak{J}$. Hence, we consider that we see the configuration

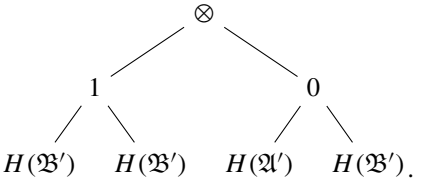


in $\mathfrak{J}$.

We denote by ω the site where the root of the configuration we study stands. At line $|\omega| + 2$, we see a block 0010, and hence line $|\omega| + 2$ is even. Therefore, $|\omega|$ is even.

Because ω lays at an even level, we can consider its source. At the source, we see 

, which yields at site ω



This immediately yields $T_{aa\tilde{\omega}}(\mathfrak{J}) = T_{ba\tilde{\omega}}(\mathfrak{J}) = T_{bb\tilde{\omega}}(\mathfrak{J}) =: \mathfrak{A}$ and $\mathfrak{B} := H(\mathfrak{A}')$. By hypothesis, $\mathfrak{A}_e = 0$ and $\mathfrak{B}_e = 1$. □

COROLLARY 5.15. (Configuration 00 at even line) *Assume that we see in X a tree* 

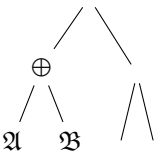
with $\mathfrak{A}_e = \mathfrak{B}_e = 0$. *Then,* $\oplus$ *stands at an odd level and* $\mathfrak{A} = \mathfrak{B}$.

Proof. Again, Lemma 5.5 and Proposition 5.9 show that it is sufficient to prove the statement for configurations 

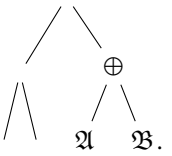
in $\mathfrak{J}$.

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At the level of the roots for $\mathfrak{A}$ and $\mathfrak{B}$, we see the block 00. This shows that this level is even. By Lemma 5.2, we only see blocks 0000 or 0010 at even levels. Hence, we must see either the configuration

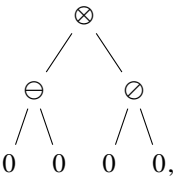


or the reverse one



Then, Lemmas 5.3 and 5.14 show that $\mathfrak{A} = \mathfrak{B}$. □

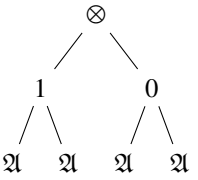
LEMMA 5.16. (0000 yields 0010 at 2^u-type, $u \geq 2$) *If a tree in X starts as*



then:

(1) *it is of type* $2 = 2^1$*;*

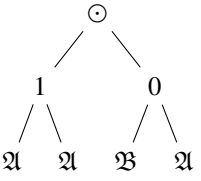
(2) *we actually see*



with $\mathfrak{A}_e = 0$*;*

(3) $\mathfrak{A}$ *is of* 2^u*-type with* $u \geq 2$.

Furthermore, we also see a configuration



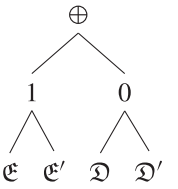
with $\mathfrak{B}_e = 1$ *in X of the same* 2^u*-type.*

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Proof. Again, by Proposition 5.9, we may only prove that the lemma holds for configurations in $\mathfrak{J}$.

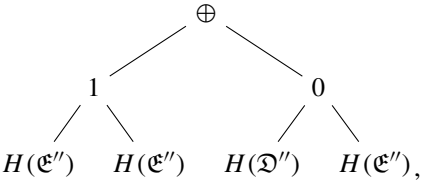
Then, the first three statements are a direct consequence of Lemma 5.3, plus the fact that a block 0000 can only appear at a level in $4\mathbb{N}$. Note that the grandfather for $\mathfrak{A}_e$ must be at an even line which is not a multiple of 4.

Let us assume that the line where the $\mathfrak{A}$ terms have their root is $2^u(2n+1)$ with $u \geq 2$. If we consider u-times the source, we are at level $2n+1$. We must see



level $(2n+1)$

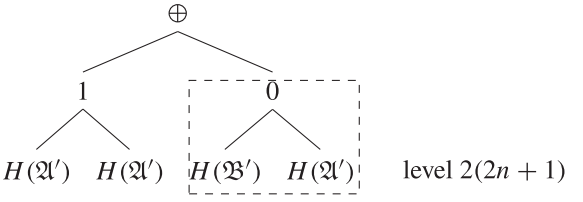
where $H^u\left(\begin{array}{c} 0 \\ \mathfrak{D} \quad \mathfrak{D}' \end{array}\right) = \mathfrak{A}$. Actually, we have a better description because the root $\oplus$ is at an even level: we see



and hence we have $\mathfrak{E} = \mathfrak{E}' = \mathfrak{D}'$.

Let us set $\mathfrak{A}' := \begin{array}{c} 0 \\ \mathfrak{D} \quad \mathfrak{D}' \end{array}$ and $\mathfrak{B}' := \begin{array}{c} 1 \\ \mathfrak{D}' \quad \mathfrak{D}' \end{array}$. This means that we see the configuration $\begin{array}{c} \oplus \\ \mathfrak{B}' \quad \mathfrak{A}' \end{array}$.

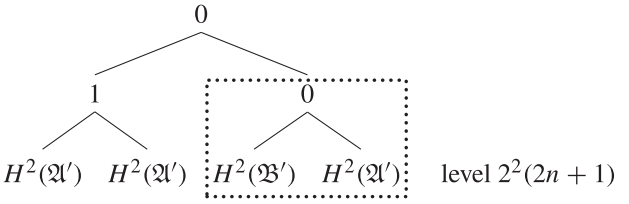
The image by H of the subtree at level $2n$ $\begin{array}{c} \oplus \\ \mathfrak{B}' \quad \mathfrak{A}' \end{array}$ is



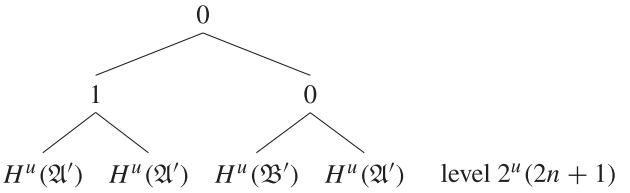
level $2(2n+1)$

Taking the image by H of the subtree in the dashed frame, we see

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In the dotted framebox, we see the same kind of configuration as previously but at level $2^2(2n+1)$. Iterating this process $u-2$ more times, we finally get the configuration:

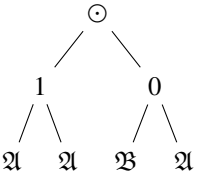


Now, remember that $H^u(\mathfrak{A}')=\mathfrak{A}$. Moreover, $\mathfrak{B}'_e=1$ and H is marked, and thus $H^u(\mathfrak{B}')_e=1$. □

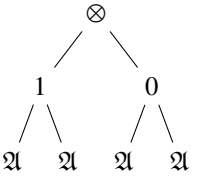
COROLLARY 5.17. *The tree $\mathfrak{J}'$ belongs to X.*

Proof. Minimality comes from the fact that $\mathfrak{J}$ contains many finite subtrees $H^u(0)$ with u as big as wanted. Lemma 5.16 applied to $\mathfrak{J}$ shows that any level where $\mathfrak{J}$ contains $H^u(0)$ with $u\geq 2$, also contains a subtree starting as $H^u(1)$. Letting $u\to+\infty$ yields that $\mathfrak{J}'=\lim_{n\to+\infty}H^n(1)$ belongs to $X=\overline{\mathcal{O}(\mathfrak{J})}$. □

LEMMA 5.18. (0010 yields 0000 at 2^u-type, $u\geq 2$) *If we see in $\mathfrak{J}$ the configuration*



with $\mathfrak{B}_e=1$ in $\mathfrak{J}$ and $\mathfrak{A}$ of 2^u-type with $u\geq 2$, then we also see the configuration



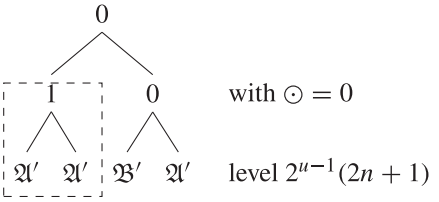
with $\mathfrak{A}_e=0$ in $\mathfrak{J}$ and at the same line.

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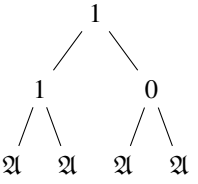
Proof. Because $\mathfrak{A}$ is of even-0-type, this means $\mathfrak{A}_e$ stands at level $2^u(2n+1)$ with



$u \geq 2$. Considering the source, we see $\mathfrak{B}' \quad \mathfrak{A}'$, with $H(\mathfrak{B}') = \mathfrak{B}$ and $H(\mathfrak{A}') = \mathfrak{A}$. Roots $\mathfrak{B}'_e = 1$ and $\mathfrak{A}'_e = 0$ are at the even level $2^{u-1}(2n+1)$. By Lemma 5.14, we only see blocks of 0000 or 0010. Hence, we actually see



Taking the H-image of the dashed frame, we get



with the $\mathfrak{A}$ terms at the level $2^u(2n+1)$. □

5.3.2. *Consequences for even trees and decidability.*

PROPOSITION 5.19. *Only one of the three following cases may happen for an even tree* $\mathfrak{A}$ *in X:*

(1) $\mathfrak{A} = H^v\left(\begin{array}{c}\mathfrak{A}_e\\ \diagup\ \diagdown\\ \mathfrak{D}\quad\mathfrak{D}\end{array}\right)$ *with* $\begin{array}{c}\mathfrak{A}_e\\ \diagup\ \diagdown\\ \mathfrak{D}\quad\mathfrak{D}\end{array}$ *odd,* $\mathfrak{D}_e = 0$, *and* $\mathfrak{D}$ *of* 2^u*-type with* $1 \leq v \leq +\infty$ *and* $u \geq 2$*;*

(2) $\mathfrak{A} = H^v\left(\begin{array}{c}1\\ \diagup\ \diagdown\\ \mathfrak{D}\quad\mathfrak{D}\end{array}\right)$ *with* $\begin{array}{c}1\\ \diagup\ \diagdown\\ \mathfrak{D}\quad\mathfrak{D}\end{array}$ *odd,* $\mathfrak{D}_e = 0$, *and* $\mathfrak{D}$ *of* 2*-type with* $1 \leq v \leq +\infty$*;*

(3) $\mathfrak{A} = H^v\left(\begin{array}{c}0\\ \diagup\ \diagdown\\ \mathfrak{E}\quad\mathfrak{D}\end{array}\right)$ *with* $\begin{array}{c}0\\ \diagup\ \diagdown\\ \mathfrak{E}\quad\mathfrak{D}\end{array}$ *odd,* $\mathfrak{D}_e = 0$, *and* $\mathfrak{E}_e = 1$, $\mathfrak{D}$ *and* $\mathfrak{E}$ *of type* 2^u, *with* $u \geq 1$ *and* $1 \leq v \leq +\infty$.

Furthermore, knowing $T_a(\mathfrak{A})$ *or* $T_b(\mathfrak{A})$ *is sufficient to determine* v.

Remark 4. We emphasize that by Lemmas 5.6 and 5.8, in the previous statement, $v = +\infty$ means $\mathfrak{A} = \mathfrak{J}$ or $\mathfrak{A} = \mathfrak{J}'$, depending on the value for $\mathfrak{A}_e$.

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Proof. We consider a sequence $\mathfrak{A}^k$ in $\mathcal{O}(\mathfrak{J})$ which converges to $\mathfrak{A}$. For simplicity, we set $\mathfrak{A}_e = \otimes$. By Proposition 5.9, we may assume that all the $\mathfrak{A}^k$ are of the form

$$H^v\left(\begin{array}{ccc} & \otimes & \\ \swarrow & & \searrow \\ \mathfrak{E}^k & & \mathfrak{D}^k \end{array}\right),$$

and $\begin{array}{ccc} & \otimes & \\ \swarrow & & \searrow \\ \mathfrak{E}^k & & \mathfrak{D}^k \end{array}$ is an odd-tree.

By Lemma 5.13, convergence for $\mathfrak{A}^k$ is equivalent to the convergence for $\mathfrak{E}^k$ to $\mathfrak{E}$ and for $\mathfrak{D}^k$ to $\mathfrak{D}$.

Furthermore, the root of $\mathfrak{E}^k$ and $\mathfrak{D}^k$ stand at even level, say $2^{u_k}(2m_k+1)$. Again by Proposition 5.9, we may assume that all are of fixed type 2^u with $u \geq 1$.

At even level, we see blocks 0000 or 0010. Hence, the possibilities are $\mathfrak{E}_e = 0 = \mathfrak{D}_e$ or $\mathfrak{E}_e = 1$ and $\mathfrak{D}_e = 0$. Note that by Corollary 5.15, $\mathfrak{E}_e = \mathfrak{D}_e$ yields $\mathfrak{E} = \mathfrak{D}$. Furthermore, $\mathfrak{E}_e = 0$ is only possible if $u \geq 2$.

Hence, the possibilities are:

(1) $u \geq 2$, $\mathfrak{A} = H^v\left(\begin{array}{ccc} & \otimes & \\ \swarrow & & \searrow \\ \mathfrak{D} & & \mathfrak{D} \end{array}\right)$;

(2) $u = 1 \otimes = 1$, $\mathfrak{A} = H^v\left(\begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ \mathfrak{D} & & \mathfrak{D} \end{array}\right)$;

(3) $u \geq 1$, $\otimes = 0$, and $\mathfrak{A} = H^v\left(\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathfrak{E} & & \mathfrak{D} \end{array}\right)$ with $\mathfrak{E}_e = 1$ and $\mathfrak{D}_e = 0$.

Remark 5. We point out that in the case $u \geq 2$, $\mathfrak{A} = H^v\left(\begin{array}{ccc} & \otimes & \\ \swarrow & & \searrow \\ \mathfrak{D} & & \mathfrak{D} \end{array}\right)$, Lemma 5.18 shows that the tree $\mathfrak{A}' := H^v\left(\begin{array}{ccc} & \bar{\otimes} & \\ \swarrow & & \searrow \\ \mathfrak{D} & & \mathfrak{D} \end{array}\right)$ with $\bar{\otimes} = 1 - \otimes$ also exists in X.

Now, we explain how we can determine the value of v. The case $v = +\infty$ is extremely simple because we must see either $\mathfrak{J}$ or $\mathfrak{J}'$. Hence, we assume that $\mathfrak{A}$ is not $\mathfrak{J}$ nor $\mathfrak{J}'$. Then, Lemma 5.11 shows that v is entirely determined by one of the branches of $\mathfrak{A}$: for any u, and for any sufficiently large n, the line n in $T_a(\mathfrak{A})$ or $T_b(\mathfrak{A})$ only contains full blocks of length $|\chi^u(10)|$. Hence, v is the unique integer u where we see blocks $\chi^u(10)$ at lines in $2^{u+1}\mathbb{N}$ (in $\mathfrak{A}$). □

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Remark 6. We emphasize that in case (3), $\mathfrak{A}$ starts as $H^v\begin{pmatrix} & 0 & \\ / & & \backslash \\ 1 & & 0 \end{pmatrix}$. Hence, lines 0 to $2^{v+1}-1$ are the same as those in $\mathfrak{J}$. Line 2^v is equal to $\chi^v(10)$. If $v \geq 2$, both halves of that word contain 0s and 1s (see Lemma 3.10).

Obviously, in cases (1) and (2), the line 2^v in $\mathfrak{A}$ only contains 0s.

5.3.3. *Branch with root* 0 *determines the brother with root* 1

LEMMA 5.20. (Configuration 10. $0 \to 1$) *Assume that we see in X a tree* $\begin{pmatrix} \otimes \\ / \quad \backslash \\ \mathfrak{A} \quad \mathfrak{B} \end{pmatrix}$, *with* $\mathfrak{A}_e = 1$ *and* $\mathfrak{B}_e = 0$. *Then,* $\mathfrak{A}$ *is entirely determined by* $\mathfrak{B}$*: if* $\mathfrak{B}$ *is of* 2^u*-type and equal to* $H^u\begin{pmatrix} & 0 & \\ / & & \backslash \\ \mathfrak{B}' & & \mathfrak{B}'' \end{pmatrix}$, *then* $\mathfrak{A} = H^u\begin{pmatrix} & 1 & \\ / & & \backslash \\ \mathfrak{B}'' & & \mathfrak{B}'' \end{pmatrix}$.

Proof. Still by Proposition 5.9, we only do the proof in the case that we see the configuration in $\mathfrak{J}$.

Let ω be a site where we see $\begin{pmatrix} \otimes \\ / \quad \backslash \\ \mathfrak{A} \quad \mathfrak{B} \end{pmatrix}$.

- First, we consider the case $|\omega|$ is even. If we consider the source of ω, we see the configuration $\begin{pmatrix} \otimes \\ / \quad \backslash \\ \mathfrak{C} \quad \mathfrak{D} \end{pmatrix}$, which yields at site ω

$$\begin{array}{ccccccccc} & & & & \otimes & & & & \\ & & & / & & \backslash & & & \\ & & 1 & & & & 0 & & \\ & / & & \backslash & & / & & \backslash & \\ H(\mathfrak{D}) & & H(\mathfrak{D}) & & H(\mathfrak{C}) & & H(\mathfrak{D}). & & \end{array}$$

Hence, we get $\mathfrak{B} = \begin{pmatrix} 0 \\ / \quad \backslash \\ H(\mathfrak{E}) \quad H(\mathfrak{D}) \end{pmatrix}$ and $\mathfrak{A} = \begin{pmatrix} 1 \\ / \quad \backslash \\ H(\mathfrak{D}) \quad H(\mathfrak{D}) \end{pmatrix}$.

The latter tree can be obtained from $\mathfrak{B}$:

(1) by exchanging the color of the root;
(2) by replacing the left-hand side subtree by a copy of the one at the right-hand side.

This corresponds to the result of the lemma with $u = 0$.

- Assume that $|\omega|$ is odd. Then we set $|\omega| + 1 = 2^u(2n+1)$. We consider the u-source for $\mathfrak{A}$ and $\mathfrak{B}$. By Lemma 5.1, they are either brothers or equal. As H is marked and $\mathfrak{A}_e = 1 \neq 0 = \mathfrak{B}_e$, the u-sources are brothers. Let us denote them by $\mathfrak{A}^u$ and $\mathfrak{B}^u$. They

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are odd-type trees at level $2n+1$. Furthermore, $\mathfrak{A}^u_e = 1$ and $\mathfrak{B}^u_e = 0$. By Lemma 5.2, their father is 0.

We use the part of Lemma 5.20 that is already proved (with father at even level). Hence, $\mathfrak{A}^u$ is obtained from $\mathfrak{B}^u$ by exchanging the color of the root and replacing $T_a(\mathfrak{B}^u)$ by $T_b(\mathfrak{B}^u)$. Then, applying H^u, we get for $\mathfrak{A}$:

(1) $\mathfrak{A}$ starts as $H^u(1)$ whereas $\mathfrak{B}$ starts with $H^u(0)$;

(2) at any slot at line 2^u, we glue $H^u(T_b(\mathfrak{B}^u))$.

Employing notation from the statement of the lemma, we have $\mathfrak{B} = H^u(\mathfrak{B}^u)$, $\mathfrak{A} = H^u(\mathfrak{A}^u)$. If $\mathfrak{B}^u = \overset{0}{\diagup\ \diagdown}\ \mathfrak{B}'\quad \mathfrak{B}''$, then $\mathfrak{A}^u = \overset{1}{\diagup\ \diagdown}\ \mathfrak{B}''\quad \mathfrak{B}''$. □

Remark 7. We emphasize that the construction of $\mathfrak{A}$ from $\mathfrak{B}$ in Lemma 5.20 is coherent with the construction of $\mathfrak{B}$ from $\mathfrak{A}$ in Lemma 5.16.

6. *Preimages and proof of Theorem B*

The aim of this section is to obtain preimages of a given tree with respect to the shift action, say, to obtain (when it is possible) T_a^{-1} and T_b^{-1}.

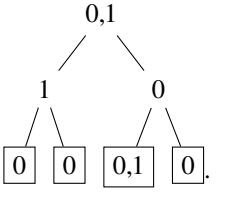
6.1. *Preimages in general.*

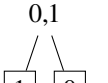
Given a tree $\mathfrak{A}$ in X, the preimages are of the form $\overset{\otimes}{\diagup\ \diagdown}\ \mathfrak{A}\quad \mathfrak{B}$ or $\overset{\otimes}{\diagup\ \diagdown}\ \mathfrak{B}\quad \mathfrak{A}$. We remind that $\mathfrak{A} \in X$ means $\mathfrak{A} = \lim_{k\to+\infty} \mathfrak{A}^k$, $\mathfrak{A}^k = T_{\omega_k}(\mathfrak{J})$.

The question of preimages is then equivalent to determining, for a given converging sequence $(\mathfrak{A}^k)$, which configurations, $\overset{\otimes}{\diagup\ \diagdown}\ \mathfrak{A}^k\quad \mathfrak{B}^k$ or $\overset{\otimes}{\diagup\ \diagdown}\ \mathfrak{B}^k\quad \mathfrak{A}^k$, we see in $\mathfrak{J}$.

We shall see that answer depends on the type of the configuration $\mathfrak{A}$: even or odd. It also depends on the values for the roots of $\mathfrak{A}^k$ and $\mathfrak{B}^k$.

From Lemmas 5.3, 5.2, and 5.14, for even type trees, we only see (the root level is the framed one)



For odd-type trees, we only see (the root level is the framed one) 

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(1) In §6.2, we deal with odd-type trees. Proposition 6.3 deals with odd trees with root equal to 0 and Proposition 6.2 deals with odd trees with root equal to 1.
(2) In §6.3, we deal with even-type trees with root equal to 0. Proposition 6.4 deals with 2^u-types with $2 \leq u < +\infty$, Proposition 6.5 deals with 2-type, and Proposition 6.6 deals with the case 2^∞-type ($\mathfrak{J}$ and $\mathfrak{J}'$).
(3) In §6.4, we deal with the even-type trees and root equal to 1. Proposition 6.8 deals with the 2^u-types $1 \leq u < +\infty$, and Proposition 6.9 deals with the case 2^u-type $u = +\infty$.

The proof of Theorem B is done in §6.5.

6.2. *Preimage of odd trees.* Let $\mathfrak{A}, \mathfrak{B} \in X$ be odd trees with $\mathfrak{A}_e = 1$ and $\mathfrak{B}_e = 0$. Then:
(1) $T^{-1}(\mathfrak{A}) = T_a^{-1}(\mathfrak{A})$;
(2) $T^{-1}(\mathfrak{B}) = T_b^{-1}(\mathfrak{B})$.

In the first case, a brother is an odd tree $\mathfrak{B}'$ with $\mathfrak{B}'_e = 0$ and such that  exists in X. In the second case, a brother is an odd tree $\mathfrak{A}'$ with $\mathfrak{A}_e = 1$ and such that  exists in X.

Because the root for odd trees determines if there are preimages by T_a or T_b, we shall, in this subsection, reserve the letter $\mathfrak{A}$ for odd trees with root 1 and $\mathfrak{B}$ for odd trees with root 0.

We emphasize that for the latter case, Lemma 5.20 shows that $\mathfrak{A}'$ is entirely determined by $\mathfrak{B}$.

We now see that the same holds in the first case. First we point out that an odd tree with root equal to 1 is of the form  with $\mathfrak{D}_e = 0$, $\mathfrak{D}$ of 2^u-type with $1 \leq u \leq +\infty$.

Furthermore, any brother for $\mathfrak{A}$ is a tree of the form . More precisely, Lemmas 5.16 and 5.18 show that the configuration  always exists but the configuration  exists if and only if $\mathfrak{D}$ is of 2^u-type with $u \geq 2$. This is formulated more explicitly in the following lemma.

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LEMMA 6.1. *Let* $\mathfrak{A} \in X$ *be an odd tree with root equal to 1. Set* $\mathfrak{A} =$  *with* $\mathfrak{D}_e = 0$, $\mathfrak{D}$ *of* 2^u*-type with* $1 \leq u \leq +\infty$.

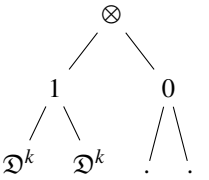
Let $\mathfrak{F}$ *be the tree obtained applying Lemma 5.20 to*  *. Set* $\mathfrak{B} :=$  *and* $\mathfrak{B}' :=$ .

Then:

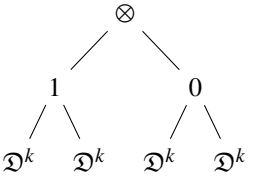
(1) *whatever* $u \geq 1$ *is, there exists a tree*  *in X;*

(2) *if (and only if)* $u \geq 2$*, there exists a tree*  *in X.*

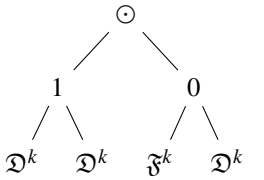
Proof. As $\mathfrak{A}$ belongs to X, we see in $\mathfrak{J}$ trees



with $\lim_{k\to+\infty} \mathfrak{D}^k = \mathfrak{D}$. The trees $\mathfrak{D}^k$ stand at 2^u-levels. By Lemma 5.16, we will actually see in $\mathfrak{J}$ trees



(this holds only for $u \geq 2$) and trees



(at the same levels) with $\mathfrak{F}^k_e = 1$ (this holds for any $u \geq 1$).

The tree $\mathfrak{F}^k$ is obtained from $\mathfrak{D}^k$ via the procedure described in Lemma 5.20. Lemma 5.13 yields that $\mathfrak{F}^k$ converges as $\mathfrak{D}^k$ converges. The link between $\mathfrak{F}^k$ and $\mathfrak{D}^k$ passes to the limit and then $\mathfrak{F}^k$ converges to $\mathfrak{F}$ (from the statement of the lemma).

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Therefore, letting $k \to +\infty$, we get:

(1) in any case, a tree $\begin{array}{c}\odot\\ \mathfrak{A}\quad\mathfrak{B}\end{array}$ in X;

(2) if and only if $u \geq 2$, a tree $\begin{array}{c}\otimes\\ \mathfrak{A}\quad\mathfrak{B}'\end{array}$ in X. □

Lemma 6.1 states (for a given $\mathfrak{A} \in X_{\text{od}}$ with root 1) what kind of trees $\begin{array}{c}\otimes\\ \mathfrak{A}\quad\mathfrak{B}\end{array}$ we can find in X. However, it does not make precise what is the value for $\otimes$. Similarly, we have seen that Lemma 5.20 states that an odd tree $\mathfrak{B}$ has a single brother $\mathfrak{A}'$ in X. Again, it does not make precise what the father may be.

Furthermore, preimages of these trees are even trees, and thus obey the rule described in Proposition 5.19. We now connect the two results and give a full description of preimages of odd trees.

PROPOSITION 6.2. *Let $\mathfrak{A}$ be an odd tree satisfying $\mathfrak{A}_e = 1$. Let v be such that $\mathfrak{A} \in T_a \circ H^v(X_{\text{od}})$. Then, we get the following.*

(1) *If $v \geq 2$, then $\mathfrak{A}$ has a unique brother $\mathfrak{B}$ given by Lemma 6.1 case (1). Furthermore, only one of the following cases may appear:*

   (a) $T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{c}0\\ \mathfrak{A}\quad\mathfrak{B}\end{array}, \begin{array}{c}1\\ \mathfrak{A}\quad\mathfrak{B}\end{array} \right\}$, *line $2^v - 1$ in $\mathfrak{A}$ only contains 0s, and $T^{2^v-1}(\mathfrak{A})$ is even of 2^{v+u}-type with $u \geq 2$;*

   (b) $T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{c}1\\ \mathfrak{A}\quad\mathfrak{B}\end{array} \right\}$, *line $2^v - 1$ in $\mathfrak{A}$ only contains 0s, and $T^{2^v-1}(\mathfrak{A})$ is even of 2^{v+1}-type;*

   (c) $T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{c}0\\ \mathfrak{A}\quad\mathfrak{B}\end{array} \right\}$, *line $2^v - 1$ in $\mathfrak{A}$ contains 0s and 1s.*

(2) *If $v = 1$, then $\mathfrak{A}$ has two brothers $\mathfrak{B}$ and $\mathfrak{B}'$ given by Lemma 6.1. Furthermore, only one of the following cases may appear:*

   (a) $T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{c}0\\ \mathfrak{A}\quad\mathfrak{B}'\end{array}, \begin{array}{c}1\\ \mathfrak{A}\quad\mathfrak{B}'\end{array}, \begin{array}{c}0\\ \mathfrak{A}\quad\mathfrak{B}\end{array} \right\}$, *and $T_a(\mathfrak{A})$ is of 2^{1+u}-type with $u \geq 2$;*

   (b) $T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{c}1\\ \mathfrak{A}\quad\mathfrak{B}'\end{array}, \begin{array}{c}0\\ \mathfrak{A}\quad\mathfrak{B}\end{array} \right\}$, *and $T_a(\mathfrak{A})$ is of 2^2-type.*

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Proof. We remind that by Proposition 5.19, $\mathfrak{A}$ determines the type of any preimage.

• If $v \geq 2$ holds, then any tree $\begin{array}{c} \mathfrak{A} \end{array}$ starts as

$$\begin{array}{ccccccccc} & & & & \otimes & & & & \\ & & & \swarrow & & \searrow & & & \\ & & 1 & & & & 0 & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ 0 & & 0 & & 1 & & 0, & & \end{array}$$

$$\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 1 & & 0. \end{array}$$

which shows that $\mathfrak{A}$ has a unique brother, and it starts as

Using notation from Lemma 6.1, this yields that $\mathfrak{B}$ is the unique brother for $\mathfrak{A}$ (case $\mathfrak{B}'$ is impossible).

Line $2^v - 1$ in $\mathfrak{A}$ corresponds to the line 2^v in $\begin{array}{cc} \mathfrak{A} & \mathfrak{B} \end{array}$. By Remark 6, this line only contains 0s if and only if we are in cases (1) or (2) from Proposition 5.19, and the line contains 0s and 1s if and only if we are in case (3) from Proposition 5.19.

Furthermore, for any $\omega \in \mathbb{F}_2^+$ with $|\omega| = 2^v - 1$, $T_\omega(\mathfrak{A})$ is equal to $H^v(\mathfrak{D})$ if we re-employ notation from Proposition 5.19. It is thus of type 2^{u+v} and this determine the type of $\mathfrak{D}$. Hence, this differentiates case (1) and case (2) (from Proposition 5.19).

Now, still for $v \geq 2$, case (1) in Proposition 5.19 corresponds to case (1a) in the statement of Proposition 6.2, case (2) in Proposition 5.19 corresponds to case (1b) in the statement of Proposition 6.2, and case (3) in Proposition 5.19 corresponds to case (1c) in the statement of Proposition 6.2.

• Let us now assume that $v = 1$ holds. Re-employing notation from Lemma 6.1, we see $\begin{array}{cc} \mathfrak{A} & \mathfrak{B} \end{array}$ and $\begin{array}{cc} \mathfrak{A} & \mathfrak{B}' \end{array}$ in X. Line 2 in $\begin{array}{cc} \mathfrak{A} & \mathfrak{B} \end{array}$ is equal to 0010, whereas line 2 in $\begin{array}{cc} \mathfrak{A} & \mathfrak{B}' \end{array}$ is equal to 0000. Hence, the tree $\begin{array}{cc} \mathfrak{A} & \mathfrak{B} \end{array}$ corresponds to case (3) from Proposition 5.19, and $\begin{array}{cc} \mathfrak{A} & \mathfrak{B} \end{array}$ corresponds to case (1) or (2) from Proposition 5.19.

If we set $\mathfrak{A} =$ $\begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ \mathfrak{G} & & \mathfrak{G} \end{array}$, because $v = 1$, $\begin{array}{c} \mathfrak{A} \end{array}$ belongs to $H(X_{\text{od}})$, and hence $\mathfrak{G}$ is of 2^k-type with $k \geq 2$. This means that if we set $\begin{array}{cc} \mathfrak{A} & \mathfrak{B}' \end{array} = H\left(\begin{array}{cc} \mathfrak{D} & \mathfrak{D} \end{array} \right)$, then $\mathfrak{D}$ is of type $k - 1$.

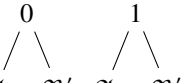
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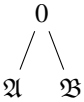


If $k-1=1$, only the configuration $\mathfrak{D}\ \ \mathfrak{D}$ exists because lines at even levels not in $4\mathbb{N}$ only contain blocks 0010. In contrast, if $k-1\geq 2$, then Lemmas 5.16 and 5.18 yield

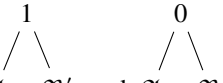


that both configurations $\mathfrak{D}\ \ \mathfrak{D}$ and $\mathfrak{D}\ \ \mathfrak{D}$ do exist.

Hence we get: if $T_a(\mathfrak{A})$ is of type $k\geq 3$, then $\mathfrak{A}$ has three preimages: ,



and $\mathfrak{A}\ \ \mathfrak{B}$.

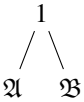
If $T_a(\mathfrak{A})$ is of type $k=2$, then $\mathfrak{A}$ has two preimages:  □

PROPOSITION 6.3. *Let $\mathfrak{B}\in X$ be an odd tree with $\mathfrak{B}_e=0$. Let $\mathfrak{A}$ be its brother (as in Lemma 5.20). Let $T_b(\mathfrak{B})$ be of type 2^k. Let v be such that*  *is of type 2^v (as in Proposition 5.19). Then, only one of the following cases may appear.*



(1) *If $\mathfrak{B}$ starts as* $0\ \ 0$, *then:*

(a) *$k\geq 3$ and $v=1$, and* $T_b^{-1}(\mathfrak{B})=\left\{\begin{array}{c}0\\ \diagup\ \diagdown\\ \mathfrak{A}\ \ \mathfrak{B}\end{array},\ \begin{array}{c}1\\ \diagup\ \diagdown\\ \mathfrak{A}\ \ \mathfrak{B}\end{array}\right\}$;

(b) *$k=2$ and $v=1$, and*  *is the unique preimage of $\mathfrak{B}$.*



(2) *If $\mathfrak{B}$ starts as* $1\ \ 0$, *then:*

(a) *if $k\geq 2$, then $v=1$ and* $T_b^{-1}(\mathfrak{B})=\left\{\begin{array}{c}0\\ \diagup\ \diagdown\\ \mathfrak{A}\ \ \mathfrak{B}\end{array}\right\}$;

(b) *if $k=1$, then $v\geq 2$ and one of the following case occurs:*

(i) *line 2^v-1 of $\mathfrak{B}$ only contains 0s and $T^{2^v-1}(\mathfrak{B})$ is of 2^{u+v}-type with $u\geq 2$. Then* $T_b^{-1}(\mathfrak{B})=\left\{\begin{array}{c}0\\ \diagup\ \diagdown\\ \mathfrak{A}\ \ \mathfrak{B}\end{array},\ \begin{array}{c}1\\ \diagup\ \diagdown\\ \mathfrak{A}\ \ \mathfrak{B}\end{array}\right\}$;

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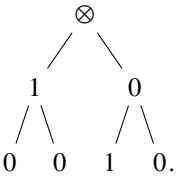
(ii) *line* $2^v - 1$ *of* $\mathfrak{B}$ *only contains 0s and* $T^{2^v-1}(\mathfrak{B})$ *is of* 2^{1+v}*-type. Then* $T_b^{-1}(\mathfrak{B}) = \left\{ \begin{array}{c} 1 \\ \mathfrak{A} \quad \mathfrak{B} \end{array} \right\}$;

(iii) *line* $2^v - 1$ *of* $\mathfrak{B}$ *contains 0s and 1s. Then* $T_b^{-1}(\mathfrak{B}) = \left\{ \begin{array}{c} 0 \\ \mathfrak{A} \quad \mathfrak{B} \end{array} \right\}$.

Proof. Any tree $\begin{array}{c} \otimes \\ \mathfrak{A} \quad \mathfrak{B} \end{array}$ is of the form $H^v\left(\begin{array}{c} \otimes \\ \mathfrak{E} \quad \mathfrak{D} \end{array} \right)$ with $\begin{array}{c} \otimes \\ \mathfrak{E} \quad \mathfrak{D} \end{array}$ odd. We remind that v is uniquely determined by $\mathfrak{B}$ (see Proposition 5.19).

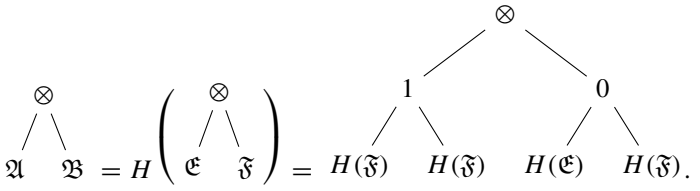
- Suppose $\mathfrak{B}$ starts (which means that it is a $\mathfrak{B}'$ if we re-employ notation from Lemma 6.1) as $\begin{array}{c} 0 \\ 0 \quad 0 \end{array}$. This is possible only if $v = 1$, as $H^2(\otimes)$ starts as $\begin{array}{ccccc} & & \otimes & & \\ & 1 & & & 0 \\ 0 & 0 & & 1 & 0 \end{array}$. Hence, we get $k = u + 1$. Furthermore:
  - either $u \geq 2$ (which is equivalent to $k \geq 3$) and then $\mathfrak{B} = T_b H\left(\begin{array}{c} 0 \\ \mathfrak{D} \quad \mathfrak{D} \end{array} \right) = T_b H\left(\begin{array}{c} 1 \\ \mathfrak{D} \quad \mathfrak{D} \end{array} \right)$ for some $\mathfrak{D}$ of type 2^u satisfying $\mathfrak{D}_e = 0$. Note that by Lemmas 5.16 and 5.18, both configurations $\begin{array}{c} 1 \\ \mathfrak{D} \quad \mathfrak{D} \end{array}$ and $\begin{array}{c} 0 \\ \mathfrak{D} \quad \mathfrak{D} \end{array}$ do exist. This means that $\mathfrak{B}$ has two preimages;
  - or $u = 1$ (that is, $k = 2$) and $\mathfrak{B} = T_b H\left(\begin{array}{c} 0 \\ \mathfrak{D} \quad \mathfrak{D} \end{array} \right)$ for some $\mathfrak{D}$ of 2^1-type with $\mathfrak{D}_e = 0$. Here, only the configuration $\begin{array}{c} 1 \\ \mathfrak{D} \quad \mathfrak{D} \end{array}$ does exist. This means that $\mathfrak{B}$ has a unique preimage with root 1.
- Suppose $\mathfrak{B}$ starts as $\begin{array}{c} 0 \\ 1 \quad 0 \end{array}$. Note that $\begin{array}{c} \otimes \\ \mathfrak{A} \quad \mathfrak{B} \end{array}$ starts as

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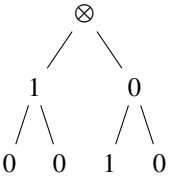
If $T_a(\mathfrak{B})$ is of 2^k-type with $k \geq 2$, then $\otimes$ stands at an even level which is not in $4\mathbb{N}$. This yields $v = 1$.

Hence, we get



This yields $\mathfrak{E}_e = 1$ and $\mathfrak{F}_e = 0$. Now, $\begin{smallmatrix}\otimes\\ 1 \quad 0\end{smallmatrix}$ with $\otimes$ at an odd level is possible in $\mathfrak{J}$ if and only if $\otimes = 0$.

Now, we deal with the case where $\mathfrak{B}$ starts as $\begin{smallmatrix}0\\ 1 \quad 0\end{smallmatrix}$ and $k = 1$. Recall that $\begin{smallmatrix}\otimes\\ \mathfrak{A} \quad \mathfrak{B}\end{smallmatrix}$ starts as



and the last line 0010 stands at a level in $2\mathbb{N} \setminus 4\mathbb{N}$. This yields that the root $\otimes$ stands at a level in $4\mathbb{N}$, and hence $v \geq 2$. We remind that v can be detected, as it is explained in Lemma 5.11. Furthermore, Proposition 5.19 states that v can be detected knowing only $\mathfrak{B}$.

Hence, we have:

– either we are in cases (1) or (2) from Proposition 5.19 with $v \geq 2$. In that case, there is a line of 0s at line 2^v of $\begin{smallmatrix}\otimes\\ \mathfrak{A} \quad \mathfrak{B}\end{smallmatrix}$;
– or we are in case (3) from Proposition 5.19 (with $v \geq 1$). In that case, the line 2^v of $\begin{smallmatrix}\otimes\\ \mathfrak{A} \quad \mathfrak{B}\end{smallmatrix}$ contains 0s and 1s.

Furthermore, to separate cases (1) and (2), we need to check what is the type 2^{u+v} of any tree $T_\omega(\mathfrak{B})$ with $\omega \in \mathbb{F}_2^+$ and $|\omega| = 2^v - 1$. □

Table 1 summarizes the possible preimages for a given odd tree.

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TABLE 1. Preimages of odd trees. All cases in the same line coexist.

$\swarrow$ T_a or T_b	$H^v\left(\begin{array}{c} 0,1 \\ / \quad \backslash \\ \mathfrak{D} \quad \mathfrak{D} \end{array}\right)$, $\mathfrak{D}$ of 2^u-type $u \geq 2$, $\mathfrak{D}_e = 0$	$H^v\left(\begin{array}{c} 1 \\ / \quad \backslash \\ \mathfrak{D} \quad \mathfrak{D} \end{array}\right)$, $\mathfrak{D}$ of 2-type, $\mathfrak{D}_e = 0$	$H^v\left(\begin{array}{c} 0 \\ / \quad \backslash \\ \mathfrak{F} \quad \mathfrak{D} \end{array}\right)$, $\mathfrak{D}_e = 0$, $\mathfrak{F}_e = 1$	Number of preimages	Case
$\mathfrak{A}$, $\mathfrak{A}_e = 1$ $T_a(\mathfrak{A})$ of $2^k - type$	$v \geq 2$, brother $\mathfrak{B}$			2	Proposition 6.2(1a)
		$v \geq 2$, brother $\mathfrak{B}$		1	Proposition 6.2(1b)
			$v \geq 2$, brother $\mathfrak{B}$	1	Proposition 6.2(1c)
	$v = 1$ and brother $\mathfrak{B}'$ $k = u + 1$		$v = 1, k = u + 1$ brother $\mathfrak{B}$	3	Proposition 6.2(2a)
		$v = 1$ and brother $\mathfrak{B}'$ $k = 2$	$v = 1, k = 2$ brother $\mathfrak{B}$	2	Proposition 6.2(2b)
$\mathfrak{B}'$, starts as $\begin{array}{ccc} & 0 & \\ / & & \backslash \\ 0 & & 0 \end{array}$, $T_a(\mathfrak{B}')$ of $2^k - type$	$k \geq 3, v = 1$			2	Proposition 6.3(1a)
		$k = 2, v = 1$		1	Proposition 6.3(1b)
$\mathfrak{B}$, starts as $\begin{array}{ccc} & 0 & \\ / & & \backslash \\ 1 & & 0 \end{array}$, $T_a(\mathfrak{B})$ of $2^k - type$	$v \geq 2$			2	Proposition 6.3(2bi)
		$v \geq 2$		1	Proposition 6.3(2bii)
			$v \geq 2$	1	Proposition 6.3(2biii)
			$v = 1$	1	Proposition 6.3(2a)

6.3. *Preimage for even-type elements with root equal to* 0

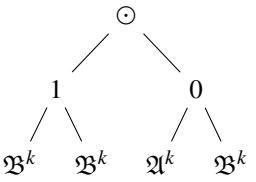
PROPOSITION 6.4. *Let* $\mathfrak{B}$ *be in* X *of* 2^u*-type with* $2 \leq u < +\infty$ *and* $\mathfrak{B}_e = 0$*. Then,* $T^{-1}(\mathfrak{B})$ *is the set*

$$\left\{ \begin{array}{ccc} 0 & 1 & 0 \\ \diagup\ \diagdown & \diagup\ \diagdown & \diagup\ \diagdown \\ \mathfrak{B}\quad\mathfrak{B}, & \mathfrak{B}\quad\mathfrak{B}, & \mathfrak{A}\quad\mathfrak{B} \end{array} \right\},$$

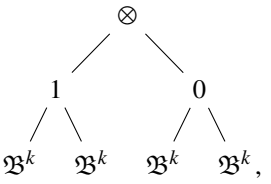
where $\mathfrak{A} = H^u\left(\begin{array}{c} 1 \\ \diagup\ \diagdown \\ \mathfrak{B}''\quad\mathfrak{B}'' \end{array} \right)$ *if* $\mathfrak{B} = H^u\left(\begin{array}{c} 0 \\ \diagup\ \diagdown \\ \mathfrak{B}'\quad\mathfrak{B}'' \end{array} \right)$.

Proof. We consider a sequence of trees $\mathfrak{B}^k := T_{\omega_k}(\mathfrak{J})$ such that $\mathfrak{B} = \lim_{k\to+\infty} \mathfrak{B}^k$. For simplicity, we assume that for every k, $|\omega_k| = 2^u(2n_k + 1)$ holds.

Lemmas 5.16 and 5.18 show that we see the configurations



and



with $\mathfrak{A}^k_e = 1$. Moreover, $\mathfrak{A}^k$ is obtained from $\mathfrak{B}^k$ following Lemma 5.16 and as explained in Lemma 5.20.

As $\mathfrak{B}^k$ goes to $\mathfrak{B}$ if $k \to +\infty$, the branches at level 2^u for $\mathfrak{B}^k$ converge. Hence, $\mathfrak{A}^k$ converges to some tree $\mathfrak{A} \in X$. This yields

$$T^{-1}(\mathfrak{B}) \supset \left\{ \begin{array}{ccc} 0 & 1 & 0 \\ \diagup\ \diagdown & \diagup\ \diagdown & \diagup\ \diagdown \\ \mathfrak{B}\quad\mathfrak{B}\ , & \mathfrak{B}\quad\mathfrak{B}\ , & \mathfrak{A}\quad\mathfrak{B} \end{array} \right\}.$$

Furthermore, configurations $\begin{array}{c} 1 \\ \diagup\ \diagdown \\ \mathfrak{B}^k\quad\mathfrak{D}^k \end{array}$ or $\begin{array}{c} 1 \\ \diagup\ \diagdown \\ \mathfrak{D}^k\quad\mathfrak{B}^k \end{array}$ with $\mathfrak{D}^k_e = 0$ and $\mathfrak{D}^k \neq \mathfrak{B}^k$ are forbidden (by Corollary 5.15). For the same reason, configurations $\begin{array}{c} 0 \\ \diagup\ \diagdown \\ \mathfrak{B}^k\quad\mathfrak{D}^k \end{array}$ with

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$\mathfrak{D}_e^k = 0$ and $\mathfrak{D}^k \neq \mathfrak{B}^k$ are also forbidden. Finally, configurations $\begin{array}{ccc} & 0 & \\ \mathfrak{B}^k & & \mathfrak{D}^k \end{array}$ with $\mathfrak{D}_e^k = 1$ are forbidden because the configuration $\begin{array}{ccc} & \otimes & \\ 0 & & 1 \end{array}$ is forbidden (see Remark 2 after the proof of Lemma 3.7 for allowed digits). □

PROPOSITION 6.5. *Let $\mathfrak{B}$ be in X 2-type and $\mathfrak{B}_e = 0$. Then, $T^{-1}(\mathfrak{B})$ is the set*

$$\left\{ \begin{array}{ccc} & 1 & \\ \mathfrak{B} & & \mathfrak{B} \end{array} , \begin{array}{ccc} & 0 & \\ \mathfrak{A} & & \mathfrak{B} \end{array} \right\},$$

where $\mathfrak{A} = H\left(\begin{array}{ccc} & 1 & \\ \mathfrak{B}'' & & \mathfrak{B}'' \end{array} \right)$ *if* $\mathfrak{B} = H\left(\begin{array}{ccc} & 0 & \\ \mathfrak{B}' & & \mathfrak{B}'' \end{array} \right)$.

Proof. The proof is almost the same as for Proposition 6.4. The unique difference is that, by Proposition 3.11, the configuration

$$\begin{array}{ccccccc} & & & \otimes & & & \\ & & 1 & & & 0 & \\ & \mathfrak{B}^k & & \mathfrak{B}^k & \mathfrak{B}^k & & \mathfrak{B}^k \end{array}$$

does not exist because the line with the $\mathfrak{B}^k$ stands at a level $2.(2n_k + 1)$, where we only see blocks of 0010. This just prohibits the configuration $\begin{array}{ccc} & 0 & \\ \mathfrak{B}^k & & \mathfrak{B}^k \end{array}$, and hence forbids one preimage compared with the case $u \geq 2$. □

PROPOSITION 6.6.

$$T^{-1}(\mathfrak{J}) = \left\{ \begin{array}{ccc} & 0 & \\ \mathfrak{J} & & \mathfrak{J} \end{array} , \begin{array}{ccc} & 1 & \\ \mathfrak{J} & & \mathfrak{J} \end{array} , \begin{array}{ccc} & 0 & \\ \mathfrak{J}' & & \mathfrak{J} \end{array} \right\}.$$

Proof. The fixed point $\mathfrak{J}$ is not periodic. Hence, there is no $\omega \neq e$ such that $T_\omega(\mathfrak{J}) = \mathfrak{J}$. This yields that there is no $\omega \in \mathbb{F}_2^+$ with $\omega \neq e$ such that $T_\omega(\mathfrak{J})$ belongs to $T^{-1}(\mathfrak{J})$.

An element $\mathfrak{C}$ in $T^{-1}(\mathfrak{J})$ is thus obtained as $\lim_{k\to+\infty} \mathfrak{C}^k$ with $\mathfrak{C}^k$ of the form $T_{\omega_k}(\mathfrak{J})$. If, say $T_a(\mathfrak{C}) = \mathfrak{J}$, then $\lim_{k\to+\infty} T_{a\omega_k}(\mathfrak{C}^k) = \mathfrak{J}$.

Conversely, if $\mathfrak{B}^k$ is of the form $\mathfrak{B}^k = T_{\alpha_k}(\mathfrak{J})$ and $\lim_{k\to+\infty} \mathfrak{B}^k = \mathfrak{J}$, and if $\mathfrak{C}^k$ is such that, say $T_a\mathfrak{C}^k = \mathfrak{B}^k$, then any accumulation point $\mathfrak{C}$ for the $\mathfrak{C}^k$ terms satisfies $T_a(\mathfrak{C}) = \mathfrak{J}$.

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To compute $T^{-1}(\mathfrak{J})$, we thus adapt the proof of Proposition 6.4. We consider a sequence of terms $\mathfrak{B}^k := T_{\omega_k}(\mathfrak{J})$ such that $\lim_{k\to+\infty} \mathfrak{B}^k = \mathfrak{J}$. We compute $T^{-1}(\mathfrak{B}^k)$ and consider the accumulation points for these sequences of trees if k goes to $+\infty$.

Note that $\mathfrak{B}^k_e = 0$ (at least for any sufficiently large k). By Lemma 5.8, if we set $|\omega_k| = 2^{u_k}(2n_k + 1)$, then $u_k \to +\infty$ as $k \to +\infty$. Hence, for any sufficiently large k, $u_k \geq 2$ and we can apply Proposition 6.4 to each $\mathfrak{B}^k$. For simplicity, we assume that the above conditions hold for every k.

Then, for each fixed k, $T^{-1}(\mathfrak{B}^k)$ is the set whose elements are $\begin{array}{ccc} & 0 & \\ \mathfrak{B}^k & & \mathfrak{B}^k \end{array}$, $\begin{array}{ccc} & 1 & \\ \mathfrak{B}^k & & \mathfrak{B}^k \end{array}$, $\begin{array}{ccc} & 0 & \\ \mathfrak{A}^k & & \mathfrak{B}^k \end{array}$, where $\mathfrak{A}^k$ is given in Lemma 5.20.

The first two elements converge respectively to $\begin{array}{ccc} & 0 & \\ \mathfrak{J} & & \mathfrak{J} \end{array}$ and $\begin{array}{ccc} & 1 & \\ \mathfrak{J} & & \mathfrak{J} \end{array}$ as k goes to $+\infty$. Now, we remind that $\mathfrak{A}^k$ starts as $H^{u_k}(1)$ and $u_k \to +\infty$. Furthermore, in the proof of Proposition 2.4, we have seen that $H^n(0)$ and $H^n(1)$ differ only from the root. Therefore, $\lim_{k\to+\infty} \mathfrak{A}^k = \mathfrak{J}'$. This shows that $\begin{array}{ccc} & 0 & \\ \mathfrak{J}' & & \mathfrak{J} \end{array}$ is also in X and in $T^{-1}(\mathfrak{J})$. □

6.4. *Preimages of even-type elements with root equal to* 1. We start with a description of even-type elements with root equal to 1.

LEMMA 6.7. *Let $u \geq 1$ be an integer. The 2^u-type elements with root equal to* 1 *in X are elements of the form $H^u\begin{pmatrix} & 1 & \\ \mathfrak{C} & & \mathfrak{C} \end{pmatrix}$ with $\mathfrak{C} = H^v\begin{pmatrix} & 0 & \\ \mathfrak{C}' & & \mathfrak{C}'' \end{pmatrix}$, $1 \leq v \leq +\infty$.*

Remark 8. The case $v = +\infty$ means $\mathfrak{C} = \mathfrak{J}$.

Proof. Because $\mathfrak{A}$ is of even type and $\mathfrak{A}_e = 1$ and H is marked, Proposition 5.12 yields that we can write $\mathfrak{A}$ under the form $\mathfrak{A} = H^u(\mathfrak{B})$ with $\mathfrak{B}_e = 1$, and $\mathfrak{B}$ is of odd type. Any 1 at an odd line is followed only by 0s and thus Corollary 5.15 yields

$$\mathfrak{B} = \begin{array}{ccc} & 1 & \\ \mathfrak{C} & & \mathfrak{C}, \end{array}$$

with $\mathfrak{C}_e = 0$.

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The tree $\mathfrak{B}$ is of odd type, and thus $\mathfrak{C}$ is of even type, say 2^v-type with $1 \leq v \leq +\infty$. Hence, $\mathfrak{C}$ can be written under the form $\begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{C}' & & \mathfrak{C}'' \end{array}$, with the convention that $v = +\infty$ means $\mathfrak{C} = \mathfrak{J}$. □

PROPOSITION 6.8. *Let $\mathfrak{A}$ be in X of 2^u-type with $1 \leq u < +\infty$ and $\mathfrak{A}_e = 1$. Set $\mathfrak{A} = H^u\left(\begin{array}{ccc} & 1 & \\ / & & \backslash \\ \mathfrak{C} & & \mathfrak{C} \end{array}\right)$ with $\mathfrak{C} = H^v\left(\begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{C}' & & \mathfrak{C}'' \end{array}\right)$, $1 \leq v \leq +\infty$. Then:*

(1) *if $v = 1$, then*

$$T^{-1}(\mathfrak{A}) = T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{A} & & \mathfrak{B} \end{array} \right\}$$

with $\mathfrak{B} = H^u\left(\begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{D} & & \mathfrak{C} \end{array}\right)$ and $\mathfrak{D} = H^v\left(\begin{array}{ccc} & 1 & \\ / & & \backslash \\ \mathfrak{C}'' & & \mathfrak{C}'' \end{array}\right)$;

(2) *if $v > 1$, then*

$$T^{-1}(\mathfrak{A}) = T_a^{-1}(\mathfrak{A}) = \left\{ \begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{A} & & \mathfrak{B} \end{array} , \begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{A} & & \mathfrak{B}' \end{array} \right\},$$

with $\mathfrak{B} = H^u\left(\begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{D} & & \mathfrak{C} \end{array}\right)$, $\mathfrak{D} = H^v\left(\begin{array}{ccc} & 1 & \\ / & & \backslash \\ \mathfrak{C}'' & & \mathfrak{C}'' \end{array}\right)$ and $\mathfrak{B}' = H^u\left(\begin{array}{ccc} & 0 & \\ / & & \backslash \\ \mathfrak{C} & & \mathfrak{C} \end{array}\right)$.

Remark 9. We remind that if $v = +\infty$, then we replace $H^v\left(\begin{array}{ccc} & 0 & \\ / & & \backslash \end{array}\right)$ by $\mathfrak{J}$ and $H^v\left(\begin{array}{ccc} & 1 & \\ / & & \backslash \end{array}\right)$ by $\mathfrak{J}'$. Note that $\mathfrak{B}'$ differs from $\mathfrak{A}$ only with the root.

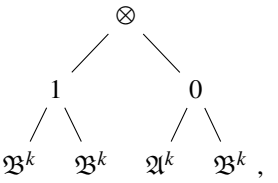
Proof. We consider a sequence of terms $\mathfrak{A}^k := T_{\omega_k}(\mathfrak{J})$, with $\mathfrak{A}^k \to \mathfrak{A}$ as $k \to +\infty$. By Proposition 5.9, we may assume that every $\mathfrak{A}^k$ is of 2^u-type. Then, with the notation of Lemma 6.7, we set $\mathfrak{A}^k = H^u\left(\begin{array}{ccc} & 1 & \\ / & & \backslash \\ \mathfrak{C}^k & & \mathfrak{C}^k \end{array}\right)$. By Lemma 5.13, $\mathfrak{C}^k \to \mathfrak{C}$. Hence, we may

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write $\mathfrak{C}^k = H^v\begin{pmatrix} & 0 & \\ \mathfrak{G}^k & & \mathfrak{H}^k \end{pmatrix}$. This makes sense if $v \neq +\infty$. If $v = +\infty$, then we set $\mathfrak{C}^k = H^{v_k}\begin{pmatrix} & 0 & \\ \mathfrak{G}^k & & \mathfrak{H}^k \end{pmatrix}$ with $v_k \to +\infty$.

The root of $\mathfrak{A}^k$ (equal to 1) is at an even line. We remind that even lines are composed by blocks 0000 or 0010. This shows that $T_b^{-1}(\mathfrak{A}^k) = \emptyset$.

By Lemma 5.14, we see a sequence of configurations



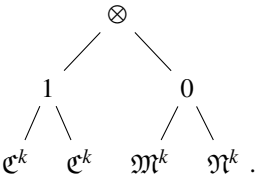
with $\mathfrak{B}^k_e = 0$.

Note that this is the unique configuration where $\mathfrak{A}^k$ appears, and this yields that $T^{-1}(\mathfrak{A}) = T_a^{-1}$ and that any preimage of $\mathfrak{A}$ has its root equal to 0.

We now see how the $\mathfrak{B}^k$ terms depend on the $\mathfrak{A}^k$ terms.

We remember the equality $\mathfrak{A}^k = H^u\begin{pmatrix} & 1 & \\ \mathfrak{C}^k & & \mathfrak{C}^k \end{pmatrix}$. Hence, we consider the u-times unsubstreetuted trees for $\mathfrak{A}^k$ and $\mathfrak{B}^k$. They have different roots since H is marked. By Lemma 5.1, they are brothers.

Hence, we see a configuration

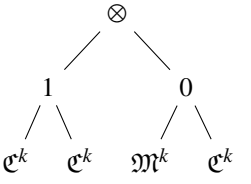


We first deal with the case $v < +\infty$. The root of $\mathfrak{C}^k$ stands at even level $2^v(2m_k + 1)$ and its value is 0. Because at even lines we only see blocks of 0010 or blocks of 0000, Lemma 5.14 yields $\mathfrak{N}^k = \mathfrak{C}^k$.

Lemmas 5.16 and 5.18 show that:

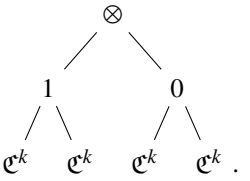
(1) either $v = 1$ and necessarily $\mathfrak{M}^k_e = 1$;

(2) or $v \geq 2$ and we see at the same level both configurations



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for some $\mathfrak{M}^k$ with $\mathfrak{M}^k_e = 1$ and



For both cases where we see some subtree $\mathfrak{M}^k$, it remains to determine how we obtain the subtree $\mathfrak{M}^k$ from the subtree $\mathfrak{C}^k$. For that, we use Lemma 5.20. We have $\mathfrak{C}^k = H^v\left(\begin{array}{c} 0 \\ \mathfrak{G}^k \quad \mathfrak{H}^k \end{array}\right)$, and thus

$$\mathfrak{M}^k = H^v\left(\begin{array}{c} 1 \\ \mathfrak{H}^k \quad \mathfrak{H}^k \end{array}\right).$$

Because $\mathfrak{C}^k$ converges to $\mathfrak{C} = \begin{array}{c} 0 \\ \mathfrak{C}' \quad \mathfrak{C}'' \end{array}$, Lemma 5.13 shows that $\mathfrak{G}^k$ converges to (say) $\mathfrak{C}'$ and $\mathfrak{H}^k$ converges to $\mathfrak{C}''$. Furthermore, the same lemma shows that $\mathfrak{M}^k$ converges to $H^v\left(\begin{array}{c} 1 \\ \mathfrak{C}'' \quad \mathfrak{C}'' \end{array}\right)$.

Going back to $\mathfrak{B}^k$, we finally get:

(1) if $v = 1$, there is a unique preimage for $\mathfrak{A}$, it is $\begin{array}{c} 0 \\ \mathfrak{A} \quad \mathfrak{B} \end{array}$ with $\mathfrak{B} = H^u\left(\begin{array}{c} 0 \\ \mathfrak{D} \quad \mathfrak{C} \end{array}\right)$, where $\mathfrak{D} = H^v\left(\begin{array}{c} 1 \\ \mathfrak{C}'' \quad \mathfrak{C}'' \end{array}\right)$;

(2) if $2 \leq v < +\infty$, then $\mathfrak{A}$ has two preimages. The same as for the case $v = 1$ plus $\begin{array}{c} 0 \\ \mathfrak{A} \quad \mathfrak{B}' \end{array}$ with $\mathfrak{B}' = H^u\left(\begin{array}{c} 0 \\ \mathfrak{C} \quad \mathfrak{C} \end{array}\right)$.

Now, we deal with the case $v_k \to +\infty$. In that case, $\mathfrak{A} = H^u\left(\begin{array}{c} 1 \\ \mathfrak{J} \quad \mathfrak{J} \end{array}\right)$. The same work as for the case $v > 1$ can be done, except that we finally consider $v \to +\infty$. Hence,

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we get two preimages for $\mathfrak{A}$. The tree $\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathfrak{A} & & \mathfrak{B} \end{array}$ with $\mathfrak{B} = H^u\left(\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathfrak{J}' & & \mathfrak{J} \end{array}\right)$ and the tree $\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathfrak{A} & & \mathfrak{B}' \end{array}$ with $\mathfrak{B}' = H^u\left(\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathfrak{J} & & \mathfrak{J} \end{array}\right)$. □

PROPOSITION 6.9. $T^{-1}(\mathfrak{J}') = T_a^{-1}(\mathfrak{J}') = \left\{\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathfrak{J}' & & \mathfrak{J} \end{array}\right\}$.

Proof. By Corollary 4.5, there is no ω such that $T_\omega(\mathfrak{J}) = \mathfrak{J}'$. Therefore, preimages of $\mathfrak{J}'$ are the accumulation points of preimages of $\mathfrak{A}^k := T_{\omega_k}(\mathfrak{J})$ such that $\lim_{k\to+\infty} \mathfrak{A}^k = \mathfrak{J}'$.

Let us consider such a sequence, and set $\omega_k = 2^{u_k}(2m_k + 1)$. By Lemma 5.8, $\lim_{k\to+\infty} u_k = +\infty$.

If we re-employ notation from the beginning of the proof of Proposition 6.8, we will see in $\mathfrak{J}$ the configuration

$$\begin{array}{ccccccc} & & & \odot & & & \\ & & \swarrow & & \searrow & & \\ & & 1 & & & & 0 \\ & \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathfrak{B}^k & & \mathfrak{B}^k & & \mathfrak{A}^k & & \mathfrak{B}^k \end{array}.$$

By Lemma 5.1, $\mathfrak{A}^k$ starts as $H^{u_k}(1)$ whereas $\mathfrak{B}^k$ starts as $H^{u_k}(0)$. Then, by Lemma 5.6, $\lim_{k\to+\infty} \mathfrak{B}^k = \mathfrak{J}$. This finishes the proof. □

6.5. *Proof of Theorem B.* The proof is immediate. It follows from Propositions 6.3, 6.2, 6.4, 6.5, 6.8, and 6.9 that for any $\mathfrak{A} \in X$,

$$p(1, \mathfrak{A}) = \#T^{-1}(\mathfrak{A}) \cap X \leq 3.$$

Hence, it yields $p(n, \mathfrak{A}) \leq 3^n$.

7. *Some more results*

Here we present some examples from grammars others than BBAB.

7.1. *Thue–Morse case.* Define the function H in such a way that

$$0 \mapsto \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 1 & & 1 \end{array} \quad \text{and} \quad 1 \mapsto \begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ 0 & & 0 \end{array}.$$

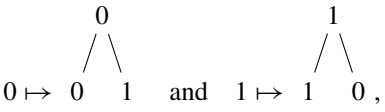
Using the recurrence, it is easy to see that fixed points for H are 'grammar-free': each one of the fixed points ${}_0\mathfrak{A}$ and ${}_1\mathfrak{A}$ has each generation with only one symbol. Thus, there is a natural projection π from ${}_0\mathfrak{A}$ and ${}_1\mathfrak{A}$ to $\{0, 1\}^{\mathbb{N}}$, commuting the dynamics. Moreover,

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$\pi \circ H(0) = 01$ and $\pi \circ H(1) = 10$. This means that $\pi \circ H$ is the classical Thue–Morse sequence.

Hence, $\overline{\{T_\omega({}_0\mathfrak{A}),\ \omega \in \mathbb{F}_2^+\}} = \overline{\{T_\omega({}_1\mathfrak{A}),\ \omega \in \mathbb{F}_2^+\}}$ is a minimal dynamical system, although it does not really exploit the tree structure.

7.2. *Proof of Theorem 1.4.* We consider the substreetution given by



equipped with the grammar ABBA. It is marked and therefore admits two fixed points, ${}_0\mathfrak{A}$ and ${}_1\mathfrak{A}$, respectively with root 0 and 1.

The renormalization equation is

$$T_a T_a H = H T_a = T_b T_b H \quad \text{and} \quad T_a T_b H = T_b T_a H = H T_b, \tag{5}$$

and then the source is given by

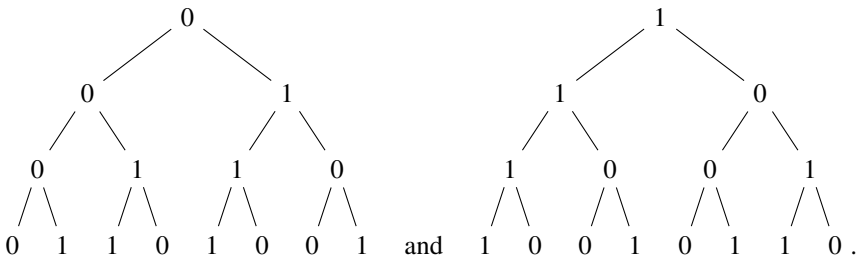
$$\begin{cases} s(aa) = s(bb) = a, \\ s(ab) = s(ba) = b. \end{cases}$$

More generally, we have

$$s(p_1 p_2 \dots p_{2n}) = s(p_1 p_2) \dots s(p_{2n-1} p_{2n}). \tag{6}$$

Using the identification $a = 0$ and $b = 1$, we can write simply $s(xy) = (x + y) \bmod (2)$.

The images by H^2 of roots 0 and 1 are



In the following, $\mathfrak{D}$ stands for ${}_0\mathfrak{A}$ or ${}_1\mathfrak{A}$. We note the following equalities:

$$H(\mathfrak{D})_e = \mathfrak{D}_e,\ H(\mathfrak{D})_a = \mathfrak{D}_e,\ H(\mathfrak{D})_b = 1 - \mathfrak{D}_e = \bar{\mathfrak{D}}_e$$

(where $\bar{x} = 1 - x$). We remark that by *identifying a with* 0 *and b with* 1, the last two expressions above can be rewritten as

$$H(\mathfrak{D})_\omega = (\mathfrak{D}_e + \omega) \bmod (2) \quad \text{where } \omega = 0, 1.$$

By induction, we get for the odd generations,

$$\mathfrak{D}_{p_1 p_2 \dots p_{2n} a} = \mathfrak{D}_{p_1 p_2 \dots p_{2n}} \quad \text{and} \quad \mathfrak{D}_{p_1 p_2 \dots p_{2n} b} = \bar{\mathfrak{D}}_{p_1 p_2 \dots p_{2n}}. \tag{7}$$

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These expressions can be rewritten as

$$\mathfrak{D}_{p_1p_2\dots p_{2n}\omega} = (\mathfrak{D}_{p_1p_2\dots p_{2n}} + \omega) \bmod (2),$$

where we still make the identification $a = 0$ and $b = 1$.

LEMMA 7.1. *Writing* $\mathfrak{D} = {}_i\mathfrak{A}$, $i = 0, 1$, *then the following holds:*

$$\begin{aligned} \textit{for all } n \geq 1,\ \mathfrak{D}_{p_1p_2\dots p_n} &= (\mathfrak{D}_e + p_1 + p_2 + \cdots + p_n) \bmod (2) \\ &= (i + p_1 + p_2 + \cdots + p_n) \bmod (2). \end{aligned}$$

Proof. The proof is by induction. For $n = 1$, it is true from the definition of H.

Assume that it is valid for n; then $\mathfrak{D}_{p_1\dots p_k} = (\mathfrak{D}_e + p_1 + p_2 + \cdots + p_k) \bmod (2)$ for any $k = 1, \dots, n$.

Now we need to show that the expression is valid for $n + 1$. If $n + 1$ is odd, then we can use equation (7) to obtain

$$\mathfrak{D}_{p_1p_2\dots p_np_{n+1}} = (\mathfrak{D}_{p_1\dots p_n} + p_{n+1}) \bmod (2) = (\mathfrak{D}_e + p_1 + \cdots + p_n + p_{n+1}) \bmod (2)$$

as wished.

If, however, $n + 1$ is even, then $n + 1 = 2k$ and so we have, from equation (6), that

$$\begin{aligned} \mathfrak{D}_{p_1p_2\dots p_{n+1}} &= \mathfrak{D}_{p_1p_2\dots p_{2k-1}p_{2k}} = \mathfrak{D}_{f(p_1p_2)f(p_3p_4)\cdots f(p_{2k-1}p_{2k})} \\ &= (\mathfrak{D}_e + f(p_1p_2) + \cdots + f(p_{2k-1}p_{2k})) \bmod (2) \\ &= (\mathfrak{D}_e + p_1 + p_2 + p_3 + p_4 + \cdots + p_{2k-1} + p_{2k}) \bmod (2) \\ &= (\mathfrak{D}_e + p_1 + p_2 + \cdots + p_n + p_{n+1}) \bmod (2) \end{aligned}$$

as claimed, concluding the proof. □

The expression above allows us to write a generation from the previous one, just using that 0 gives rise to 01 (on the subsequent generation) and 1 gives rise to 10 (also on the subsequent generation). This gives us an easy recursive way to write the two fixed points of H in this case. It is interesting to notice that each new line corresponds to a step in the construction of the Thue–Morse sequence.

The fixed points ${}_0\mathfrak{A}$ and ${}_1\mathfrak{A}$ (whose roots are respectively 0 and 1) can be obtained writing the generation $n + 1$ from the generation n, and in this case, we have $T_a({}_0\mathfrak{A}) = T_b({}_1\mathfrak{A}) = {}_0\mathfrak{A}$ and $T_b({}_0\mathfrak{A}) = T_a({}_1\mathfrak{A}) = {}_1\mathfrak{A}$, showing that X is the periodic orbit $\{{}_0\mathfrak{A}, {}_1\mathfrak{A}\}$.

Despite the orbit being periodic, the ball $B({}_0\mathfrak{A}, 1)$ is not syndetic (see [**11**, Definition 5.1]) since for any n, $T_{ba^n}({}_0\mathfrak{A}) = {}_1\mathfrak{A} \notin B({}_0\mathfrak{A}, 1)$. Therefore, the system is not minimal.

7.3. *Proof of Theorem 1.5.* We give here an example of a periodic tree whose orbit does not support an invariant measure. The tree is not obtained by a substreetution but via another way: each line is obtained from the previous one by applying some map. The choice of the map depends on the parity of the line.

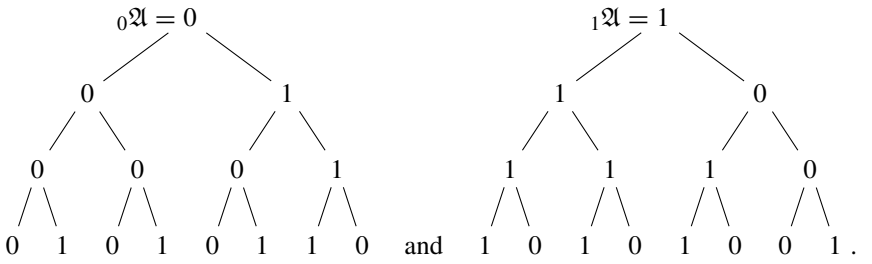
We set

$$\begin{cases} \mathcal{T}_1(0) = 01 \quad \text{and } \mathcal{T}_1(1) = 10, \\ \mathcal{T}_2(01) = 0001 \quad \text{and } \mathcal{T}_2(10) = 1110. \end{cases}$$

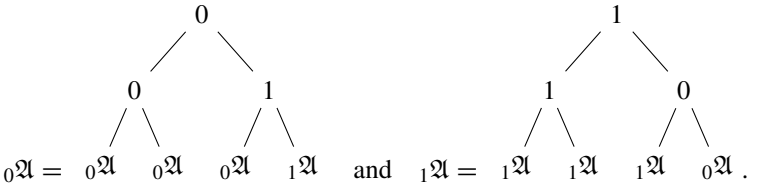
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We call ${}_0\mathfrak{A}$ and ${}_1\mathfrak{A}$ the two colored trees obtained by the following process with roots 0 and 1: an even line $2k$ is obtained applying $\mathcal{T}_2$ to the line $2k-1$ (gluing digit by pair); a line $2k+1$ is obtained by applying $\mathcal{T}_1$ to each digit of the line $2k$.

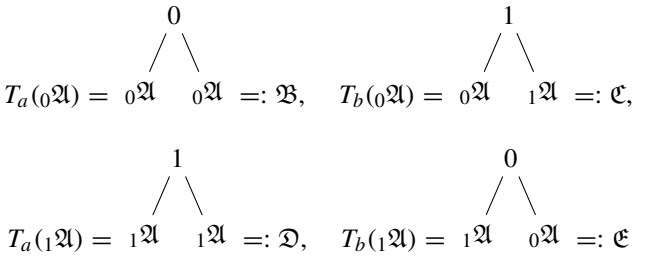
The fixed points start as



As for each even line, we apply $\mathcal{T}_1$, which depends only on one digit, then the subtree only depends on that site. Hence, at each even line, at each site with color 0, there is a ${}_0\mathfrak{A}$ and at each site of color 1, there is a ${}_1\mathfrak{A}$. This means that we have



Hence,



and

$$T_a(\mathfrak{B}) = T_b(\mathfrak{B}) = {}_0\mathfrak{A}, \quad T_a(\mathfrak{C}) = {}_0\mathfrak{A}, \quad T_b({}_0\mathfrak{A}) = {}_1\mathfrak{A}, \quad T_a(\mathfrak{D}) = T_b(\mathfrak{D}) = {}_1\mathfrak{A},$$
$$T_a(\mathfrak{E}) = {}_1\mathfrak{A}, \quad T_b(\mathfrak{E}) = {}_0\mathfrak{A}.$$

Then the orbit of ${}_0\mathfrak{A}$ is $\mathcal{O} = \{{}_0\mathfrak{A}, {}_1\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\}$ and $T_a(\mathcal{O}) \cup T_b(\mathcal{O}) = \mathcal{O}$ (see Figure 4).

We remind the reader that a measure μ on X is said to be an invariant probability measure if for every Borel set A, we have $\mu(A) = \mu(T_a^{-1}(A)) = \mu(T_b^{-1}(A))$. In particular, $\mu(\mathfrak{C}) = \mu(T_a^{-1}(\mathfrak{C})) = \mu(\emptyset) = 0$; similarly, we get $\mu(\mathfrak{B}) = \mu(\mathfrak{D}) = \mu(\mathfrak{E}) = 0$. Finally, $\mu({}_0\mathfrak{A}) = \mu(T_a^{-1}({}_0\mathfrak{A})) = \mu(\mathfrak{B} \cup \mathfrak{C}) = 0$ and $\mu({}_1\mathfrak{A}) = \mu(T_a^{-1}({}_1\mathfrak{A})) = \mu(\mathfrak{D} \cup \mathfrak{E}) = 0$, showing that we do not have an invariant probability on the periodic orbit $\mathcal{O}$.

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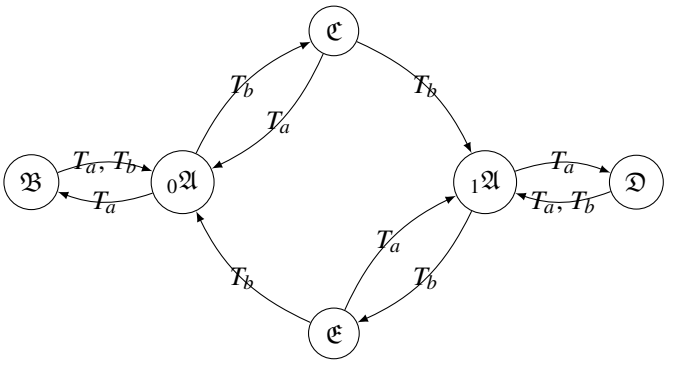


FIGURE 4. The graph for $\mathbb{F}_2^+$-action on $\mathcal{O}$.

We remind the reader that the classical Krylov–Bogolubov theorem (see [16] or [14, Theorem 4.1.1]) states that a continuous action of an amenable group G on a compact metrizable space X admits at least one invariant Borel probability measure; but here, the semi-group under consideration, $\mathbb{F}_2^+$, is non-amenable.

7.4. *Colored quasi-periodic tilings for hyperbolic disk.* In [3], it is proved that there does not exist a primitive substitutive tiling of the hyperbolic plane $\mathbb{H}^2$. Tilings studied there are geometric ones. Here, we show how our colored trees may generate colored tilings for the hyperbolic plane. The construction is done in the hyperbolic disk.

In the following, quasi-periodic means it is not periodic but repetitive: any patch is seen infinitely many times with bounded number of iterations to see repetition of the patch.

The following example of 'ping-pong' in the hyperbolic disk has kindly been given to us by M. Peigné.

We consider in $\mathbb{D}^2$ the isometry h_1 which maps $-\frac{1}{2}$ to $\frac{1}{2}$ and let ± 1 be fixed (we use the canonical embedding of $\mathbb{D}^2$ into $\mathbb{C}$). We also consider the isometry h_2 which maps $-i/2$ to $i/2$ and let $\pm i$ be fixed.

The Dirichlet domain (see [8]) is obtained by considering the intersection of the half-planes

$$\{z \in \mathbb{D}^2, d(z, 0) \leq d(z, \gamma 0)\},$$

with $\gamma = h_i^{\pm 1}$, $i = 1, 2$.

We consider the half-planes $\{z \in \mathbb{D}^2, d(z, 0) \leq d(z, \gamma 0)\}$, with $\gamma = h_i^{\pm 1}$, $i = 1, 2$. They are denoted by $\mathcal{P}_w$, $\mathcal{P}_e$, $\mathcal{P}_s$, and $\mathcal{P}_n$ as in Figure 5. We set $\mathcal{P}_0 := \mathbb{D}^2 \setminus \bigcup_{i=w,e,s,n} \mathcal{P}_i$. Hence, $\mathcal{P}_0$ is the Dirichlet domain with which we deal.

Furthermore, we identify $0 \sim e$, $a \sim h_1$, and $b \sim h_2$.

Now, for any given $\mathfrak{A} \in X$, we construct a colored tiling for $\mathbb{D}^2$ in the following way: for each $\omega \in \mathbb{F}_2^+$, the fundamental domain that contains ω in $\mathbb{D}^2$ (up to identification) has the color of the site $\mathfrak{A}_\omega$. Each half-plane $\omega.b(\mathcal{P}_w)$ (respectively $\omega.a(\mathcal{P}_s)$) has the same color as $\mathfrak{A}_{\omega.b}$ (respectively $\mathfrak{A}_{\omega.a}$).

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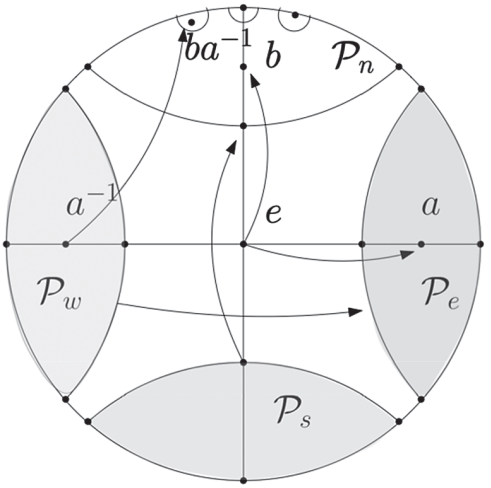


FIGURE 5. Colored tiling from ping-pong in $\mathbb{D}$.

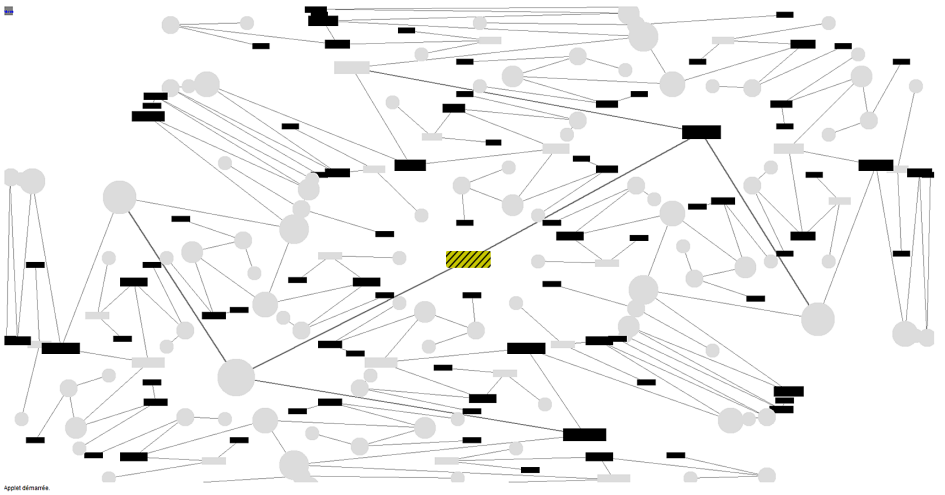


FIGURE 6. The Jacaranda tree: black = 1, grey = 0, a-follower = rectangle, b-follower = disk, root = yellow + hatching.

This tiling only colors images $\omega.\mathcal{P}_0$ with $\omega \in \mathbb{F}_2^+$. We can also decide to color all different $\omega^{-1}.\mathcal{P}_0$ via the same principle. We can also use another tree $\mathfrak{B} \in X$, up to the condition that the color for $\mathcal{P}_0$ has to be fixed (either $\mathfrak{A}_e$ or $\mathfrak{B}_e$).

7.5. *Picture of Jacaranda tree.* The next picture has kindly been computed by F. Flouvat. We point out that the tree structure increases exponentially fast with the number of sites and this is a brake to compute the fixed point on a large number of generations. Graphic representation is also a challenge. Figure 6 used the algorithm introduced in [18].

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REFERENCES

[1] A. Baraviera, R. Leplaideur and A. Lopes. The potential point of view for renormalization. *Stoch. Dyn.* **12**(4) (2012), 1250005.
[2] S. Beckus, T. Hartnick and F. Pogorzelski. Symbolic substitution systems beyond abelian groups. *Preprint*, 2021, arXiv:2109.15210.
[3] N. Bédaride and A. Hilion. Geometric realizations of 2-dimensional substitutive tilings. *Q. J. Math.* **64**(4) (2013), 955–979.
[4] I. Benjamini and Y. Peres. Markov chains indexed by trees. *Ann. Probab.* **22**(1) (1994), 219–243.
[5] J. Berstel, L. Boasson, O. Carton and I. Fagnot. A first investigation of Sturmian trees. *STACS 2007: Proceedings 24th Annual Symposium on Theoretical Aspects of Computer Science (Aachen, Germany, February 22–24, 2007)*. Ed. W. Thomas and P. Weil. Springer, Berlin, 2007, pp. 73–84.
[6] H. Bruin and R. Leplaideur. Renormalization, thermodynamic formalism for quasi-crystals in subshifts. *Comm. Math. Phys.* **231** (2013), 209–247.
[7] H. Bruin and R. Leplaideur. Renormalization, freezing phase transition and Fibonacci quasicrystals. *Ann. Sci. Éc. Norm. Supér. (4)* **48**(fascicule 3) (2015), 739–763.
[8] F. Dal'Bo. *Geodesic and Horocyclic Trajectories (Universitext)*. Springer-Verlag, London, 2011; EDP Sciences, Les Ulis.
[9] W. Damm. The IO- and OI-hierarchies. *Theoret. Comput. Sci.* **20** (1982), 95–207.
[10] J. de Vries. *Elements of Topological Dynamics*. Springer, Dordrecht, 1993.
[11] D. B. Ellis, R. Ellis and M. Nerurkar. The topological dynamics of semigroup actions. *Trans. Amer. Math. Soc.* **353**(4) (2001), 1279–1320.
[12] P. Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics (Lecture Notes in Mathematics, 1794)*. Springer, Berlin, 2002.
[13] H. Furstenberg and B. Weiss. Markov processes and Ramsey theory for trees. *Combin. Probab. Comput.* **12** (2003), 547–563.
[14] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 2010.
[15] D. H. Kim, B. Lee, S. Lim and D. Sim. Quasi–Sturmian colorings on regular trees. *Ergod. Th. & Dynam. Sys.* **40**(12) (2020), 3403–3419.
[16] N. Krylov and N. Bogolubov. La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire. *Ann. of Math. (2)* **38** (1937), 65–113.
[17] C. Müllner and R. Yassawi. Automorphisms of automatic shifts. *Ergod. Th. & Dynam. Sys.* **41** (2021), 1530–1559.
[18] Q. V. Nguyen and M. L. Huang. Space optimized tree: a connection+enclosure approach for the visualization of large hierarchies. *Inform. Vis.* **2** (2003), 3–15.
[19] K. Petersen and I. Salama. Entropy on regular trees. *Discrete Contin. Dyn. Syst.* **40**(7) (2020), 4453–4477.
[20] F. Przytycki. Conical limit set and Poincaré exponent for iterations of rational functions. *Trans. Amer. Math. Soc.* **351**(5) (1999), 2081–2099.
[21] F. Przytycki, J. Rivera-Letelier and S. Smirnov. Equality of pressures for rational functions. *Ergod. Th. & Dynam. Sys.* **24**(3) (2004), 891–914.
[22] M. Queffelec. *Substitution Dynamical Systems—Spectral Analysis (Lecture Notes in Mathematics, 1294)*. Springer, Berlin, 2010.
[23] U. A. Rozikov. *Gibbs Measures on Cayley Trees*. World Scientific, Singapore, 2013.
[24] F. Spitzer. Markov random fields on an infinite tree. *Ann. Probab.* **3**(5) (1975), 387–398.

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