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Microislands and transport in tokamaks

E. MINARDI

Istituto di Fisica del Plasma del CNR, EUR-ENEA-CNR Association,
Via R. Cozzi, 53 20125 Milano Italy
(minardi@ifp.cnr.it)

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Abstract. Kadomtsev proposed a self-generation mechanism of microislands (whose half-width is smaller than the Larmor radius of the ions) based on momentum and energy exchange between the electrostatic field of the islands and the ions of the background tokamak plasma. In this paper we study the motion of the ions intersecting, within a Larmor rotation, the electric field in the interspace between adjacent microisland chains. After appropriate average over the ensemble of ions and island chains, the interaction with the electric field gives rise to a net slow cumulative drift of the assembly of ions in the outward direction of the major radius. This effect is the consequence of nonlinearity and toroidal geometry in the equation of motion. The outward motion is associated with a decrease of the perpendicular plasma kinetic energy and a concomitant increase of the island's electrostatic energy, which can be described consistently by a 'pumping term' in the energy integral of the magnetohydrodynamic reduced equations for the islands. The pumping term can be contrasted by anomalous dissipation due to resonant interaction between the background electrons and the island waves. A stationary system of self-sustained microisland chains can then be formed in the tokamak plasma preferentially in the region $q > 2$ of the safety factor. Thus, the plasma acquires a spongy structure, where the sponginess (that is to say, the density of packing of the island structures) is measured by the ratio, lower than one, between the island width and the distance between adjacent rational surfaces. This structure determines an electron heat conductivity, which is strictly related to the sponginess and to the macroscopic equilibrium. Examples of scalings of the electron confinement time, predicted in zero-dimensional approximation, are given for typical tokamak conditions.

1. Introduction

Microislands, that is to say, islands whose width is much smaller than the ion Larmor radius but larger than the electron Larmor radius, were originally studied by Smolyakov [1] and Kadomtsev [2] who considered that they could be a possible alternative source of anomalous transport in tokamaks (see also Rebut and Brusati [3], White and Romanelli [4]). The study was based on the reduced magnetohydrodynamic (MHD) equations combined with the adiabatic assumption on the ions. Recently, the effects of anomalous perpendicular resistivity and anomalous perpendicular electron heat conductivity on the microisland structure were studied by Mikhailovskii et al. [5].

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A prerequisite for microislands to be relevant for transport is the existence of some kind of self-generation mechanism, which could preserve the islands against dissipation, so as to maintain equilibrium between the formation of the islands and the diffusion effects. Possible mechanisms of self-excitation have been suggested by Kadomtsev [2] and Chatenet et al. [6].

Kadomtsev considered the exchange of momentum between the ions and the electric field of densely packed microisland chains and suggested that in toroidal geometry this could give rise to a systematic slow drift of the ions in the outward direction of the major radius. According to Kadomtsev and with a reasoning based on energy conservation and adiabatic invariance, energy can be pumped into the islands by this process. Moreover, the pumping could be regulated by the transport generated by the islands themselves through an automatic control of the density of packing of the island chains and of the related electric field.

As a first step for formulating a clear mathematical basis of the Kadomtsev's intuitions in a previous paper [7] we have studied the interaction of a suitably defined ensemble of ions with the electric field of a chain of microislands with the purpose of elucidating the dynamical mechanism of pumping. In a subsequent paper [8], the conditions for equilibrium between island pumping and anomalous dissipation were considered. In the present paper we revisit the previous work from the beginning, revise and clarify it in some significant conceptual and mathematical aspects, and present a theoretical frame that, while incorporating the essential ideas of Kadomtsev, provides an inedited point of view for the treatment of the transport processes in tokamaks.

The content of the paper is divided in the following sections. In Sec. 2 the equations describing the islands are reviewed in order to have the basis for the calculation in Sec. 3 of the electric field in the interspace between island chains localized on adjacent rational surfaces. The ratio S_{p} between the island width and the distance between adjacent rational surfaces emerges as the controlling parameter of the electric and magnetic configuration of the islands. In a plasma volume where $S_{\mathrm{p}} < 1$, the microisland chains maintain their individuality, but they are densely packed and the plasma acquires a spongy structure. The density of packing is measured by $S_{\mathrm{p}} < 1$ and for this reason S_{p} is called the *sponginess*.

In Sec. 4 the equation of motion of an ion intersecting within a Larmor gyration the electrostatic field in the interspace, is solved in tokamak geometry taking into account the nonlinear terms in the electric field. The calculation involves a number of Larmor gyrations associated with a time shorter than the pulse duration of the travelling system of microislands so that this system can be considered at rest during this time. Also, under conditions to be specified (Sec. 4.1), the adiabatic drifts of the ion can be neglected during the interaction time. Thus the ion–chain interaction gives rise to a slow shift in the outward direction of the major radius that cumulates systematically in the successive gyrations and in the successive pulses of the travelling islands (Secs 4.2 and 4.3). This result holds also for an assembly of ions in interaction with a random system of island chains, as is shown by introducing the concept of a 'test ion' (Sec. 4.4), namely an ion representative of the whole assembly. Indeed, after performing the appropriate ensemble averages, the effect on the motion of the random electric field persists, as a consequence of nonlinearity combined with toroidicity (Sec. 4.5). However, in contrast to the assumption in previous papers, [2, 8] the conservation of the magnetic moment is broken by the interactions during the gyromotion and is replaced by the conservation

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of the mean square value of the Larmor radius, averaged over many gyrations (Sec. 4.6).

In Sec. 5 the energy conservation in the isolated system background plasma-island chains is discussed. The ambipolar ion and electron outward motion is associated with a loss of perpendicular kinetic energy, which is compensated for by an increase of the electrostatic energy in the interspace between chains and consequent sustenance of the islands through the continuity equation. The energy increase can be contrasted by anomalous dissipation due to resonant interaction between the background electrons and the island waves, as discussed in Sec. 6. A stationary structure of microisland chains can then be formed controlled by the value of the sponginess and by the parameters of the zero-order plasma equilibrium.

The ion outward drift, when essential for the process of self-excitation and self-sustenance of the microisland structure, is too small to contribute to transport. However, transport can arise from exchange of momentum between the background electrons moving parallel to the unperturbed magnetic field and the waves of microisland chains. In Sec. 7 a simple random walk argument is applied for deriving an electron diffusivity related to the microisland structure and the effects on transport are discussed on the basis of a simple zero-dimensional treatment. One finds a strict interdependence between sponginess, transport and magnetic equilibrium. Finally, in Sec. 8 the predictive capabilities of the theory are illustrated by calculating the scalings of the electron confinement time in some practical cases.

2. Basic equations of the microislands

2.1. The reduced MHD description and the energy integral

The microislands are described by the reduced set of MHD equations combined with the assumption that the ions are in equilibrium with the Boltzmann distribution (up to first order in the electrostatic potential) as a consequence of the large scale of the Larmor radius with respect to the island width [2, 9]. Then the ion density can be written in the form

$$n_{\mathrm{i}} = n_0\left(1 - \frac{x}{L} - \frac{e\varphi}{T_{\mathrm{i}}}\right). \tag{2.1}$$

Here $x = r - r_{\mathrm{s}}$, r_{s} is the radial location of a rational surface, T_{i} is the temperature of the ions, $L^{-1} = (dn_{\mathrm{i}}/n_{\mathrm{i}}\,dx)_{r_{\mathrm{s}}}$ is the scale of the ion density inhomogeneity and e is the ion electric charge.

The electrons, which in contrast to the ions are not adiabatic, sustain the parallel current density $j_{\parallel}$ and follow the continuity equation

$$\frac{\partial n_{\mathrm{e}}}{\partial t} + \frac{\mathbf{E}\times\mathbf{B}}{B^2}\nabla n_{\mathrm{e}} = \frac{1}{e}\nabla_{\parallel} j_{\parallel}, \tag{2.2}$$

where n_{e} is constrained by the quasi-neutrality condition $n_{\mathrm{e}} = n_{\mathrm{i}} \equiv n$. The electron temperature is taken as uniform along $\mathbf{B}$

$$\mathbf{B}\cdot\nabla T_{\mathrm{e}} = 0, \tag{2.3}$$

while the electron pressure gradient along $\mathbf{B}$ is in equilibrium with the electric field and the friction force with ions

$$\frac{1}{n}\nabla(n_{\mathrm{e}}T_{\mathrm{e}})\cdot\mathbf{B} = -e\mathbf{E}\cdot\mathbf{B} + \eta\mathbf{j}\cdot\mathbf{B}. \tag{2.4}$$

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The magnetic and the electric fields in tokamak geometry are expressed conveniently by the relations

$$\mathbf{B} = B_\phi \mathbf{e}_\phi + \mathbf{e}_\phi \times \nabla\psi, \quad \mathbf{E} = -\nabla\varphi + \frac{1}{c}\mathbf{e}_\phi \frac{\partial\psi}{\partial t}, \tag{2.5}$$

where $\mathbf{B}$ and $\mathbf{E}$ satisfy the pre-Maxwell equations

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0, \quad \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \tag{2.6}$$

As is known [2, 9], the system of equations above can be reduced to two equations for ψ and φ

$$\begin{gathered} \frac{e}{T_\mathrm{i}}\left(\frac{\partial\varphi}{\partial t} - \frac{v_* \tau}{r}\frac{\partial\varphi}{\partial\theta}\right) = -\frac{c}{4\pi e n_0}\nabla \cdot \left(\frac{\mathbf{B}}{B}\Delta_\perp \psi\right), \\ \frac{\partial\psi}{\partial t} + \frac{v_*}{r}\frac{\partial\psi}{\partial\theta} = c\frac{(1+\tau)}{\tau}\frac{\mathbf{B}}{B} \cdot \nabla\varphi + c\eta\frac{\mathbf{j}\cdot\mathbf{B}}{B}, \end{gathered} \tag{2.7}$$

where $\tau = T_\mathrm{i}/T_\mathrm{e}$ and $v_* = cT_\mathrm{e}/eLB$. These equations admit an energy integral of the form [2, 9]

$$\begin{aligned} &\frac{1}{4\pi}\int_S \frac{\partial\psi}{\partial t}(\mathbf{e}_\phi \times \mathbf{B}) \cdot d\mathbf{S} + \frac{\partial}{\partial t}\int_V \left(\frac{(\nabla\psi)^2}{8\pi} + \frac{n_0}{2}(T_\mathrm{e} + T_\mathrm{i})\left(\frac{e\varphi}{T_\mathrm{i}}\right)^2\right) dV \\ &\quad = -\int_V \eta j^2 \, dV, \end{aligned} \tag{2.8}$$

where $d\mathbf{S}$ is directed outwards from V.

2.2. Solution for the microislands

We neglect for the moment the dissipation term in η and consider a stationary system of islands described by the helical coordinate $y = r(\theta - n\phi/m) - uv_* t$ and rotating with velocity uv_*. Let us introduce the new independent variables

$$\Psi(x, y) = \frac{r_\mathrm{s}}{B_\theta S}\left(\psi - \frac{nB_\phi r^2}{2mR}\right), \quad \Phi(x, y) = \frac{1+\tau}{\tau}\frac{e\varphi L}{T_\mathrm{e}}, \tag{2.9}$$

where $S = r_\mathrm{s} q'(r_\mathrm{s})/q(r_\mathrm{s})$ is the magnetic shear at the rational surface characterized by the poloidal and toroidal numbers m and n.

The normalized flux Ψ can be split into an equilibrium part and a part $\tilde{\Psi}$ pertaining to the island

$$\Psi = -\frac{x^2}{2} + \tilde{\Psi}(x, y). \tag{2.10}$$

Introducing this into (2.7) one obtains the following equations in the limits $(B_\theta/B)^2 \ll 1$, $|\nabla B/B| \ll |\nabla n/n|$, $\beta = 8\pi nT/B^2 \ll 1$ and $(r/R)^2 \ll 1$ (see [2]):

$$\begin{gathered} (1-u)\frac{\partial\Psi}{\partial y} + x\frac{\partial\Phi}{\partial y} - (\nabla\tilde{\Psi} \times \nabla\Phi)_z = 0, \\ -(\tau+u)\frac{\partial\Phi}{\partial y} - \frac{1}{\gamma_0}\left(x - \frac{\partial\tilde{\Psi}}{\partial x}\right)\frac{\partial}{\partial y}\Delta_\perp\tilde{\Psi} - \frac{1}{\gamma_0}\frac{\partial\tilde{\Psi}}{\partial y}\frac{\partial}{\partial y}\Delta_\perp\tilde{\Psi} = 0, \end{gathered} \tag{2.11}$$

where $\gamma_0 = 4\pi n_0 T_\mathrm{e}(r_\mathrm{s}/LSB_\theta)^2/(1+\tau)$.

The first equation in (2.11) is satisfied identically by the relation

$$\Phi = P(\Psi) + (u-1)x \tag{2.12}$$

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with

$$P(\Psi) = \begin{cases} (1-u)(2(\Psi-\Psi_s))^{1/2} & \text{for } \Psi < \Psi_s, \\ 0 & \text{for } \Psi > \Psi_s, \end{cases} \tag{2.13}$$

where Ψ_s is the value of Ψ on the separatrix, to be determined later. Taking (2.13) into account, one can see that the second equation in (2.11) is satisfied outside the island ($\Psi < \Psi_s$) by the relation

$$\Delta_\perp \tilde{\Psi} = -x\gamma_0(u+\tau)P'(\Psi) + \text{constant}, \tag{2.14}$$

while inside the island ($\Psi > \Psi_s$) one has

$$\Delta_\perp \tilde{\Psi} = G(\Psi), \tag{2.15}$$

where $G(\Psi)$ is an arbitrary function of Ψ. Thus, recalling (2.10) and (2.13) the following equations for Ψ are obtained

$$\Delta_\perp \Psi = \begin{cases} \Gamma\left(1 - \dfrac{x}{(2(\Psi-\Psi_s))^{1/2}}\right) - 1, & (\Psi < \Psi_s), \\ G(\Psi) - 1, & (\Psi > \Psi_s), \end{cases} \tag{2.16}$$

where the constant in (2.14) has been determined in order that the island current density vanishes at infinity and $\Gamma = \gamma_0(u+\tau)(u-1)$.

The simplest choice for $G(\Psi)$ is the linear function $G(\Psi) = -k^2\Psi$. Recalling (2.10) this gives the following equation for $\tilde{\Psi}$

$$\Delta_\perp \tilde{\Psi} + k^2\tilde{\Psi} = \frac{k^2x^2}{2}. \tag{2.17}$$

Choosing a solution even with respect to x, one finds

$$\tilde{\Psi}(x,y) = \frac{w^2}{4}\cos ky + O\left(\frac{k^2x^4}{24}\right) \tag{2.18}$$

so that one recovers in the limit $(kx)^2x^2/6 \ll w^2$ the usual form of $\tilde{\Psi}$ where $k = m/r_s$ and w is the half-width of the island described by (2.10) and (2.18) with separatrix at $\Psi_s = -w^2/4$.

In the outside region of the island $\tilde{\Psi}$ depends on the velocity u through the parameter Γ. In order to determine u as well as $\tilde{\Psi}$ we assume that the magnetic energy pertaining to the island, which is proportional to $(1/8\pi)\int_{V_{out}} (\nabla\tilde{\Psi})^2\, dV$, is minimum. According to the Dirichlet principle, this implies that $\Delta_\perp\tilde{\Psi} = 0$ and consequently $\Gamma = 0$ so that $u = 1$ (rotation in the electron diamagnetic direction) or $u = -\tau$ (rotation in the ion diamagnetic direction). The solution for $\tilde{\Psi}$ vanishing at infinity is $\tilde{\Psi} = (kx)^{1/2}K_\nu(kx)\cos ky$ where $\nu = \frac{1}{2}$ and $K_\nu(kx)$ is the modified Bessel function that behaves asymptotically as $\exp(-kx)/(kx)^{1/2}$.

3. The electric field in the interspace between island chains

We consider a toroidal plasma filled by densely packed filaments localized on adjacent rational surfaces and looking in the poloidal cross section of the torus is like island chains centred on these surfaces. In terms of the parameter $S_p \equiv 2w/d$, where $d \approx 1/kS$ is the separation distance between adjacent rational surfaces, one

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has that for $S_p \geqslant 1$ the islands overlap, their individuality is lost and the magnetic field becomes chaotic. In the case $S_p < 1$ considered presently, the individuality of the islands is preserved but the volume covered by the densely packed chains exhibits a kind of spongy structure. The *sponginess* is measured by $S_p < 1$: the closer the chains are, the higher the sponginess.

We are interested in the poloidal component of the electric field in the inter-space between two adjacent chains. As is seen from (2.12) and (2.13) the poloidal electric field vanishes inside the islands and it exists outside the islands only in the case $u = -\tau$, that is to say, for island chains rotating in the ion diamagnetic direction.

Let us consider an island located on a rational surface at $z = 0$, where $z = (r - r_s)/w$ and calculate the normalized electric potential Φ_1 outside the island. Noting that $kw = S_p/2S$, putting $\xi = ky = m\theta - n\phi + k\tau v_* t$ and taking into account (2.10), one has the relations

$$\Phi_1 = (1+\tau)w\big(\Delta_1^{1/2} - z\big), \tag{3.1}$$

where

$$\Delta_1(z,\xi) = \frac{1}{2} + z^2 - \frac{e^{-zS_p/2S}}{2}\cos\xi. \tag{3.2}$$

This potential should be superimposed to that generated by the island chain centred at the adjacent rational surface with poloidal number $m+1$ located at the distance $d = 2w/S_p$ from $z = 0$. In the following we consider the inter-space comprised in the interval $0 \leqslant \xi \leqslant 2\pi$ or $0 \leqslant \theta \leqslant 2\pi/m$ for one chain and $0 \leqslant \theta \leqslant 2\pi/(m+1)$ for the adjacent one. As m is a very large number we can neglect the phase difference arising in the interval above from the different θ-dependence. However, in general there will be an intrinsic random phase difference between the two chains. Thus, the potential generated at the same point z by the adjacent chain is obtained from (3.1) and (3.2) after the substitution $z \to z - d/w = z - 2/S_p$:

$$\Phi_2 = (1+\tau)w\left(\Delta_2^{1/2} - \left(z - \frac{2}{S_p}\right)\right), \tag{3.3}$$

$$\Delta_2(z,\xi,\delta) = \frac{1}{2} + \left(z - \frac{2}{S_p}\right)^2 - \frac{e^{-|z-2/S_p|S_p/2S}}{2}\cos(\xi+\delta), \tag{3.4}$$

where δ is the phase difference between the adjacent island chains. The separatrix of the two chains is given by $\Delta(z_1,\xi) = 0$ and $\Delta_2(z_2,\xi,\delta) = 0$, respectively, and when the two chains are considered independently, the potential Φ in the interspace $z_1 \leqslant z \leqslant z_2$ will be the sum of Φ_1 and Φ_2. Thus, the poloidal electric field $E_Y = -\partial\varphi/\partial(r\theta) = -(T_i/(1+\tau)Le)k\partial\Phi/\partial\xi$ is expressed by the equality

$$E_Y(z,\xi,\delta) = \frac{T_i kw}{4Le}\Phi_\xi(z,\xi,\delta) = \frac{T_i S_p}{8LeS}\Phi_\xi(z,\xi,\delta), \tag{3.5}$$

where

$$\Phi_\xi(z,\xi,\delta) = \frac{e^{-z(S_p/2S)}}{\Delta_1^{1/2}}\sin\xi + \frac{e^{-|z-2/S_p|(S_p/2S)}}{\Delta_2^{1/2}}\sin(\xi+\delta). \tag{3.6}$$

The formulas above show that the sponginess S_p is an important controlling parameter of the electrostatic configuration.

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A quantity that will be of interest in the following (Sec. 7) is the total electrostatic energy in the interspace, averaged over the phase difference δ

$$U(S_\mathrm{p}, S) = \frac{1}{8\pi}\int_0^{2\pi} R\,d\phi \int_0^{2\pi} r_\mathrm{s}\,d\theta \int_0^{2\pi} \frac{d\delta}{2\pi} \int_{z_1(\xi)}^{z_2(\xi,\delta)} E^2(z,\xi,\delta) w\,dz$$

$$= \frac{R r_\mathrm{s} w}{4}\int_0^{2\pi} d\xi \int_0^{2\pi} \frac{d\delta}{2\pi} \int_{z_1(\xi)}^{z_2(\xi,\delta)} E^2(z,\xi,\delta)\,dz, \tag{3.7}$$

where $E^2 = E_X^2 + E_Y^2$. One sees from the expression of Φ that U is controlled by the sponginess and by the magnetic shear.

4. How the ion–chain interaction in toroidal geometry generates a slow systematic outward drift of the ions along the major radius

4.1. Ion–chain interaction

We wish to study the interaction of the ions with the electric field generated by the microislands. We start by considering the simplest case of a single ion whose guiding centre at an initial instant of time is situated just at the central point of the minimum distance between the separatrices of the two adjacent chains. This distance is $\Delta w = d - 2w = 2w(1 - S_\mathrm{p})/S_\mathrm{p}$. We assume that the Larmor radius of the ion is larger than the island half-width w and larger than the interspace $\Delta w/2$. During one Larmor gyration the ion will intersect twice the interspace between island chains, interacting twice with the electric field. We shall ignore, for simplicity, the interactions in adjacent interspaces, which occur when the Larmor radius is sufficiently large. The ion will also interact with the electric field inside the islands, which is a uniform field in the x-direction, according to (2.12) and (2.13). The ion will experience this field in opposite directions during one Larmor rotation and therefore no exchange of momentum is expected in the average at least in lowest order of the electric field. The effects on the ion motion that we are going to study are due entirely to second-order terms in the non-uniform y-component of the electric field, which exists only outside the islands. Thus, for simplicity, we shall treat only the interaction with the electric field in the interspace, without considering explicitly the field inside the island, although it can be included formally in the following calculations through the appropriate definition of the x-component of the electric field.

We consider a number N of Larmor gyrations such that $t_N = 2\pi N/\omega_\mathrm{c}$ (where $\omega_\mathrm{c} = eB_0/Mc$) is shorter than the pulse duration $\pi/\tau k v_*$ of the travelling system of microislands, so that this system can be considered to be at rest for $t < t_N$. Also, we neglect the parallel motion of the guiding centre at zero order during t_N. Then, let $\theta_0(t) = v_\phi t/qR_0 + \theta$ describe the poloidal component of the parallel motion, where v_ϕ is the toroidal velocity. If N and v_ϕ are such that $t_N = 2\pi N/\omega_\mathrm{c} \ll qR_0/v_\phi$, the value $\theta_0(t_N)$ at t_N can be identified with θ. Similarly, let $\Delta r_N = -\rho_\mathrm{i}^2 \omega_\mathrm{c} t_N \sin\theta/2R_0 = -\pi N \rho_\mathrm{i}^2 \sin\theta/R_0$ be the radial shift of the ion at $t = t_N$ due to the inhomogeneity of the toroidal magnetic field. This shift can be neglected with respect to the half-width w if $N \sin\theta \ll R_0 w/\pi\rho_\mathrm{i}^2$, which is well satisfied for ions localized around $\sin\theta = 0$, which are just the ions for which the effects we are going to discuss will be important.

We now proceed to calculate the effect of N interactions of the ion with the electric field, noting that, notwithstanding the limitations above, N can still be

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a large number, $N \gg 1$. We take local rectangular coordinates whose origin is at the position of the guiding centre r_g at the initial instant of time, with the X-axis oriented in the outward radial direction and the Y-axis in the poloidal direction. The ion rotates remaining in the X, Y plane within the N rotations, in accordance with our approximation scheme. We assume that the effect of the electrostatic field on the ion motion is much larger than the effect of the magnetic perturbation of the islands, so the magnetic field is taken as unperturbed. Thus the two components of the equation of motion of the ion in the X, Y plane are

$$\begin{aligned} X'' &= \frac{eE_X}{M} + \omega_c Y' \left(1 - \frac{\cos\hat{\theta}(t)}{R_0}(r_g + X(t))\right) - \Omega_\theta v_\phi, \\ Y'' &= \frac{eE_Y}{M} - \omega_c X' \left(1 - \frac{\cos\hat{\theta}(t)}{R_0}(r_g + X(t))\right). \end{aligned} \tag{4.1}$$

Here $\Omega_\theta = eB_\theta/Mc$ B_θ is the poloidal component of the magnetic field ($B_\theta^2 \ll B_0^2$) and $\hat{\theta}(t) = Y(t)/r_g + \theta$ describes the poloidal displacement including the contribution of the gyro-motion.

Let $t_{n-1} = \pi(n-1)/\Omega_{n-1}$ be the time just before the nth crossing and $\Delta t_n = \Delta w/v_{\perp,n}$ the duration of the crossing. Here $v_{\perp,n}$ is the perpendicular ion velocity and Ω_{n-1} is the cyclotron frequency at the time t_{n-1} (apart from small corrections of higher order in ρ_i/r_g to be considered later)

$$\Omega_{n-1} = \omega_c \left(1 - \frac{r_g}{R_0}\cos(\hat{\theta}(t_{n-1}))\right), \tag{4.2}$$

where r_g is the position of the guiding centre at $t = t_0$.

In the interval

$$\frac{\pi(n-1)}{\Omega_{n-1}} + \Delta t_n \leqslant t \leqslant \frac{\pi n}{\Omega_n} \tag{4.3}$$

the ion rotates freely after the interaction at the time $t_{n-1} + \Delta t_n$ until at the time $\pi n/\Omega_n$ it starts a new interaction while crossing the interspace at a distance $\Delta Y = 2\rho_i$ from the location of the previous interaction.

Equations (4.1) are integrated during each interaction taking as initial conditions at the nth interaction the values of X, X', Y, Y', which result from the previous interactions starting from $X_0 = 0$, $Y_0' = r_g\theta_0' = v_\phi r_g/qR_0$. The values of X_n and Y_n are then expressed in terms of the previous $n-1$ contributions [7].

The effect we are going to discuss arises from the terms $Y'X\cos\hat{\theta}/R_0$ and $X'X\cos\hat{\theta}/R_0$ of (4.1). It is therefore a combined toroidal and nonlinear effect. The second term gives rise to a quadratic term in X in the expression of Y_n'. With the approximation $t_n = \pi n/\Omega_n^{(0)} \cong \pi n/\Omega_{n-1}^{(0)}$ one finds [7]

$$Y_n'(t) = A_n - \Omega_{n-1}X_n(t) + \tfrac{1}{2}\omega_c\gamma_{n-1}X_n^2(t) + Y_0' + c_n. \tag{4.4}$$

Here $\gamma_{n-1} = \cos\hat{\theta}(t_{n-1})/R_0$, c_n is determined in order that $Y_0' + c_n$ describes the poloidal component of the parallel motion at the time $t = t_{n-1}$, namely $Y_0' + c_n = v_\phi\Omega_\theta/\Omega_{n-1}$; A_n describes the exchange of momentum between the ion and the electric field after n interactions

$$A_n = \sum_{j=1}^{n} \alpha_j \quad \text{where } \alpha_j = \frac{e}{M}\int_{t_{n-1}}^{t_{n-1}+\Delta t_n} E_Y(z_j(t'), \xi_j)\, dt' = \frac{ew}{M}\int_{z_{1,j}}^{z_{2,j}} \frac{E_Y(z, \xi_j)}{v_{\perp,j}}\, dz. \tag{4.5}$$

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The expression (4.4) for Y' is substituted in the first equation of (4.1) to obtain an equation for X up to order X^2

$$X_n'' + \Omega_{n-1}^2 X_n = \frac{eE_X}{M} + \Omega_{n-1} A_n - \omega_c \gamma_{n-1} A_n X_n + \frac{3}{2}\omega_c \Omega_{n-1} \gamma_{n-1} X_n^2 - \omega_c \gamma_{n-1} X_n (Y_0' + c_n). \tag{4.6}$$

This equation is treated introducing an expansion with respect to the formal parameter $\varepsilon = \rho_i / r_g$

$$\begin{aligned} X &= X^{(0)} + \varepsilon X^{(1)} + \cdots, \\ Y &= Y^{(0)} + \varepsilon Y^{(1)} + \cdots. \end{aligned} \tag{4.7}$$

Also the cyclotron frequency Ω_{n-1} is redefined according to the expansion

$$\Omega_{n-1} = \Omega_{n-1}^{(0)} + \varepsilon \Omega_{n-1}^{(1)} + \varepsilon \omega_{n-1}^{(1)} + \cdots, \tag{4.8}$$

where, up to first order in ε one has

$$\begin{aligned} \Omega_{n-1}^{(0)} + \varepsilon \Omega_{n-1}^{(1)} &= \omega_c \left(1 - \frac{r_g(t)}{R_0} \cos\left(\theta_0(t) + \varepsilon \frac{\rho_i}{r_g(t_n)} \cos \omega_c t\right)\right)_{t=t_{n-1}} \\ &= \left(\omega_c \left(1 - \frac{r_g(t)}{R_0} \cos \theta_0(t)\right) + \varepsilon \omega_c \frac{\rho_i}{R_0} \sin \theta_0(t) \cos \omega_c t\right)_{t=t_{n-1}}. \end{aligned} \tag{4.9}$$

The first-order frequency shift $\omega_{n-1}^{(1)}$ will be determined in order to eliminate the secular terms and recover the periodic Larmor motion in the limit of vanishing electric field and vanishing drifts (Sec. 4.3).

4.2. The zero-order solution $X_n^{(0)}$

The zero order is defined by the equation

$$X_n''^{(0)} + \left(\Omega_{n-1}^{(0)}\right)^2 X_n^{(0)} = \frac{eE_X}{M} + \Omega_{n-1}^{(0)} A_n. \tag{4.10}$$

This equation can be integrated in the interval $\pi(n-1)/\Omega_{n-1}^{(0)} \leqslant t \leqslant \pi n/\Omega_{n-1}^{(0)}$ (which includes the interval Δt_n of the crossing) obtaining

$$\begin{aligned} X_n^{(0)}(\tau) = &\left(X_{n-1}^{(0)} - \frac{1}{\left(\Omega_{n-1}^{(0)}\right)^2} \int_{\tau_{n-1}}^{\tau_{n-1}+\Delta\tau_n} \frac{eE_X(z_n(\tau'), \xi_n)}{M} \sin \tau' \, d\tau'\right) \cos \tau \\ &+ \left(\left(\frac{dX^0}{d\tau}\right)_{n-1} + \frac{1}{\left(\Omega_{n-1}^{(0)}\right)^2} \int_{\tau_{n-1}}^{\tau_{n-1}+\Delta\tau_n} \frac{eE_X(z_n(\tau'), \xi_n)}{M} \cos \tau' \, d\tau'\right) \sin \tau \\ &+ \frac{A_n}{\Omega_{n-1}^{(0)}}, \end{aligned} \tag{4.11}$$

where $\tau_{n-1} = \Omega_{n-1}^{(0)} t_{n-1}$. Up to first order in Δt_n one has

$$\begin{aligned} X_n^{(0)}(\tau) = &X_{n-1}^{(0)} \cos \tau \\ &+ \left(\left(\frac{dX^{(0)}}{d\tau}\right)_{n-1} + \frac{(-1)^{n-1}}{\Omega_{n-1}^{(0)}} \int_{\tau_{n-1}}^{\tau_n + \Delta\tau_{n-1}} \frac{eE_X(z_n(\tau'), \xi_n)}{M} \, d\tau'\right) \sin \tau \\ &+ \frac{A_n}{\Omega_{n-1}^{(0)}}, \end{aligned} \tag{4.12}$$

where $X_0^{(0)} = 0$ and $(dX^{(0)}/d\tau)_0 = v_{\perp,0}/\Omega_0^{(0)}$.

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We first consider the coefficients $X_{n-1}^{(0)}$ of $\cos\tau$. These are successively

$$\begin{aligned} 0 \leqslant t\Omega_0^{(0)} \leqslant \pi &: X_0^{(0)} = 0, \\ \pi \leqslant t\Omega_1^{(0)} \leqslant 2\pi &: A_1/\Omega_0^{(0)}, \\ 2\pi \leqslant t\Omega_2^{(0)} \leqslant 3\pi &: A_1/\Omega_0^{(0)} + A_2/\Omega_1^{(0)}, \\ 3\pi \leqslant t\Omega_3^{(0)} \leqslant 4\pi &: -A_1/\Omega_0^{(0)} - A_2/\Omega_1^{(0)} + A_3/\Omega_2^{(0)}. \end{aligned} \tag{4.13}$$

In general, after $N = 2n$ gyrations, recalling that $(\Omega_{n-1}^{(0)} - \Omega_{n+1}^{(0)})/\Omega_n^{(0)} \ll 1$, one can write

$$(N-1)\pi \leqslant t\Omega_{N-1}^{(0)} \leqslant N\pi : -(-1)^{N/2}\left(\frac{\alpha_1 - \alpha_3}{\Omega_2^{(0)}} + \cdots + \frac{\alpha_{N-3} - \alpha_{N-1}}{\Omega_{N-2}^{(0)}}\right). \tag{4.14}$$

This expression vanishes because all α_{2n-1} describe the interaction of the ion at the same point with the same electric field, the shift of the guiding centre and the change in the Larmor radius remaining imperceptible with respect to the island size during the N gyrations.

For the remaining terms of (4.12) one has, at $n = 1$ and 2,

$$\begin{aligned} X_1^{(0)}(\tau) &= \left(\frac{v_{\perp,0}}{\Omega_0^{(0)}} + \frac{\beta_1}{\Omega_0^{(0)}}\right)\sin\tau + \frac{A_1}{\Omega_0^{(0)}}, \\ X_2^{(0)}(\tau) &= \left(v_{\perp,0}\Omega_0^{(0)} + \frac{\beta_1}{\Omega_0^{(0)}} - \frac{\beta_2}{\Omega_1^{(0)}}\right)\sin\tau + \frac{A_2}{\Omega_1^{(0)}}, \end{aligned} \tag{4.15}$$

where

$$\beta_j = \int_{\tau_{n-1}}^{\tau_{n-1}+\Delta\tau_n} \frac{eE_X(z_j(\tau'),\xi_j)}{M}\,d\tau' = \frac{ew}{M}\int_{z_{1,j}}^{z_{2,j}} \frac{E_X(z,\xi_j)}{v_{\perp,j}}\,dz. \tag{4.16}$$

Note that the velocity before the nth intersection is the negative of the velocity after the intersection $n-1$. In general, one has

$$X_n^{(0)} = \rho_n \sin\Omega_{n-1}^{(0)} t + \frac{A_n}{\Omega_{n-1}^{(0)}}, \tag{4.17}$$

where

$$\rho_n = \rho_0 + B_n \tag{4.18}$$

with

$$\rho_0 = v_{\perp,0}/\Omega_0^{(0)} \quad \text{and} \quad B_n = \sum_{j=1}^{n} (-1)^{j-1}\beta_j/\Omega_{j-1}^{(0)}. \tag{4.19}$$

4.3. The first-order correction $X_n^{(1)}$

Substitution of $X = X^{(0)} + \varepsilon X^{(1)}$ into (4.6) gives an equation for $X^{(1)}$ of the form $X_n''^{(1)} + (\Omega_{n-1}^{(0)})^2 X_n^{(1)} =$ inhomogeneous terms [7]. In order to avoid secular terms, the coefficient of $\sin\Omega_{n-1}^{(0)} t$ in the right-hand side is put equal to zero by choosing $\omega_{n-1}^{(1)}$ as follows

$$\omega_{n-1}^{(1)} = -\Omega_{n-1}^{(1)} - \frac{\gamma_{n-1}}{2}(Y_0' + C_n) + \gamma_{n-1}A_n = -\frac{\gamma_{n-1}}{2}v_\phi\frac{\Omega_\theta}{\Omega_{n-1}^{(0)}} + \gamma_{n-1}A_n. \tag{4.20}$$

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In the limit of vanishing interaction ($A_n = 0$, $B_n = 0$) the solution should be periodic and should vanish for $t = t_{n-1}$. Thus, we find

$$X_n^{(1)}(t) = -v_\phi \frac{\gamma_{n-1}}{2\omega_c^2} \frac{\Omega_\theta}{\Omega_{n-1}^{(0)}} A_n + \frac{3}{2\omega_c^2}\gamma_{n-1}A_n^2 + \frac{3}{4}\gamma_{n-1}\rho_0^2 + \frac{3}{4}\gamma_{n-1}(2\rho_0 B_n + B_n^2)$$
$$+ \frac{\gamma_{n-1}\rho_n^2}{4}\cos 2\Omega_{n-1}^{(0)}t - (-1)^{n-1}\gamma_{n-1}\rho_0^2 \cos \Omega_{n-1}^{(0)}t. \tag{4.21}$$

4.4. The 'test' ion

Up to now we have considered a single ion in interaction with the electric field whose guiding centre is situated in the interspace Δw. However, in reality there is a whole ensemble of ions uniformly distributed in Δw along ξ, namely along two adjacent rational surfaces, which are interacting with the electric field. The ions of the ensemble will experience different values of the electric field depending on their random position ξ with respect to the field. Conversely, in a point of view in which a single ion experiences an electric field with random position and phase, the ion becomes a representative of all ions of the ensemble. That is to say, the ion becomes a 'test' ion when the electric field in the solutions (4.17) and (4.21) is replaced by appropriate averages over the phase shift δ and the position ξ of the field.

Let us call the location of the first interaction point 1 and the location of the second point 2. Let us put $\mathrm{E}_{\alpha,j} = \int_{z_1(\xi_j)}^{z_2(\xi_j,\delta)} E_\alpha(z, \xi_j, \delta)\, dz$ (where $\alpha = X$ or $\alpha = Y$) and let $\mathrm{E}_{\alpha,j} = \mathrm{E}_{\alpha,1}$ when j is odd and $\mathrm{E}_{\alpha,j} = \mathrm{E}_{\alpha,2}$ when j is even. The two points are physically equivalent, thus

$$\overline{\mathrm{E}_{\alpha,1}} = \overline{\mathrm{E}_{\alpha,2}} \quad \text{and} \quad \overline{\mathrm{E}_{\alpha,1}^2} = \overline{\mathrm{E}_{\alpha,2}^2}. \tag{4.22}$$

Although these averages do not depend on point 1 or point 2, the electric fields at the two points are correlated by a Larmor diameter $2\rho_n$ with $\xi_2 - \xi_1 = 2k\rho_n$ and the correlation $\overline{\mathrm{E}_{\alpha,2}\mathrm{E}_{\alpha,1}}$ will not vanish.

It is shown in Appendix A that $\overline{A_n} = 0$ and that

$$2\rho_0 \overline{B_n} + \overline{B_n^2} = 0 \tag{4.23}$$

up to second order in $E_{\alpha,j}$. Then the average of (4.21) becomes

$$\overline{X_n^{(1)}(t)} = \frac{3}{2\omega_c^2}\gamma_{n-1}\overline{A_n^2} + \frac{3}{4}\gamma_{n-1}\rho_0^2 + \frac{\gamma_{n-1}\rho_0^2}{4}\cos 2\Omega_{n-1}^{(0)}t - (-1)^{n-1}\gamma_{n-1}\rho_0^2 \cos \Omega_{n-1}^{(0)}t. \tag{4.24}$$

The first term in the right-hand side describes a shift of the ion in the outward radial direction that cumulates after n half-gyrations, while the sum of the other terms is a periodic function of the half-gyration (the n-dependence of γ is imperceptible). Strictly speaking this result holds for a number N of gyrations such that the shift remains smaller than Δw. For a larger number of gyrations the guiding centre moves off-centre of Δw, contrary to the assumption of Sec. 4.1. In this case the interspace will not be crossed perpendicularly to the ξ-axis at ξ_n, but the ion will follow a trajectory $\xi(t)$ inside Δw. This implies that the integrals in the expressions (4.5) and (4.16) for the α_j and β_j should be replaced by an integral along the trajectory of the form

$$\int_{t_{j-1}}^{t_{j-1}+\Delta t_j} E_\alpha(z(t), \xi(t), \delta)\frac{dz}{dt}\, dt.$$

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This would complicate the calculations considerably. Our presupposition, supported by calculations in a comparable context [10], is that the result (4.24), in particular the existence of a cumulative shift, is not essentially modified for off-centre ions and that the test ion in the centre is sufficiently representative of the assembly of all ions after averaging over the off-centre positions of the guiding centres. Truly, after N interactions the ion can no longer be considered as fixed to the same position in the interspace Δw, but it will eventually drift radially to an adjacent interspace where it will be submitted to the same cumulative interactions. Meanwhile, the drifted ion will be replaced in its old position by a new ion to which the outward drift will be equally impressed by the cumulative interactions and in fact, in view of its systematic character, impressed to all ions moving in the spongy region, irrespective of their actual position.

4.5. Ensemble averages

The cumulative shift has been expressed by (A 6) in Appendix A in terms of ξ and δ-averages as follows:

$$\begin{aligned}\bar{X} &= \frac{3}{2}\frac{\cos\theta(t_{n-1})}{R_0}\overline{A_n^2}\\ &= 3\frac{\cos\theta(t_{n-1})}{R_0}\frac{e^2w^2}{M^2v_{\perp,0}^2}\frac{N^2}{\omega_c^2}\overline{\left(\mathrm{E}_{Y,1}^2+\mathrm{E}_{Y,1}\mathrm{E}_{Y,2}\right)}\\ &= 3\frac{\cos\theta(t_{n-1})}{R_0}\frac{e^2w^2N^2}{M^2v_{\perp,0}^2\omega_c^2}\frac{1}{4\pi^2}\int_0^{2\pi}d\delta\int_0^{2\pi}d\xi\Bigg(\left(\int_{z_1(\xi)}^{z_2(\xi,\delta)}E_Y(z,\xi,\delta)\,dz\right)^2\\ &\quad+\int_{z_1(\xi)}^{z_2(\xi,\delta)}E_Y(z,\xi,\delta)\,dz\int_{z_1(\xi+2k\rho)}^{z_2(\xi+2k\rho,\delta)}E_Y(z,\xi+2k\rho,\delta)\,dz\Bigg).\end{aligned}\tag{4.25}$$

Here N can be taken as the number of gyrations $\pi\omega_c/2\pi\tau kv_*$ that the test ion performs after interacting with the travelling islands during a single pulse $\pi/\tau kv_*$. The velocity acquired after this interaction is then $\overline{X'}=(\tau kv_*/\pi)X$. Expressing the electric field with the help of (3.5) one obtains the following expression for the microscopic velocity acquired in the time $t_{n-1}=\pi/\tau kv_*$:

$$\overline{X'}=\frac{\cos(\theta(\pi/\tau kv_*))}{R_0}\frac{3}{512\pi^3}\frac{w^4\omega_c k}{L}(D+D_{\mathrm{corr}}),\tag{4.26}$$

where

$$\begin{aligned}D&=\frac{v_{\mathrm{th}}^2}{v_{\perp,0}^2}\int_0^{2\pi}d\delta\int_0^{2\pi}d\xi\left(\int_{z_1(\xi)}^{z_2(\xi,\delta)}\Phi_\xi(z,\xi,\delta)\,dz\right)^2\\ D_{\mathrm{corr}}(2k\rho)&=\frac{v_{\mathrm{th}}^2}{v_{\perp,0}^2}\int_0^{2\pi}d\delta\int_0^{2\pi}d\xi\Bigg(\int_{z_1(\xi)}^{z_2(\xi,\delta)}\Phi_\xi(z,\xi,\delta)\,dz\\ &\quad\times\int_{z_1(\xi+2k\rho)}^{z_2(\xi+2k\rho,\delta)}\Phi_\xi(z,\xi+2k\rho,\delta)\,dz\Bigg)\end{aligned}\tag{4.27}$$

and $v_{\mathrm{th}}^2=2T_{\mathrm{i}}/M$. These expressions are averaged over the Maxwellian distribution of the perpendicular velocity

$$\bar{D}=\frac{2}{v_{\mathrm{th}}^2}\int_{v_{\perp,\min}}^{\infty}e^{-v_{\perp,0}^2/v_{\mathrm{th}}^2}Dv_{\perp,0}\,dv_{\perp,0},\tag{4.28}$$

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$$\overline{D_{\rm corr}} = \frac{2}{v_{\rm th}^2}\int_{v_{\perp,\min}}^{\infty} e^{-v_{\perp,0}^2/v_{\rm th}^2} D_{\rm corr}(2kv_{\perp,0}/\omega_{\rm c}) v_{\perp,0}\, dv_{\perp,0}. \tag{4.29}$$

Here $v_{\perp,\min}$ follows from the basic limitation $\rho = v_\perp/\omega_{\rm c} > \Delta w/2 = w(1-S_{\rm p})/S_{\rm P}$. Thus, $v_{\perp,\min} = aw\omega_{\rm c}(1-S_{\rm p})/S_{\rm P}$ where a is a number slightly larger than one, say $a = 1.2$.

Finally, one should average over the Maxwellian distribution of the toroidal velocity v_ϕ, which enters through $\theta(\pi/\tau k v_*) = (v_\phi/qR_0)(\pi/\tau k v_*) + \theta$. One has [11]

$$\frac{1}{\pi^{1/2} v_{\rm th}}\int_{-\infty}^{+\infty} e^{-v_\phi^2/v_{\rm th}^2} \cos((v_\phi/qR_0)(\pi/\tau k v_*) + \theta)\, dv_\phi = C_1 \cos\theta, \tag{4.30}$$

where $C_1 = \exp -(\pi L\omega_{\rm c}/qR_0 k v_{\rm th})^2$.

In conclusion, the assembly of ions is subject to a microscopic drift velocity in the outward direction of the major radius described collectively by the expression

$$\overline{X'} = \frac{\cos\theta}{R_0} C_1 \frac{3}{512\pi^3}\frac{w^4\omega_{\rm c} k}{L}(\bar{D} + \overline{D_{\rm corr}}). \tag{4.31}$$

4.6. Constant of motion

The relations (4.18) and (4.23) imply that $\overline{\rho_n^2} = \rho_0^2$ up to second order in $\mathrm{E}_{\alpha,j}$. Thus, the Larmor radius changes after a single interaction, but its mean square value is conserved after averaging over many interactions. This conservation law replaces the conservation of the magnetic moment, which is in fact broken by the interactions within a Larmor rotation. One can verify that $\overline{\rho_n^2}$ can be expressed in terms of the mean square perpendicular velocity of the ion. Subtracting the part arising from the parallel motion from (4.4), one has

$$Y'_{\perp,n} = A_n - \Omega_{n-1}X_n + \tfrac{1}{2}\omega_{\rm c}\gamma_{n-1}X_n^2. \tag{4.32}$$

We now calculate $v_{\perp,n}^2 = Y_{\perp,n}'^2 + X_n'^2$ neglecting terms of order ρ/R_0 and $A_n^2/R_0^2\omega_{\rm c}^2$. Only $X_n^{(0)}$ given by (4.17) contributes to the calculation. We finally obtain

$$\frac{\overline{v_{\perp,n}^2}}{\left(\Omega_{n-1}^{(0)}\right)^2} = \overline{\rho_n^2} = \rho_0^2. \tag{4.33}$$

5. The pumping term

Recalling that $B_\phi \approx 1/R$, and expressing $\overline{v_{\perp,n}^2}$ in terms of temperature, (4.33) takes the form

$$T_{{\rm i},n} = T_{{\rm i},0}\left(1 - 2\frac{r_{\rm g} + \overline{X}_n}{R_0}\cos\theta\right). \tag{5.1}$$

Then, the cumulative drift involves a change per unit time of the perpendicular kinetic energy of the ions described by the equality

$$T_{\rm i}' = -T_{{\rm i},0}\frac{\overline{2X'}}{R_0}\cos\theta. \tag{5.2}$$

To see the change of the total kinetic energy of the ions in the interspace, we put $\overline{X'} = V\cos\theta$ and integrate over all ions in a volume $V_{\rm int}$ averaged over the phase

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shift δ. The integral can be written as follows

$$
\begin{aligned}
&-\int_{V_{\rm int}} n_0 T_{\rm i} \frac{2V(\cos\theta)^2}{R_0}\, dV_{\rm int} \\
&\quad = -\int_0^{2\pi} R_0\, d\phi \int_0^{2\pi} r\, d\theta \int_0^{2\pi} \frac{d\delta}{2\pi} n_0 T_{\rm i} \int_{z_1(\xi)}^{z_2(\xi,\delta)} \frac{2V(\cos\theta)^2}{R_0} w\, dz \\
&\quad = -\int_0^{2\pi} R_0\, d\xi \int_0^{2\pi} \frac{d\delta}{2\pi} n_0 T_{\rm i} \int_{z_1(\xi)}^{z_2(\xi,\delta)} w\, dz \int_0^{2\pi} r\, d\theta \frac{2V\cos\theta}{r}\left(1+\frac{r}{R_0}\cos\theta\right) \\
&\quad = -\int_{V_{\rm int}} n_0 T_{\rm i} \frac{2V\cos\theta}{r}\, dV_{\rm int}. \qquad (5.3)
\end{aligned}
$$

This relation can be transformed further to the more expressive form

$$
\begin{aligned}
-\int_{V_{\rm int}} n_0 T_{\rm i} \frac{2V\cos\theta}{r}\, dV_{\rm int} &= -\int_{V_{\rm int}} n_0 T_{\rm i} \frac{2\overline{X'}}{r}\, dV_{\rm int} \\
&= -\int_{n_0 T_{\rm i}} \frac{2}{r} \frac{\partial r \overline{X'}}{\partial r}\, dV_{\rm int} \\
&= -2 n_0 T_{\rm i} \int_{S_{\rm int}} \overline{X'}\, dS, \qquad (5.4)
\end{aligned}
$$

where $S_{\rm int}$ is the surface enclosing $V_{\rm int}$. Equation (5.4) gives the fraction of the ion perpendicular kinetic energy lost in $V_{\rm int}$ per unit time associated with the outward microscopic ion flux $n_0\overline{X'}$ across $S_{\rm int}$. However, in view of the quasi-neutrality, the ion flux should be accompanied by a similar flux of electrons, which will convey a loss $-n_0 T_{\rm e} \int_{S_{\rm int}} X'\, dS$ of electron perpendicular kinetic energy. This amounts to saying that the electrons, in contrast to the ions, conserve the magnetic moment in their quasi-neutral outward motion (as the loss above follows from $v_{\perp,{\rm e}}^2 \approx B \approx R^{-1}$). In conclusion one has a total loss $\Delta K'$ of plasma perpendicular kinetic energy per unit time in the interspace, given by the equality

$$
\Delta K' = -\int_{V_{\rm int}} (2n_0 T_{\rm i} + n_0 T_{\rm e}) \frac{V\cos\theta}{r}\, dV_{\rm int}. \qquad (5.5)
$$

Note that the parallel ion and electron kinetic energy remain unaffected in the process above because the parallel velocity does not change within the N gyrations, in accordance with our approximation scheme (Sec. 4).

In our calculation of the ion–chain interaction, the electrostatic field of the islands was treated as a given external field. However, the system ions–island chains form an isolated system and the energy lost by the ions should be found in the islands. This means that the ions should react back to the field, which can no longer be considered as a given external field, but is modified in order that the energy integral (2.8) of the islands includes a positive term $P_+ = -\Delta K'$ (a pumping term) in the right-hand side, thereby reestablishing the energy balance with the background plasma. This can be accomplished formally by adding a source term $\tilde{S}$ to the right-hand side of the continuity equation (2.7) of the electrons

$$
\frac{n_0 e^2}{T_{\rm i}} \left(\frac{\partial \varphi}{\partial t} - \frac{v_* \tau}{r} \frac{\partial \varphi}{\partial \theta} \right) = -\nabla_\parallel j_\parallel + e\tilde{S}. \qquad (5.6)
$$

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With this modification the energy integral (2.8) contains an additional term

$$\frac{1}{4\pi}\int_S \frac{\partial\psi}{\partial t}(\mathbf{e}_\phi \times \mathbf{B})\cdot d\mathbf{S} + \frac{\partial}{\partial t}\int_V \left(\frac{(\nabla\psi)^2}{8\pi} + \frac{n_0}{2}(T_\mathrm{e}+T_\mathrm{i})\left(\frac{e\varphi}{T_\mathrm{i}}\right)^2\right) dV$$
$$= \int_V (T_\mathrm{e}+T_\mathrm{i})\frac{e\varphi}{T_\mathrm{i}}\tilde{S}\, dV - \int_V \eta j^2\, dV \tag{5.7}$$

and the additional term becomes equal to P_+ when we put $\tilde{S} = n_0(e\varphi/T_\mathrm{i})V_\mathrm{D}\cos(\theta)/r$ and V_D is determined in order to satisfy the relation

$$\int_{V_\mathrm{int}} n_0(T_\mathrm{e}+T_\mathrm{i})\left(\frac{e\varphi}{T_\mathrm{i}}\right)^2 \frac{V_\mathrm{D}\cos\theta}{r}\, dV_\mathrm{int} = \int_{V_\mathrm{int}} (2n_0T_\mathrm{i} + n_0T_\mathrm{e})\frac{V\cos\theta}{r}\, dV_\mathrm{int} = P_+. \tag{5.8}$$

The formalism can be interpreted by saying that while the outward motion driven by the ions is quasi-neutral, some electrons are left behind in the interspace, so that a source of negative charge appears in the continuity equation of the electrons. Thus, the electrostatic energy in V_int is increased at the expense of the perpendicular kinetic energy of the ions and of the electrons.

6. Stationary system of microislands

The source term $\tilde{S}$ gives rise in linear approximation to an exponential growth of the islands [7,8] and is therefore the source of the instability that leads to the formation of the microislands. The driving effect of $\tilde{S}$ can be contrasted by the dissipative term, provided that it is sufficiently strong. As first observed by Kadomtsev [2] (see also [8]), classical resistivity is too weak to contrast the island growth and one is led to consider the effect of anomalous dissipation arising from the resonant interaction between the electrons of the background plasma and the electrons carrying the current inside the island wave. The electrons of the background with parallel velocity v_e, near the phase velocity $\tau v_* k/k_\parallel$ of the wave, exchange momentum with the wave at a rate $\upsilon_\mathrm{a} = v_e|k_\parallel|$ (where $k_\parallel = -kSx/qR_0$), equivalent to an anomalous collision frequency. An anomalous resistivity can correspondingly be defined by the equality [2]

$$\eta = \frac{m_\mathrm{e}\upsilon_\mathrm{a}}{n_0e^2} = \frac{m_\mathrm{e}v_\mathrm{e}}{n_0e^2}\frac{kS|x|}{qR_0}, \tag{6.1}$$

where $|x| < w$. The process involves the bulk of the electrons provided that $v_\mathrm{e} \approx \tau v_* k/|k_\parallel| \leqslant v_\mathrm{th}$. Combining this with $|x| < w$ and putting $\varepsilon_\mathrm{L} = L/R_0$ one has the condition

$$\frac{cm_\mathrm{e}^{1/2}}{2^{1/2}e}\frac{\tau T_\mathrm{e}^{1/2}q}{\varepsilon_\mathrm{L}BSw} \leqslant \left|\frac{x}{w}\right| < 1, \tag{6.2}$$

which should be satisfied for the resonant process to be efficient.

As is seen from (5.7) a stationary situation is realized when the pumping term equilibrates exactly the dissipation term, $P_- \equiv \int \eta\tilde{j}^2\, dV$. As there is no diffuse current in the inter-space (a fact related to the minimum of the magnetic energy there, as we know from Sec. 2) the equilibration can only occur through transfer of power by the Poynting flux between the power pumped in the interspace and the power dissipated inside the islands. Albeit both adjacent island chains are involved in the power transfer on the same footing, one can assume that the energy pumped in one interspace is equal in the average to the power dissipated inside one chain.

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Implicit in this assumption is that the adjacent chains are independent and have the same structure because the numbers m, n are very large and the difference between the chains can be neglected. Also, we have assumed that the test ion interacts with the electric field in a single interspace, neglecting for simplicity the interactions in adjacent interspaces when the Larmor radius is sufficiently large.

To proceed with the calculation of the dissipation term, we observe that, by presupposing the system of microislands as stationary, one can express $\tilde{j}$ with the relations (2.10) and (2.15) valid in the stationary situation. Thus

$$\tilde{j} = \frac{c}{4\pi}\frac{B_\theta S}{r}G(\Psi) = -\frac{c}{4\pi}\frac{B_\theta S}{r}k^2\Psi = \frac{c}{8\pi}\frac{B_\theta S k^2}{r}\left(x^2 - \frac{w^2}{2}\cos\xi\right). \tag{6.3}$$

It follows that

$$P_- \equiv \int_\Omega \eta\tilde{j}^2\, d\Omega$$

$$= \frac{c^2 m_\mathrm{e}}{(4\pi)^2 4e^2 n_0}\frac{v_{\mathrm{e,th}}k^5 w^6 B_\theta^2 S^3}{qr}\int_0^{2\pi} d\phi\int_0^{2\pi} d\theta\int_{-z_1(\xi)}^{+z_1(\xi)} |z|\cdot\left(z^2 - \frac{1}{2}\cos\xi\right)^2 dz. \tag{6.4}$$

By equating this to P_+ one arrives with little algebra to an expression for the half-width under stationary conditions

$$w = 1.737\times 10^{-20}\frac{q^3 n^2 T_\mathrm{e}^{1/2}(1+2\tau)}{S^3 B\varepsilon_\mathrm{L} A_\mathrm{i} k^4}\frac{C_2(S_\mathrm{p})}{I_-(S_\mathrm{p})}. \tag{6.5}$$

Here A_i is the mass number of the ion and

$$C_2(S_\mathrm{p}, k) = C_1\frac{\bar{D}(S_\mathrm{p}) + \bar{D}_\mathrm{corr}(S_\mathrm{p}, k)}{I_-(S_\mathrm{p})}\int_0^{2\pi} d\xi\int_0^{2\pi}\frac{d\delta}{2\pi}(z_2(\xi,\delta,S_\mathrm{p}) - z_1(\xi, S_\mathrm{p})), \tag{6.6}$$

$$I_-(S_\mathrm{p}) = 2\int_0^{2\pi} d\xi\int_0^{z_1(\xi,S_\mathrm{p})} z\left(z^2 - \frac{1}{2}\cos\xi\right)^2 dz, \tag{6.7}$$

where the dependence on the sponginess of quantities related to the microisland structure has been indicated explicitly.

In view of this dependence, the half-width is not yet determined by (6.5). This equation depends on the macroscopic magnetic configuration (B, q, S), on the density and its in-homogeneity $(n, \varepsilon_\mathrm{L})$ on the electron and ion temperature (T_e, τ) and on the actual structure of the system of microislands (S_p, k). This structure then remains undetermined even when the macroscopic equilibrium of the plasma is given. The point here is that the macroscopic equilibrium is expected to be determined by transport, which is determined in turn by the microislands themselves, so that the microisland structure and the macroscopic equilibrium cannot be considered separately, but form an integral system. It is therefore essential to investigate the relations between microislands, transport and macroscopic equilibrium. We are going to face this question in the next section.

7. Transport in zero-dimensional approximation

Let us consider the random walk of the electrons moving in the spongy structure formed by the stationary system of microislands. In their motion parallel to the

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unperturbed magnetic field the electrons exchange momentum with the island waves at the average rate $\nu_a = |k_\parallel| v_{e,th}$. As there is no parallel electric field inside the islands (see Sec. 2) the average width w of the islands can be taken as the mean free path of the random walk. Thus, from $\chi_e \approx \nu_a w^2$ one obtains the electron diffusion coefficient [2]

$$\chi_e = \chi_0 \frac{w^3 k S v_{e,th}}{q R_0} \tag{7.1}$$

where χ_0 is a form factor of order unity and $k_\parallel(x)$ has been evaluated at $x \approx w$. Using the relation $w = S_p/2kS$, χ_e takes the form

$$\chi_e = 2.343 \times 10^8 \chi_0 \frac{S_p^3 T_{e,kev}^{1/2}}{q R_0 k^2 S^2}, \tag{7.2}$$

where the temperature is measured in kev and the other quantities in gaussian units.

The dependence on k can be eliminated by noting that the equilibrium condition (6.5) can be written as a relation between S_p and k. Thus

$$k^3 = 1.39 \times 10^{-2} \frac{n_{13}^2 T_{e,kev}^{1/2} (1+2\tau) q^3}{S_p B_T \varepsilon_L A_i S^2} \frac{C_2(S_p, k)}{I_-(S_p)}, \tag{7.3}$$

where B_T is measured in Tesla and $n_{13} = n \times 10^{-13}$. It follows that the sponginess is the only parameter of the microisland structure that characterizes the effects on the diffusivity, together with the parameters of the macroscopic equilibrium.

In a realistic theory all quantities above are functions of space and evolve in time so that transport will be described by partial differential equations. The mathematics are of course very difficult from the analytical point of view, but one can attempt to gain a first insight into the relations between sponginess, transport and equilibrium on purely dimensional grounds with a zero-dimensional approach.

We define

$$\tau_E \approx \frac{s^2}{\chi_e} \approx \frac{3}{2} \frac{n(T_e + T_i) 2\pi^2 s^2 R_0}{P}, \tag{7.4}$$

where P is the power supplied to the plasma in the volume $2\pi^2 s^2 R_0$. Combining this with (7.2) we obtain an expression for the temperature and for the thermal Larmor radius in terms of the sponginess

$$T_{e,kev}(S_p) = A_1 \frac{k^{4/3}}{S_p^2}, \quad \rho_{th} = 0.456 \cdot (A_1 \tau A_i)^{1/2} \frac{k^{2/3}}{B_T S_p}$$

$$\text{with } A_1 = 0.201 \cdot \left(\frac{(P/\chi_0) q S^2}{n_{13}(1+\tau)} \right)^{2/3}, \tag{7.5}$$

where P is measured in MW. These relations introduce a dependence on the sponginess (ρ_{th} enters in $C_2(S_p, k)$, see (6.7) and (4.27)) that must be taken into account in the equilibrium condition (7.3), which becomes

$$k^{7/3} = 1.39 \times 10^{-2} \frac{n_{13}^2 A_1^{1/2} (1+2\tau) q^3}{S_p^2 B_T A_i \varepsilon_L S^2} \frac{C_2(S_p, k)}{I_-(S_p)}. \tag{7.6}$$

This relation governs the structure of the system of microislands and its relation with transport. The structure is determined by assigning the magnetic zero-order

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equilibrium (B_T, q, S), the density and its characteristic length of inhomogeneity (n_{13}, ε_L), the temperature ratio $\tau = T_i/T_e$ and the sponginess S_p, while the electron temperature and k are given by (7.5) and (7.6) and the half-width by $w = S_p/2kS$. Numerical inspection shows that (7.6) admits solutions in a whole range of sponginess $S_{p,min} \leqslant S_p \leqslant S_{p,max}$ and that the associated values of k and w can be compatible with the requirements for the validity of the theory. Thus, in order to determine univocally the microisland structure one needs a more stringent limitation on the sponginess. This is provided by the observation that the electrostatic energy (3.7) in the interspace is a decreasing function of the sponginess. The maximum sponginess $S_{p,max}$ is then associated with the minimum of the electrostatic energy of the system of microislands. One can also verify that the maximum sponginess corresponds to the maximum of the diffusivity (7.2) (and to the minimum of the electron temperature (7.5)). One is then led naturally to the choice $S = S_{p,max}$, with which the structure of the microislands and the confinement time (7.4) are completely determined once the zero-order equilibrium is known.

The existence of the upper limit $S_{p,max} < 1$ prevents the island chains from collapsing into a chaotic system.

8. Microisland structure and transport effects

For typical tokamak parameters the calculated value of the half-width w is of the order of the millimetre and k (cm^{-1}) is larger than and near unity. The cumulative drift velocity (4.31) is never larger than a fraction of millimetre per second. Notwithstanding that the effect of the microstructure on the Larmor rotation is imperceptible, its consequence on transport, described by the electron diffusivity (7.2), is relevant. The value of χ_e (with $\chi_0 = 1$) is of the order of $10^4\ cm^2\ s^{-1}$ and increases with sponginess for fixed values of the other parameters.

The maximum sponginess increases when one of the parameters n_{13}, P, q increases. Thus, an upper limit exists for these quantities, or a lower limit for the total current, beyond which one has $S_{p,max} > 1$, the microislands collapse and the system becomes chaotic. In the present paper we limit ourselves to qualitative observations and do not attempt to determine the limiting value of one parameter as a function of the others. Experimentally, typical stable tokamak discharges correspond in general to $S_p < 1$.

A stringent limitation to the lowest admissible value of S_p arises when one admits with Kadomtsev [2] that w should be larger than the collisionless screening thickness $\Lambda \approx (c^2 m_e/4\pi n_e e^2)^{1/2}$ of the current. In turn, this implies a lower limit for n_{13}, P, q, the limit of one parameter being lower when the values of the others are higher. In general, the limitations $n_{13} > 2.5$, $P > 2.5$ MW, $q > 2$ are sufficient (but not necessary) for the formation of the microislands. The limitation $q > 2$ indicates that the microislands are preferentially formed outside the so-called confinement region.

One can try to establish a power law of the confinement time by varying one parameter while keeping the others fixed, but this is jeopardized by the rather complicated correlations between different parameters implied by the equilibrium condition (7.6) and in fact, strictly speaking, a general power law does not exist. However, the arbitrariness of the parameters is reduced and the situation simplified when one assumes that the magnetic zero-order equilibrium is a state of stationary magnetic entropy [12, 13] and that the variation of the parameters is constrained

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by the consistency of the equilibrium profiles with this state. For instance, in accordance with the profile consistency, the intensity of auxiliary power P has no influence on the profile of the magnetic equilibrium characterized by the parameters q and S. A consequence that is significant for our purposes is the invariance of the ratio S/q with respect to profile consistent changes of the magnetic equilibrium and in particular with respect to changes of the current in the limit when the Ohmic power can be neglected with respect to the auxiliary power. Indeed, combining (6.2) and (10.2) of [13], one can easily derive in this limit and in the zone where $q(r) > 1$, the relation (in cylindrical circular geometry)

$$\frac{S(r)}{q(r)} = C_0 \frac{s^2}{r^2} \int_0^{r/s} \frac{P(x)}{P(1)} x\, dx - \beta, \tag{8.1}$$

where $P(x)$ is the total power deposited in $r \leqslant xs$ and C_0, β are invariant with respect to certain parameter transformations (including the total current) of the magnetic zero-order equilibrium (see [13]). Thus, in the parameter scanning of (7.6) the current must be varied by keeping S/q fixed.

We give here some examples of the variability of the scalings with the parameters (B(T), P(MW), I(MA), R(m) and s(m)).

Case 1 $B = 3$; $R = 3$; $\tau = 1$; $A_\mathrm{i} = 2$

Current ($P = 5;\ n_{13} = 5;\ S/q = 0.40$) :

$\varepsilon_\mathrm{L} = 0.3;\ 1.66 \leqslant q \leqslant 3.33;\ 1.50 \leqslant I \leqslant 3;\ 0.380 \leqslant S_\mathrm{p} \leqslant 0.782$: $\tau_\mathrm{E} \approx I^{1.10}$

$\varepsilon_\mathrm{L} = 0.5;\ 2.22 \leqslant q \leqslant 4;\ 1.25 \leqslant I \leqslant 2.25;\ 0.449 \leqslant S_\mathrm{p} \leqslant 0.794$: $\tau_\mathrm{E} \approx I^{0.78}$

Density ($P = 5;\ I = 2.5$):

$\varepsilon_\mathrm{L} = 0.3;\ S = 0.80;\ 5 \leqslant n_{13} \leqslant 10;\ 0.460 \leqslant S_\mathrm{p} \leqslant 0.538$: $\tau_\mathrm{E} \approx n_{13}^{0.25}$

$\varepsilon_\mathrm{L} = 0.5;\ S = 1.33;\ 2.50 \leqslant n_{13} \leqslant 10;\ 0.576 \leqslant S_\mathrm{p} \leqslant 0.774;\ \tau_\mathrm{E} \approx n_{13}^{0.28}$

Power ($I = 2;\ n_{13} = 5;\ S = 1$) :

$\varepsilon_\mathrm{L} = 0.3;\ 2.50 \leqslant P \leqslant 10;\ 0.524 \leqslant S_\mathrm{p} \leqslant 0.646$: $\tau_\mathrm{E} \approx P^{-0.86}$

$\varepsilon_\mathrm{L} = 0.5;\ 3.75 \leqslant P \leqslant 10;\ 0.485 \leqslant S_\mathrm{p} \leqslant 0.559$: $\tau_\mathrm{E} \approx P^{-0.79}$

Magnetic field ($P = 5;\ n_{13} = 5$) :

$\varepsilon_\mathrm{L} = 0.3;\ S = 1;\ I = 2;\ 2.50 \leqslant B \leqslant 4.50;\ 0.546 \leqslant S_\mathrm{p} \leqslant 0.668$: $\tau_\mathrm{E} \approx B^{0.27}$

$\varepsilon_\mathrm{L} = 0.5;\ S = 0.50;\ I = 1.5;\ 2.50 \leqslant B \leqslant 4;\ 0.477 \leqslant S_\mathrm{p} \leqslant 0.553$: $\tau_\mathrm{E} \approx B^{0.17}$

Case 2 $B = 2.3$; $R = 1.65$; $s = 0.40$; $\varepsilon_\mathrm{L} = 0.3$; $\tau = 1$; $A_\mathrm{i} = 2$

Current ($P = 1.6;\ n_{13} = 7.5;\ S/q = 0.333$):

$1.50 \leqslant q \leqslant 3;\ 0.375 \leqslant I \leqslant 0.75;\ 0.333 \leqslant S_\mathrm{p} \leqslant 0.699$: $\tau_\mathrm{E} \approx I^{1.13}$

Density ($P = 1.6;\ I = 0.5;\ S = 0.75$):

$2.50 \leqslant n_{13} \leqslant 10;\ 0.402 \leqslant S_\mathrm{p} \leqslant 0.548$: $\tau_\mathrm{E} \approx n_{13}^{0.30}$

Power ($I = 0.5;\ n_{13} = 7.5;\ S = 0.75$):

$0.50 \leqslant P \leqslant 2.00;\ 0.432 \leqslant S_\mathrm{p} \leqslant 0.531$: $\tau_\mathrm{E} \approx P^{-0.82}$

Magnetic field ($P = 1.6;\ n_{13} = 5;\ I = 0.5;\ S = 0.75$):

$1.60 \leqslant B \leqslant 2.30;\ 0.416 \leqslant S_\mathrm{p} \leqslant 0.469$: $\tau_\mathrm{E} \approx B^{0.22}$

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Change of scaling may occur in the neighbourhood of the admissible range $w \geqslant \Lambda$, $S_{\mathrm{p}} \leqslant 0.9$. The general trend of the scalings is qualitatively comparable with the observations [14].

9. Conclusions

A stationary structure of microisland chains can form in a tokamak plasma in a self-consistent process in which the energy pumped into the islands through the electrostatic interaction with the ions crossing the islands within a Larmor rotation, is equilibrated by anomalous dissipation. Thus, the plasma acquires a spongy structure, where the sponginess S_{p}, measured by the ratio between the island width and the distance between adjacent rational surfaces, is the controlling parameter of the electric and magnetic field of the islands and of their size, and involves a structure of the order of the millimetre or less. This structure can be disregarded when electric and magnetic excitations are considered on a scale much larger than the spongy background of the plasma, but is relevant for conditioning the electron transport in the region $q > 2$, outside the so-called confinement zone, where the spongy structure is preferentially formed. The electron diffusivity is given by (7.2) where k and S_{p} are related by (7.6), which is the condition for the existence of stationary islands. This condition admits an upper limit $S_{\mathrm{p,max}}$ for the sponginess, which is lower than one, preventing, on the one hand, the island chains of collapsing into a chaotic system. On the other hand, the maximum sponginess $S_{p,\mathrm{max}}$ is that which is realized physically, as it corresponds to the minimum of the electrostatic energy of the system of islands and to maximum diffusivity. Yet the diffusivity does not depend only $S_{\mathrm{p,max}}$, but on the whole set of parameters of the macroscopic magnetic equilibrium with which the microislands form an integral system. It follows that the confinement time is correlated with all parameters of the magnetic equilibrium at hand and cannot be synthesized by a single power law valid in all cases. The quantification of the effects of the correlations and the practical consequences on confinement deserve further study.

Appendix A

A.1. Calculation of $\overline{A_n}$

Applying (4.16), (4.18), (4.19) and (4.33) one has

$$\rho_n = \frac{v_{\perp,n}}{\Omega^{(0)}_{n-1}} = \frac{v_{\perp,0}}{\Omega^{(0)}_{0}} + \frac{ew}{M}\sum_{j=1}^{n}(-1)^{j-1}\frac{E_{X,j}}{v_{\perp,j}\Omega^{(0)}_{j-1}}. \tag{A 1}$$

Thus, in lowest order (identifying $\Omega^{(0)}_{n-1}$ with $\Omega^{(0)}_{0}$) the perpendicular velocity can be expressed as

$$v_{\perp,j} = v_{\perp,0}\left(1 + \sum_{k=1}^{j}(-1)^{k-1}\frac{ew}{Mv^2_{\perp,0}}E_{X,k}\right) \tag{A 2}$$

and the α_j (4.5) take the form

$$\alpha_j = \frac{ew}{Mv_{\perp.0}}E_{Y,j}\left(1 - \frac{ew}{Mv^2_{\perp,0}}\sum_{k=1}^{j}(-1)^{k-1}E_{X,k}\right) \tag{A 3}$$

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up to second order in $E_{\alpha,j}$. The sum $A_n = \sum_{j=1}^{n} \alpha_j$ where $n = 2N$, contains N terms with the form

$$\alpha_j + \alpha_{j+1} = \frac{ew}{Mv_{\perp,0}}(E_{Y,j} + E_{Y,j+1}) - \frac{e^2w^2}{M^2v_{\perp,0}^3}\left(E_{Y,j}\sum_{k=1}^{j}(-1)^{k-1}E_{X,k} + E_{Y,j+1}\sum_{k=1}^{j+1}(-1)^{k-1}E_{X,k}\right). \quad (A\,4)$$

Putting $j = 2q + 1$ the nonlinear term becomes

$$-\frac{e^2w^2}{M^2v_{\perp,0}^3}\left(E_{Y,1}\sum_{k=1}^{2q+1}(-1)^{k-1}E_{X,k} + E_{Y,2}\sum_{k=1}^{2q+2}(-1)^{k-1}E_{X,k}\right)$$
$$= -\frac{e^2w^2}{M^2v_{\perp,0}^3}(E_{Y,1}q(E_{X,1} - E_{X,2}) + E_{Y,2}(q+1)(E_{X,1} - E_{X,2}) + E_{X,1}E_{Y,1}) \quad (A\,5)$$

We now sum up the N terms from $q = 0$ to $N - 1$ and perform the averages, obtaining

$$\overline{A_n} \approx \frac{ew}{Mv_{\perp,0}}N\overline{(E_{Y,1} + E_{Y,2})} - \frac{e^2w^2}{M^2v_{\perp,0}^3}\frac{N^2}{2}\overline{(E_{Y,1}E_{X,2} - E_{Y,2}E_{X,1})} \quad (A\,6)$$

where in the nonlinear term N has been neglected with respect to N^2.

The mixed terms in (A 6) are functions of $k\rho$ and of the phase shift δ. For instance,

$$F(\xi, k\rho, \delta) \equiv E_{Y,1}(\xi, \xi + \delta)E_{X,2}(\xi + 2k\rho, \xi + 2k\rho + \delta) \quad (A\,7)$$

Now we observe that for each ion crossing the interspace at ξ_j and $\xi_j + 2k\rho$ there is another ion crossing the interspace at ξ_j and $\xi_j - 2k\rho$, the two ions having opposite velocities in ξ_j. Our test ion should represent the combined effect of the two ions, which means that $+2k\rho$ and $-2k\rho$ should be treated symmetrically. Similarly the phase shift δ is a random quantity and one should not distinguish between positive and negative δ. Then we write the mixed terms in symmetric form

$$F_{Y\,1,X\,2}(\xi) = \tfrac{1}{4}(F(\xi, k\rho, \delta) + F(\xi, -k\rho, \delta) + F(\xi, k\rho, -\delta) + F(\xi, -k\rho, -\delta)). \quad (A\,8)$$

Noting that $E_Y(-\xi_j, -\xi_j - \delta) = -E_Y(\xi_j, \xi_j + \delta)$ and $E_X(-\xi_j, -\xi_j - \delta) = E_X(\xi_j, \xi_j + \delta)$ one has that $F_{Y\,1,X\,2}(\xi)$ is a periodic anti-symmetric function of ξ whose ξ-average vanishes, as well as the average of $E_{Y,\alpha}$ and of A_n.

A.2. Calculation of $2\rho_0\overline{B_n} + \overline{B_n^2}$ *and of* $\overline{A_n^2}$

We write B_n (4.19) in the form

$$B_n = \frac{1}{\Omega_0^{(0)}}\sum_{j=1}^{n}(-1)^{j-1}\frac{ew}{Mv_{\perp,0}}E_{X,j}\left(1 - \frac{ew}{Mv_{\perp,0}^2}\sum_{k=1}^{j}(-1)^{k-1}E_{X,k}\right)$$
$$= \frac{ew}{\Omega_0^{(0)}Mv_{\perp,0}}N(E_{X,1} - E_{X,2}) - \frac{e^2w^2}{\Omega_0^{(0)}v_{\perp,0}^3}\sum_{j=1}^{n}(-1)^{j-1}E_{X,j}\sum_{k=1}^{j}(-1)^{k-1}E_{X,k} \quad (A\,9)$$

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where $n = 2N$. According to (4.22), the first term does not contribute to the average. Let us divide the nonlinear term into even and odd contributions

$$\sum_{j=1}^{2N}(-1)^{j-1}E_{X,j}\sum_{k=1}^{j}(-1)^{k-1}E_{X,k}$$

$$= -\sum_{q=1}^{N}E_{X,2q}\sum_{k=1}^{2q}(-1)^{k-1}E_{X,k}+\sum_{q=1}^{N}E_{X,2q-1}\sum_{k=1}^{2q-1}(-1)^{k-1}E_{X,k}. \tag{A 10}$$

One has

$$-\sum_{q=1}^{N}E_{X,2q}\sum_{k}^{2q}(-1)^{k-1}E_{X,k} = -\sum_{q=1}^{N}E_{X,2}q(E_{X,1}-E_{X,2})$$

$$\approx \frac{N^2}{2}(E_{X,2}^2 - E_{X,1}E_{X,2}), \tag{A 11}$$

$$\sum_{q=1}^{N}E_{X,2q-1}\sum_{k}^{2q-1}(-1)^{k-1}E_{X,k} = \sum_{q=1}^{N}E_{X,1}(qE_{X,1}-(q-1)E_{X,2})$$

$$\approx \frac{N^2}{2}(E_{X,1}^2 - E_{X,1}E_{X,2}). \tag{A 12}$$

Combining the two terms above, one finally obtains

$$\overline{B_n} = -\frac{e^2w^2}{\Omega_0^{(0)}v_{\perp,0}^3}N^2\overline{(E_{X,1}^2 - E_{X,1}E_{X,2})} \tag{A 13}$$

We now consider the average of B_n^2, as follows by squaring (A 9) up to second order in $E_{X,j}$:

$$\overline{B_n^2} = \frac{2e^2w^2}{(\Omega_o^{(0)})^2M^2v_{\perp,0}^2}N^2\overline{(E_{X,1}^2 - E_{X,1}E_{X,2})}. \tag{A 14}$$

Comparison with (A 12) gives $2\rho_0\overline{B_n} + \overline{B_n^2} = 0$.

The calculation of $\overline{A_n^2}$ follows the same line:

$$\overline{A_n^2} = \frac{2e^2w^2}{M^2v_{\perp,0}^2}N^2\overline{(E_{Y,1}^2 + E_{Y,1}E_{Y,2})}. \tag{A 15}$$

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References

[1] Smolyakov, A. I. 1989 *Sov. J. Plasma Phys.* **15**, 667.
[2] Kadomtsev, B. B. 1991 *Nucl. Fusion* **31**, 1301.
[3] Rebut, P.-H. and Brusati, M. 1986 *Plasma Phys. Control Fusion* **28**, 113.
[4] White, R. B. and Romanelli, F. 1989 *Phys. Fluids* B **1**, 977.
[5] Mikhailvskii, A. B., Kovalishen, E. A., Shirokov, M. S., Konovalov, S. V., Tsypin, V. S., Kamenets, F. F., Ozeki, T. and Takizuka, T. 2004 *Phys. Plasmas* **11**, 666.

https://doi.org/10.1017/S0022377805004216 Published online by Cambridge University Press

[6] Chatenet, J. H., Luciani, J. F. and Garbet, X. 1996 *Phys. Plasmas* **34**, 4628.

[7] Minardi, E. 1994 *Nucl. Fusion* **34**, 1567.

[8] Minardi, E. and Lazzaro, E. 1998 *Nucl. Fusion* **38**, 1161.

[9] Smolyakov, A. I. 1993 *Plasma Phys. Control Fusion* **35**, 657.

[10] Boschetto, F. M. Autoalimentazione delle isole magnetiche nel plasma di un tokamak per mezzo dell'interazione tra ioni e catene di isole. Thesis, Politecnico di Milano, Academic Year 1993–1994 (unpublished).

[11] Abramowitz, M. and Stegun, I. A. 1969 *Handbook of Mathematical Functions*. Washington, DC: National Bureau of Standards, p. 302.

[12] Minardi, E. 2005 *J. Plasma Phys.* **71**, 53.

[13] Minardi, E., Lazzaro, E., Sozzi, C. and Cirant, S. 2003 *Nucl. Fusion* **43**, 369.

[14] ITER Physics Expert Groups on Confinement and Transport and Confinement Modelling Database, ITER Physics Basis Editors, ITER EDA. 1999 *Nucl. Fusion* **39**, 2175.

https://doi.org/10.1017/S0022377805004216 Published online by Cambridge University Press