ASSESSMENTS OF STRUCTURE-DEPENDENT INTEGRATION METHODS WITH EXPLICIT DISPLACEMENT AND VELOCITY DIFFERENCE EQUATIONS

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ABSTRACT

A family of structure-dependent integration methods has been proposed by Gui *et al*. for time integration. Although it has desirable numerical properties, such as unconditional stability, explicit formulation and second-order accuracy, it has some adverse properties, such as a poor capability to capture structural nonlinearity, an overshoot in a high frequency steady- state response and a weak instability in the high frequency response of nonzero initial conditions. The causes of these adverse properties are explored. A poor capability to capture structural nonlinearity may originate from the convergence rate of 1 in velocity error. This family method has an overshoot in a high frequency steady-state response and this overshoot can be eliminated by adding a load-dependent term into the displacement difference equation. It is also analytically verified that the family method generally has no weak instability. However, the special member with $\lambda = 4$, i.e., CR explicit method, is shown to have a weak instability. Thus, it must be prohibited from practical applications although many applications of this method were found in the literature.

Keywords: Forced vibration response, Local truncation error, Overshooting, Weak instability.

1. INTRODUCTION

A step-by-step integration method is a very powerful technique for either conducting the linear elastic or nonlinear dynamic analysis. It consists of two difference equations for displacement and velocity increments and equation of motion or its asymptotic form. For a conventional integration method, the coefficients of the two difference equations are some specific scalar constants. Whereas, these scalar constants have been replaced by structure-dependent matrices for the newly developed integration methods and thus they are referred as structure-dependent integration methods. An unconditionally stable explicit structure-dependent method was first found in 2002 [1]. In this method, the difference equation for displacement increment is explicit and structure dependent. Whereas, the difference equation for velocity increment remains implicit and structure independent. Some structure-dependent integration methods of this type were developed subsequently for better numerical properties [2-5]. The second type of the structure-dependent integration methods were developed later [6-8], where both the two difference equations are explicit and structure dependent. Some of the above-mentioned structure-dependent integration methods are further enhanced with controllable high-frequency numerical damping [4-5].

Although structure-dependent integration methods can simultaneously integrate unconditional stability and explicit formulation together and then it is very computationally efficient for nonlinear dynamic analysis, some adverse numerical properties were found. Thus, it is of need to reevaluate the currently available structure-dependent integration methods so that the misuse of a structure-dependent integration method can be avoided. Since an overshoot in the high-frequency forced vibration response has been experienced for the first type of structure-dependent integration methods, it has been thoroughly evaluated and a remedy has been proposed to remove the adverse overshoot [9]. Hence, it is important to assess the second type of structure-dependent integration methods whether it also has such an adverse overshoot property since many applications of this type of integration methods have been found in the literature [10-11]. The CR explicit Method (CRM) [6] and the family method developed by Gui *et al*. [8], which will be abbreviated as GM for brevity, are classified as the second type of structure-dependent integration methods. Consequently, both GM and CRM will be reevaluated in this work. Notice that CRM is a special member of GM. In addition to the adverse overshoot property in the high-frequency forced vibration response, other adverse

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DOI : 10.1017/jmech.2017.49
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https://doi.org/10.1017/jmech.2017.49 Published online by Cambridge University Press

properties may also experience for GM and CRM, such as the lack of capability to capture structural nonlinearity and a weak instability in the high frequency responses for nonzero initial conditions. The root causes of these adverse properties will be explored herein and are confirmed by numerical examples.

2. FORMULATION

For a linear elastic, single degree of freedom system, the general formulation of GM [8] can be expressed as:

$$\begin{aligned} & ma_{i+1} + cv_{i+1} + kd_{i+1} = f_{i+1} \\ & d_{i+1} = d_i + (\Delta t)v_i + \alpha(\Delta t)^2 a_i \ , \\ & v_{i+1} = v_i + \alpha(\Delta t)a_i \end{aligned}$$

$$\alpha = \frac{1}{1+\xi\Omega_0 + \frac{1}{\lambda}\Omega_0^2} = \frac{m}{m + \frac{1}{2}(\Delta t)c_0 + \frac{1}{\lambda}(\Delta t)^2 k_0} \tag{1}$$

where m, c and k are the mass, viscous damping coefficient and stiffness, respectively; and d_{i+1}, v_{i+1}, a_{i+1} and f_{i+1} correspond to the displacement, velocity, acceleration and external force at the end of the $(i+1) - th$ time step. In addition, ξ is a viscous damping ratio; $\Omega_0 = \omega_0(\Delta t)$ and $\omega_0 = \sqrt{k_0 / m}$ is the natural frequency of the system determined from the initial stiffness k_0, which is generally different from the stiffness k as shown in the first line since it may vary for a nonlinear system. Similarly, $c_0 = 2\xi\omega_0 m$, which is assumed that the viscous damping coefficient is determined from the initial structural properties. In general, c is taken to be c_0 for a nonlinear system although it can be updated with the variation of the stiffness for a nonlinear system. The applications of c_0, k_0 and Δt to replace ξ and Ω_0 is very important for improving computational efficiency. This is mainly because that an eigen-analysis of the analyzed system to obtain its eigenvalues is very time consuming for a matrix of a large order. Notice that λ is a free parameter to control the numerical properties of GM and CRM [6] is a member of GM if $\lambda = 4$ is adopted. In general, GM can have unconditional stability for linear elastic systems for $\lambda \le 4$ while it becomes conditionally stable for $\lambda > 4$.

3. CAPABILITY FOR CAPTURING STRUCTURAL NONLINEARITY

Since the capability to capture structural nonlinearity is closely related to the performance of an integration method in the solution a nonlinear system, thus the calculated result for a highly nonlinear system can be applied to reveal the capability of the integration method. For this purpose, the following nonlinear system is considered:

$$\ddot{u} + \frac{u}{|1+u|} = 0 \tag{2}$$

where u and $\ddot{u}$ are the displacement and acceleration, respectively. The stiffness $1/(1+|u|)$ will be softening after the system deforms. In addition, a large initial displacement or initial velocity will lead to a significant reduction of the stiffness. The period of the system is $T_0 = 2\pi$ sec based on the initial stiffness. Equation (3) is a very good example to numerically assess the nonlinear performance of an integration method since it will be drastically softened after is deforms.

Since the two Chang explicit methods will be also applied to solve Eq. (3) for comparison, its formulation can be generally expressed as:

$$\begin{aligned} & ma_{i+1} + cv_{i+1} + kd_{i+1} = f_{i+1} \\ & d_{i+1} = d_i + \beta_1(\Delta t)v_i + \beta_2(\Delta t)^2 a_i + \beta_3 \frac{1}{m}(\Delta t)^2 (f_{i+1} - f_i), \\ & v_{i+1} = v_i + (\Delta t)\left[(1-\gamma)a_i + \gamma a_{i+1}\right] \end{aligned}$$

$$\begin{aligned} & \beta_1 = (1 + 2\gamma\xi\Omega_0)/D \\ & \beta_2 = \left[\frac{1}{2} - 2\left(\beta - \frac{1}{2}\gamma\right)\xi\Omega_0\right]/D \\ & \beta_3 = \beta / D \end{aligned} \tag{3}$$

where $D = 1 + 2\gamma\xi\Omega_0 + \beta\Omega_0^2$. The case of $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$ is referred as CEM1 [1] while CEM2 [2] is used to denote the case of $\beta = \gamma = \frac{1}{2}$. The free vibration responses to the initial conditions of $u(0) = 0$ and $\dot{u}(0) = 200$ are calculated and the results are plotted Fig. 1. The result obtained from the Newmark explicit method (NEM) [12] with $\Delta t = 0.01$ sec is considered as a reference solution for comparison. The three members of GM are adopted for this study and GM1, GM2 and GM3 are introduced to represent the members of $\lambda =$ 2, 4 and 11.5, respectively. Notice that GM2 also represents CRM. In addition, NEM and the two Chang explicit methods, which are referred as CEM1 [1] and CEM2 [2] with $\Delta t = 0.3$ sec are also applied to compute the responses for comparisons. It is seen in Fig. 1(a) that the calculated results obtained from NEM, CEM1 and CEM2 are reliable and have roughly the same accuracy. However, the results obtained from GM1, GM2 and GM3 as shown in Fig. 1(b) exhibit significant period distortion when compared to the reference solution and the amount of the period distortion increase with the decrease of λ value. It is evident that CRM and GM show a relatively poor capability to capture structural nonlinearity when compared to the other integration methods.

Apparently, CEM1, CEM2 and GM1 to GM3 all are structure-dependent integration methods. However, they exhibit different capability to capture structural nonlinearity. Hence, the cause of this difference will be explored next. Since a convergence rate can be considered as an indicator of the accuracy of an integration

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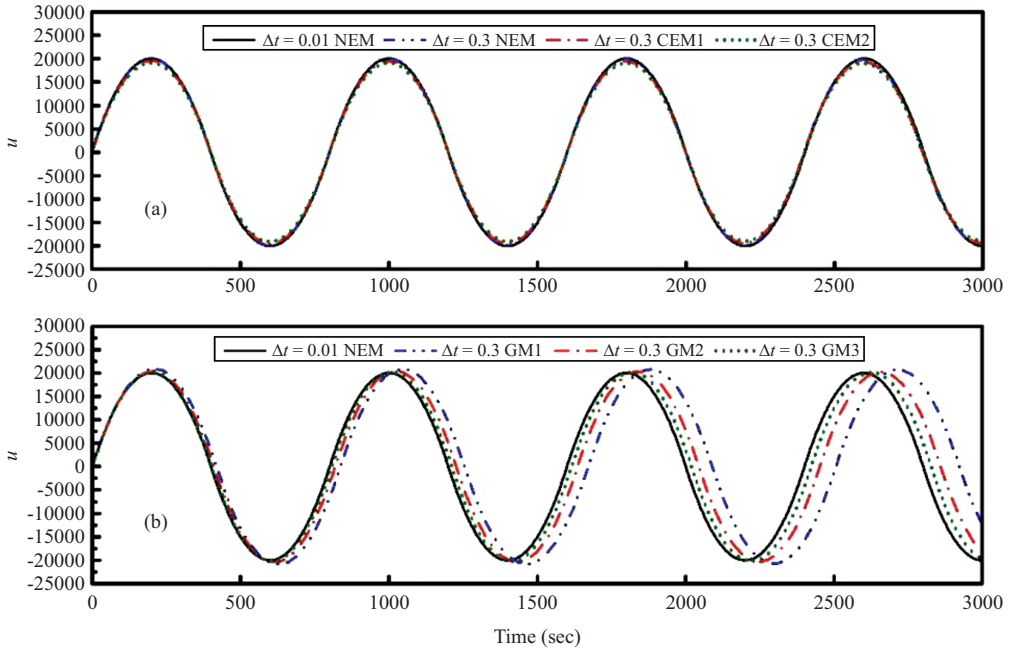


Fig. 1 Free vibration responses to a highly nonlinear system.

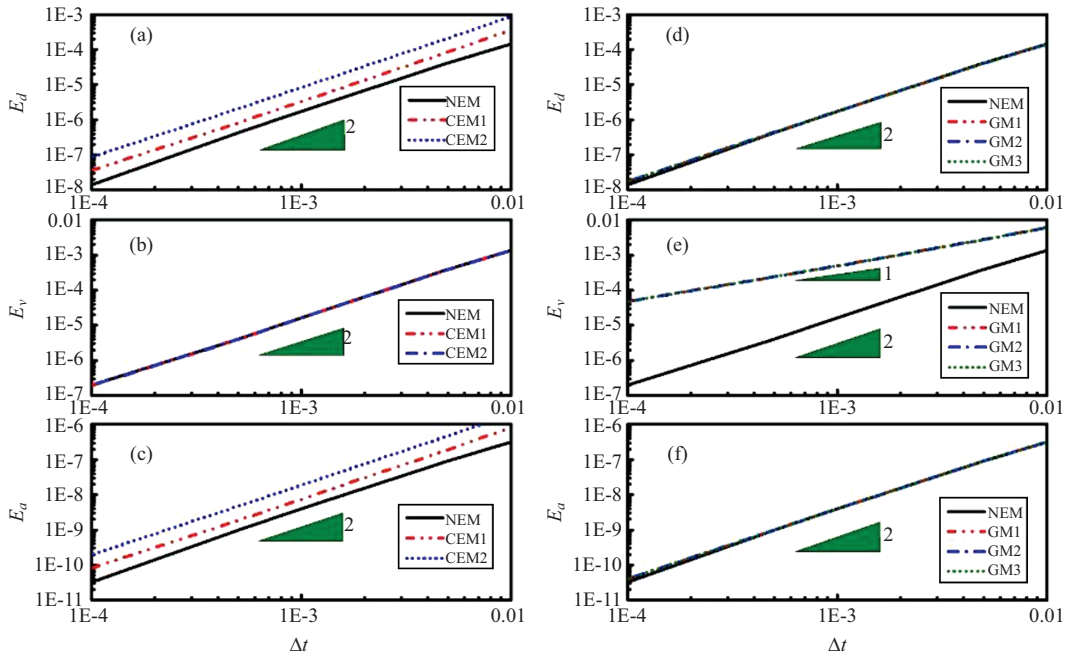


Fig. 2 Convergence rate of displacement, velocity and acceleration.

method, to identify the convergence rate of the integration method might reveal the root cause of this difference. A technique has been used to identify the convergence rate of displacement, velocity and acceleration for an integration method. In general, an integration method is applied to solve a problem with a series of small different time steps and then the results are used to evaluate the convergence rate of this integration method. In general, the displacement error E_d, velocity error E_v and acceleration error E_a at a given time can be obtained from:

$$E_d = \left|d_i - u(t_i)\right|, \quad E_v = \left|v_i - \dot{u}(t_i)\right|, \quad E_a = \left|a_i - \ddot{u}(t_i)\right| \quad (4)$$

where d_i, v_i and a_i are the calculated displacement, velocity and acceleration at the end of the time of t_i while $u(t_i)$, $\dot{u}(t_i)$ and $\ddot{u}(t_i)$ are the exact displacement, velocity and acceleration at the time of t_i although they are also obtained from a very small time step. The plot of each error versus step size on a log-log scale can be used to estimate the convergence rate of each error.

In this numerical study, NEM, CEM1, CEM2 and GM1 to GM3 are used to solve Eq. (2) and the time to calculate the response errors is specified at $\Delta t = 0.1$ sec. For each integration method, variations of E_d, E_v and E_a with Δt are shown on a log-log scale in Fig. 2. The convergence rate of the specific error for a given integra-

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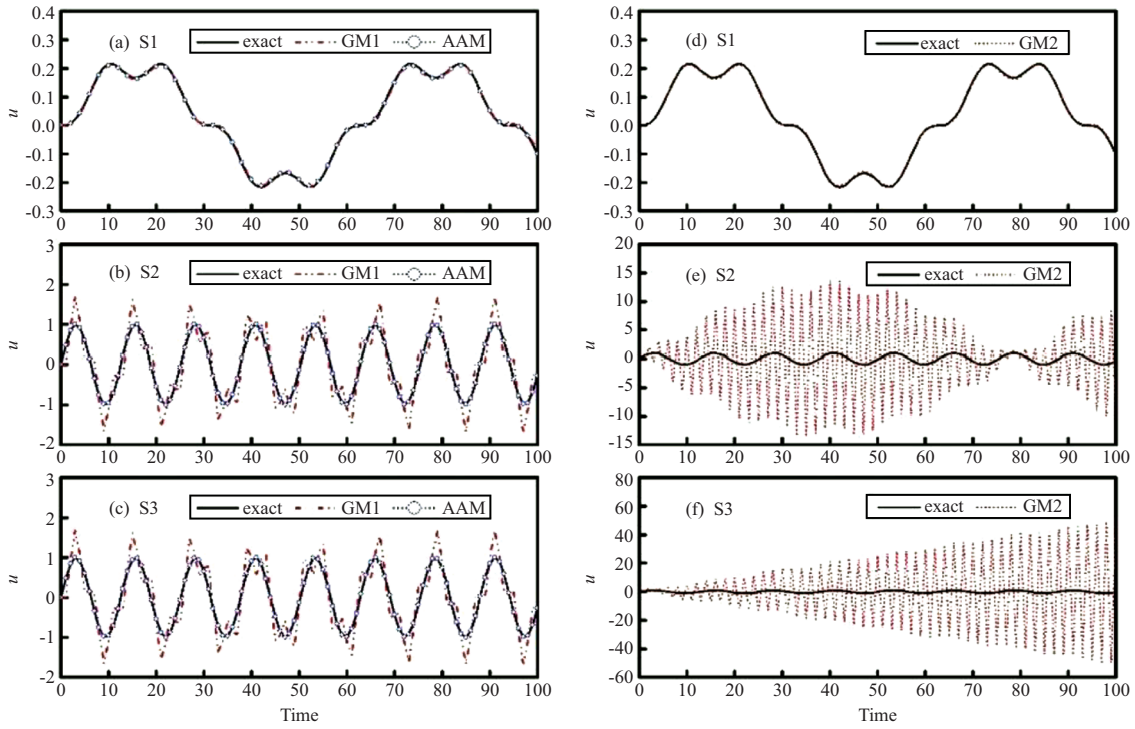


Fig. 3 Forced vibration responses to a linear elastic SDOF system for using GM1, GM2 and AAM.

tion method is equal to the slope of the corresponding curve. All the six integration methods can have a convergence rate of 2 for E_d, E_v and E_a except that for GM1 to GM3 they can only have a convergence rate 1 for E_v. As a result, a convergence rate of 1 for E_v for GM1 to GM3 might be responsible for the poor capability to capture the structural nonlinearity since their convergence rates for E_d and E_a are the same as those of NEM, CEM1 and CEM2. Notice that the convergence rate is numerically determined in this study. This is because that there is no way to analytically evaluate the convergence rate of velocity and acceleration based on the currently available evaluating techniques, although an analytical assessment of the convergence rate of displacement has been well constructed and is very commonly used. In order to overcome this difficulty, an intensive study is under way.

4. HIGH-FREQUNCY OVERSHOOT IN FORCED VIBRATION RESPONSE

An adverse overshoot phenomenon in the forced vibration response of a high frequency mode can be revealed by using an integration method to solve the equation of motion for a linear elastic system under a specific dynamic loading. As an example, the following equation of motion is considered:

$$m\ddot{u} + k_0 u = k_0 \sin\left(\bar{\omega} t\right) \tag{5}$$

Since the dynamic load is a simple, analytical function, a theoretical solution of this equation can be obtained from the fundamental theory of structural dynamics and is found to be:

$$u(t) = \frac{1}{1-\theta^2}\left[\sin\left(\bar{\omega} t\right) - \theta \sin\left(\omega_0 t\right)\right] \tag{6}$$

where $\bar{\omega}$ is the vibration frequency of the applied sine loading and $\theta = \bar{\omega}/\omega_0$ is defined as a frequency ratio. It is straightforward to find that the response is dominated by the steady-state response for a high frequency mode or a very small θ value. Whereas, it is controlled by the transient response for a low frequency mode or a large θ value. For simplicity, m = 1 and $\bar{\omega} = 1/2$ are generally taken to simulate the following three different systems:

$$\text{Sl: } k_0 = 10^{-2}, \quad \text{S2: } k_0 = 10^4, \quad \text{S3: } k_0 = 10^6 \tag{7}$$

The natural frequencies for S1, S2 and S3 are found to be 10^{-1}, 10^2 and 10^3 rad/sec, respectively. This is intended to have different natural frequencies to explore the relationship between the natural frequency and the adverse overshooting behavior. The θ values corresponding to S1, S2 and S3 are 5, 1/200 and 1/2000. This implies that the total response of S1 is dominated by the transient response while for S2 and S3 it is controlled by the steady-state response.

The constant average acceleration method (AAM), GM1 and GM2 are used to solve Eq. (5) with Δt = 1 sec. The results obtained from GM1, GM2 and AAM are plotted in Fig. 3. Notice that NEM and GM3 are inappropriate for this study since they are only conditionally stable. It is seen in this figure that AAM can always

https://doi.org/10.1017/jmech.2017.49 Published online by Cambridge University Press

give very accurate solutions for all the three systems. Whereas, GM1 and GM2 can only give reliable solutions for S1 while for S2 and S3 an overshoot behavior is found. These phenomena can be thoroughly explained next. Since the value of $\Delta t/T_0$, where $T_0 = 2\pi/\omega_0$, is as small as 0.016, the transient response of S1 can be accurately integrated for using $\Delta t = 1$ sec and thus the three integration methods all lead to reliable solutions. On the other hand, the total response of either S2 or S3 is dominated by the stead-state response and thus an accurate integration of this response will result in an accurate total response. It has been shown [13] that a harmonic loading can be accurately represented if $\Delta t / \overline{T}$, where $\overline{T} = 2\pi / \overline{\omega}$, is chosen to be smaller than 1/12. For this case, the value of $\Delta t / \overline{T}$ is found to be $1/4\pi$, which is smaller than 1/12, and thus the applied sine load can be very accurately represented. Consequently, it is anticipated that the total response either for S2 or S3 can be reliably obtained for the three integration methods. However, only AAM leads to a reliable solution for S2 and S3 while either GM1 or GM2 results in an overshooting response. It is evident that both GM1 and GM2 will exhibit an adverse overshoot in the steady-state response of a high frequency mode. This overshoot behavior is drastically different from that was found by Goudreau and Taylor [14], where an overshoot is found in the early transient response and it will disappear subsequently if the Wilson' θ method is used to compute the response. Hence, the cause of this overshoot must be explored.

5. LOCAL TRUNCATION ERROR

A local truncation error of an integration method is defined as the error committed in each time step by replacing differential equation with difference Eq. [15]. The dynamic loading is included in the differential equation and thus in the difference equation. Hence, the cause of this new type of overshoot might be revealed by a local truncation error derived from a forced vibration response. As a result, the local truncation error for GM is found to be:

$$E = \frac{1}{D}\frac{1}{6}\xi\Omega_0(\Delta t)\dot{u}_i + \frac{1}{D}\frac{1}{\lambda}\Omega_0^2\ddot{u}_i - \frac{1}{D}\frac{1}{12}\left[\Omega_0^2\ddot{u}_i - \frac{1}{m}(\Delta t)^2\ddot{f}_i\right] + O\left[(\Delta t)^3\right] \tag{8}$$

where $D = 1 + \xi\Omega_0 + \frac{1}{\lambda}\Omega_0^2$. In this equation, the third term on the right-hand side of this equation can be replaced by using the relation $\Omega_0^2\ddot{u}_i - \frac{1}{m}(\Delta t)^2\ddot{f}_i = -2\xi\Omega_0(\Delta t)\dddot{u}_i - (\Delta t)^2\ddot{\ddot{u}}_i$, which is the second derivative of the equation of motion. As a result, Eq. (7) becomes:

$$E = \frac{1}{D}\frac{1}{6}\xi\Omega_0(\Delta t)\dot{u}_i + \frac{1}{D}\frac{1}{\lambda}\Omega_0^2\ddot{u}_i + \frac{1}{D}\frac{1}{12}2\xi\Omega_0(\Delta t)\dddot{u}_i + \frac{1}{D}\frac{1}{12}(\Delta t)^2\ddot{\ddot{u}}_i + O\left[(\Delta t)^3\right] \tag{9}$$

It is revealed by this equation that the local truncation error of GM is dominated by the quadratic term of Ω_0^2. This indicates that the overshooting behavior experienced in the steady-state response of a high frequency mode might be due to this dominant error term since it is quadratically proportional to Ω_0^2.

This result indicates that there will be no overshoot if the dominant error term can be eliminated from the local truncation error. For this purpose, one can add a load-dependent term into the difference equation for displacement increment, i.e., the second line of Eq. (1) is modified to be:

$$d_{i+1} = d_i + (\Delta t)v_i + \alpha(\Delta t)^2 a_i + p_{i+1} \tag{10}$$

This concept originates from introducing a load-dependent term p_{i+1} into the difference equation so that the dominant error term can be cancelled out by using the relation $\Omega_0^2\ddot{u}_i - \frac{1}{m}(\Delta t)^2\ddot{f}_i = -2\xi\Omega_0(\Delta t)\dddot{u}_i - (\Delta t)^2\ddot{\ddot{u}}_i$. As a result, the load-dependent term is found to be:

$$p_{i+1} = \frac{1}{\lambda m D}(\Delta t)^2(f_{i+1} - f_i) \tag{11}$$

After introducing the load-dependent terms into the difference equation for displacement increment for GM, its local truncation error becomes:

$$E = \frac{1}{D}\frac{1}{6}\xi\Omega_0(\Delta t)\dot{u}_i - \frac{1}{D}\frac{1}{\lambda} - \frac{1}{12}2\xi\Omega_0(\Delta t)\dddot{u}_i - \frac{1}{D}\frac{1}{\lambda} - \frac{1}{12}(\Delta t)^2\ddot{\ddot{u}}_i + O\left[(\Delta t)^3\right] \tag{12}$$

Clearly, there is no error term that is quadratically proportional to Ω_0. Hence, it is expected that there will be no overshoot in the steady-state response of a high frequency mode for GM with the load-dependent term. In order to confirm the feasibility of this load-dependent term, Eq. (5) is solved again for the three different systems and the numerical results are plotted in Fig. 4. Clearly, the results obtained from GM1 and GM2 with the load-dependent term are overlapped together with the exact solutions in each plot of this figure. Hence, it is evident that the proposed remedy can effectively remove the adverse overshoot in the steady-state response of a high frequency mode.

6. WEAK INSTABILITY

To detect whether a weak instability will experience

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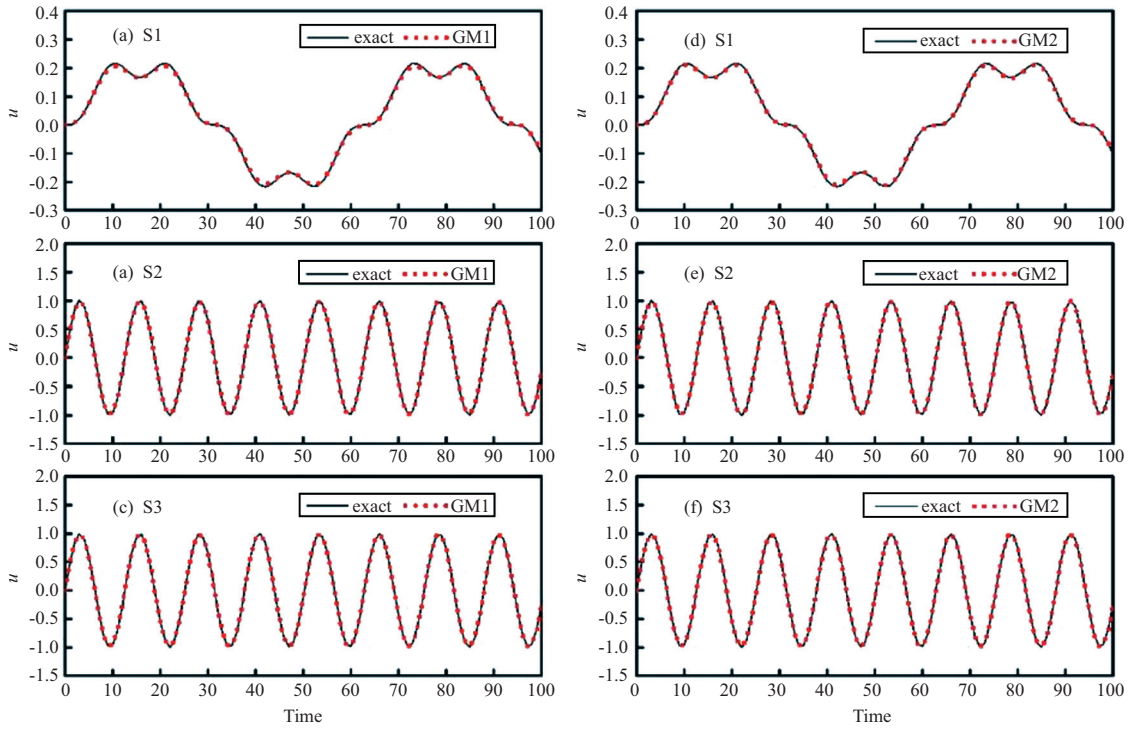


Fig. 4 Forced vibration responses to a SDOF system for using GM1 and GM2 with a load-dependent term.

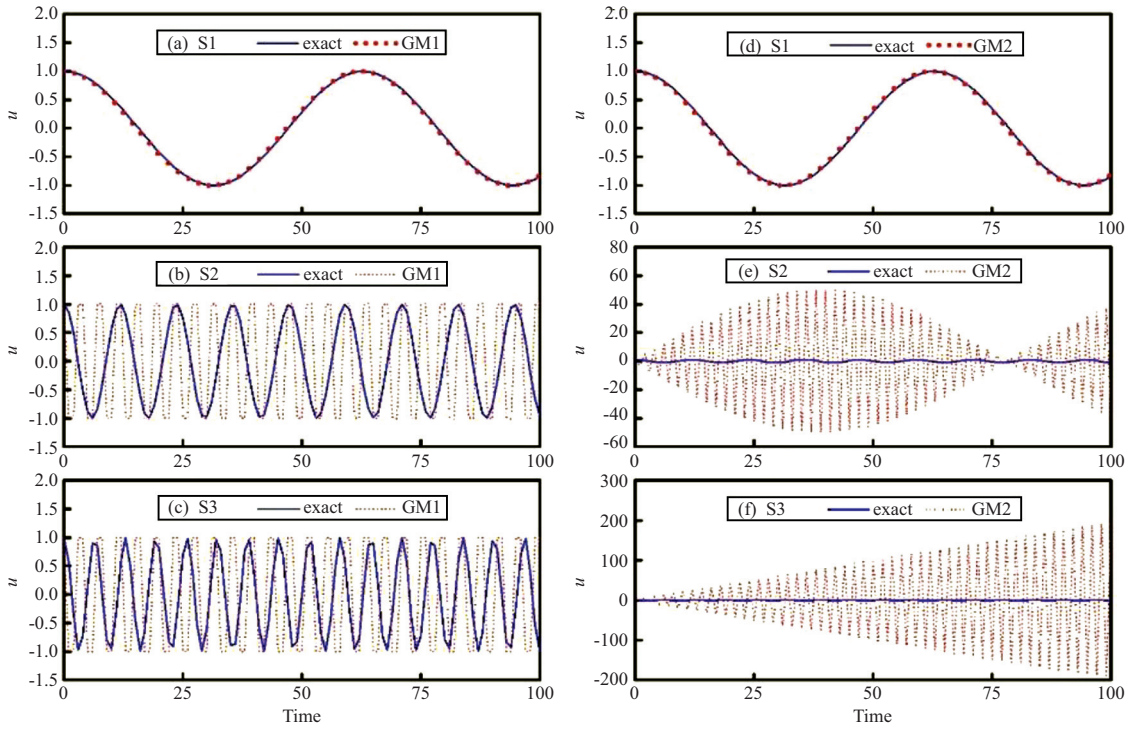


Fig. 5 Responses to linear elastic SDOF systems with initial displacement for using GM1 and GM2.

for GM in solving a system with high frequency modes, the free vibration response to a nonzero initial condition is calculated. Thus, Eq. (5) with zero dynamic load is adopted for this study. Hence, the theoretical solution is found to be $u(t) = \cos(\omega_0 t)$ for the initial conditions of $u(0) = 1$ and $\dot{u}(0) = 0$. Both GM1 and GM2 (CRM) are applied to solve Eq. (5) for zero dynamic load for the three different systems. The time step of $\Delta t = 0.1$ sec is chosen to carry out time integration and the results are shown in Fig. 5. In Figs. 5(a) and 5(d), both GM1 and

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GM2 (CRM) give accurate results. This is because that the value of $\Delta t/T_0$ is as small as 0.016 for S1. On the other hand, although the calculated results obtained from GM1 are inaccurate due to the large values of $\Delta t/T_0$ for S2 and S3 there is no overshoot in the free vibration responses as shown in Figs. 5(b) and 5(c). Whereas, significant overshoot behaviors are found in Figs. 5(e) and 5(f) for the results obtained from GM2 (CRM).

Figure 5 reveals that the results obtained from GM1 and GM2 (CRM) show different phenomena in the responses of high frequency modes. Thus, it is of interest to explore the cause of this difference. For this purpose, the numerical solution calculated by using GM will be analytically derived and is shown next. At first, using the amplification matrix **A** of GM [8] for zero damping, the three eigenvalues are found to be:

$$\lambda_{1,2}=\frac{1+\left(\frac{1}{\lambda}-\frac{1}{2}\right)\Omega_0^2}{1+\frac{1}{\lambda}\Omega_0^2}\pm\frac{\Omega_0\sqrt{1+\left(\frac{1}{\lambda}-\frac{1}{4}\right)\Omega_0^2}}{1+\frac{1}{\lambda}\Omega_0^2}i=e^{\pm i\bar{\Omega}_0},\quad \lambda_3=0 \tag{13}$$

where $\lambda\le 4$; $\lambda_{1,2}$ are the two principal roots and λ_3 is a spurious root. The corresponding eigenvector matrix is:

$$\mathbf{\Phi}=\begin{bmatrix} 1 & 1 & 0\\ \frac{\frac{1}{2}\Omega_0^2}{1+\frac{1}{\lambda}\Omega_0^2}+i\frac{\Omega_0\sqrt{1+\left(\frac{1}{\lambda}-\frac{1}{4}\right)\Omega_0^2}}{1+\frac{1}{\lambda}\Omega_0^2} & \frac{\frac{1}{2}\Omega_0^2}{1+\frac{1}{\lambda}\Omega_0^2}-i\frac{\Omega_0\sqrt{1+\left(\frac{1}{\lambda}-\frac{1}{4}\right)\Omega_0^2}}{1+\frac{1}{\lambda}\Omega_0^2} & \frac{1}{1+\frac{1}{\lambda}\Omega_0^2}\\ -\Omega_0^2 & -\Omega_0^2 & 1\end{bmatrix} \tag{14}$$

In general, GM has three different eigenvalues and thus three linearly independent eigenvectors. Thus, the matrix **A** is diagonalizable and can be expressed as $\mathbf{A}=\mathbf{\Phi\Lambda\Phi}^{-1}$, where $\mathbf{\Lambda}$ is a diagonal matrix and its diagonal term λ_i, i = 1,2,3, is an eigenvalue of **A**. The free vibration response can be obtained from $\mathbf{X}_n=\mathbf{A}^n\mathbf{X}_0=\mathbf{\Phi\Lambda}^n\mathbf{\Phi}^{-1}\mathbf{X}_0$, where $\mathbf{X}_n=[d_n\quad(\Delta t)v_n\quad(\Delta t)^2a_n]^{\mathrm{T}}$. The initial value of a_0 can be determined from $(\Delta t)^2a_0=-2\xi\Omega_0(\Delta t)v_0-\Omega_0^2d_0$ for the given d_0 and v_0. As a result, d_n obtained from GM in a mathematical form is found to be:

$$d_n=\cos\left(n\bar{\Omega}_0\right)d_0+\frac{\sin\left(n\bar{\Omega}_0\right)}{\sin\bar{\Omega}_0}\left(\Delta t\right)v_0,\quad \bar{\Omega}_0=\sin^{-1}\left[\Omega_0\sqrt{1+\left(\frac{1}{\lambda}-\frac{1}{4}\right)\Omega_0^2}\Big/\left(1+\frac{1}{\lambda}\Omega_0^2\right)\right] \tag{15}$$

where $\bar{\Omega}_0$ can be derived from Eq. (13). On the other hand, the free vibration response to an undamped single degree of freedom system can be analytically obtained. As a result, the theoretical solution is found to be:

$$d_n=\cos\left(n\Omega_0\right)d_0+\frac{\sin\left(n\Omega_0\right)}{\Omega_0}\left(\Delta t\right)v_0 \tag{16}$$

Comparing Eqs. (15) to (16), it is found that the coefficients of d_0 and $(\Delta t)v_0$ for GM are almost the same as those of the theoretical solution. In general, the slight difference is between the true Ω_0 and the calculated $\bar{\Omega}_0$ in a numerical procedure. In addition, a slight difference is also found in the denominator of the coefficient of $(\Delta t)v_0$ although $\sin\bar{\Omega}_0\approx\Omega_0$ is found for a small Ω_0.

Equation (15) reveals that GM can generally give a very good approximate solution for the free vibration response. However, a difficulty might experience for the case of $\lambda=4$ for a large value of Ω_0. Notice that the case of $\lambda=4$ for GM is referred as GM2 or CRM herein. Equation (13) reveals that the two complex conjugate eigenvalues will become two identical, real roots in the limit $\Omega_0\to\infty$ for $\lambda=4$. In fact, $\lambda_{1,2}$ becomes -1 as $\Omega_0\to\infty$. In addition, the two corresponding eigenvectors becomes the same. Hence, the matrix **A** in the limit $\Omega_0\to\infty$ has no three linearly independent eigenvectors and thus it cannot be diagonalized. This implies that the result of Eq. (15) is not applicable to GM2 (CRM) in the limit $\Omega_0\to\infty$.

The numerical solution of d_n obtained from GM2 (CRM) in a mathematical form must be derived in a different way. As an alternative, a non-singular matrix $\mathbf{\Psi}$ can be found to transform the matrix **A** into a Jordan canonical form such that $\mathbf{A}=\mathbf{\Psi J\Psi}^{-1}$ and then $\mathbf{A}^n=\mathbf{\Psi J}^n\mathbf{\Psi}^{-1}$. In fact, $\mathbf{\Psi}$, **J** and $\mathbf{J}^n$ are found to be:

$$\mathbf{\Psi}=\begin{bmatrix}1 & 1 & 0\\ 2 & 3 & 0\\ -\Omega_0^2 & -\Omega_0^2 & 1\end{bmatrix},\quad \mathbf{J}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{J}^n=\begin{bmatrix}(-1)^n & n(-1)^{n-1} & 0\\ 0 & (-1)^n & 0\\ 0 & 0 & 0\end{bmatrix} \tag{17}$$

Hence, $\mathbf{X}_n=\mathbf{A}^n\mathbf{X}_0=\mathbf{\Psi J}^n\mathbf{\Psi}^{-1}\mathbf{X}_0$ can be used to derive the free vibration response for using GM2 (CRM). As a result, the numerical solution d_n obtained from GM2 (CRM) in a mathematical form is found to be:

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$$d_n = (2n+1)(-1)^n d_0 - n(-1)^n (\Delta t) v_0 \qquad (18)$$

where d_n generally increases with the increase of the total conducted time steps. This equation is drastically different from the theoretical solution. In fact, the absolute coefficients of d_0 and $(\Delta t)v_0$ increase with the increase of n. In Eq. (16), the coefficient of d_0 is $\cos(n\Omega_0)$, varying from –1 to 1; and that for $(\Delta t)v_0$ is $\sin(n\Omega_0)/\Omega_0$, which will decrease to zero for a large Ω_0. Thus, the difference between Eqs. (16) and (18) will become significant for a large Ω_0. In fact, either the coefficient of d_0 or $(\Delta t)v_0$ can conclude that GM2 (CRM) will experience a weak instability property for high frequency modes. Consequently, it is verified by this analytical study that GM2 (CRM) has an adverse weak instability property and it will experience in the response to nonzero initial conditions for the systems with high frequency modes. These analytical results are highly consistent with the results shown in Fig. 5.

7. CONCLUSIONS

A family of structure-dependent integration methods was published recently. In order to avoid the misuse of this family method, some adverse properties of this family method are assessed and addressed in this work. The three adverse properties are: (1) a poor capability to capture structural nonlinearity; (2) an overshoot in the high frequency steady-state response; and (3) a weak instability property for any nonzero initial condition for the special member GM2 (CRM). The cause of a poor capability to capture structural nonlinearity may be due to a convergence rate of 1 for velocity error. An overshoot in a high frequency steady-state response is due to an incomplete formulation of the displacement difference equation. This difficulty can be eliminated by introducing a load-dependent term into the displacement difference equation and thus this modification must be adopted for the general form of this family method. A member of this family method, which is also known as CR method, has a weak instability. This is because that it has linearly dependent eigenvectors for a high frequency mode and thus the amplification matrix cannot be diagonalized. Hence, its applications to conduct time integration must be prohibited.

ACKNOWLEDGEMENTS

The author is grateful to acknowledge that this study is financially supported by the National Science Council, Taiwan, R.O.C., under Grant No. NSC-105-2211-E-027-029.

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(Manuscript received November 24, 2016,
accepted for publication June 19, 2017.)

https://doi.org/10.1017/jmech.2017.49 Published online by Cambridge University Press