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An expansion formula for Fox's H-function

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1. *Introduction.* The object of this paper is to establish an integral involving Fox's H-function and employ it to obtain an expansion formula for the H-function involving Bessel functions.

The H-function introduced by Fox ((4), p. 408), will be represented and defined as follows:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\,z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)}\,ds, \quad (1{\cdot}1)$$

where an empty product is interpreted as 1, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; e's and f's are all positive, L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j-f_j s)$, $j=1,2,\ldots,m$ lie on the right-hand side of the contour and those of $\Gamma(1-a_j+e_j s)$, $j=1,2,\ldots,n$ lie on the left-hand side of the contour.

Recently Braaksma (1) has discussed the asymptotic expansion and analytic continuation of the H-function.

2. *The integral.* The integral to be evaluated is

$$\int_0^\infty x^{-u}J_w(x)\,J_v(x)\,H_{p,q}^{m,n}\left[zx^{2h}\left|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right]dx=2^{-u}H_{p+4,q+1}^{m+1,n+1}$$
$$\times\left[2^{2h}z\left|\begin{matrix}\left(\frac{1+u-w-v}{2},h\right),(a_1,e_1),\ldots,(a_p,e_p),\left(\frac{u+w+v+1}{2},h\right),\left(\frac{u-w+v+1}{2},h\right),\\ \left(\frac{u+w-v+1}{2},h\right);\\ (u,2h),(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right], \quad (2{\cdot}1)$$

where h is a positive number and

$$\sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_j\leqslant 0,\quad \sum_{j=1}^{n}e_j-\sum_{j=n+1}^{p}e_j+\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_j\equiv k>0,\quad |\arg z|<\tfrac{1}{2}k\pi,$$

$$\Re(w+v-u+2hb_i/f_i)>-1\quad (i=1,2,\ldots,m),$$

$$\Re(2ha_j/e_j-u)<2h\quad (j=1,2,\ldots,n).$$

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Proof. Expressing the H-function in the integrand as a Mellin–Barnes type integral (1·1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\,z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)}\int_0^\infty x^{-u+2hs}J_w(x)\,J_v(x)\,dx\,ds.$$

Evaluating the inner-integral with the help of the formula ((3), p. 342, (24)), we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\,2^{-u+2hs}\,\Gamma(u-2hs)\,\Gamma[\frac{1}{2}(w+v-u+2hs+1)]_z s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)\,\Gamma[\frac{1}{2}(u-2hs+w+v+1)]}\,ds.$$
$$\times\,\Gamma[\tfrac{1}{2}(u-2hs-w+v+1)]\,\Gamma[\tfrac{1}{2}(u-2hs+w-v+1)]$$

On applying (1·1), the integral is established.

3. *The expansion formula.* The expansion formula to be established is

$$x^{-u}J_w(x)\,H_{p,q}^{m,n}\left[zx^{2h}\middle|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right]=2^{-u}\sum_{s=0}^{\infty}rH_{p+4,q+1}^{m+1,n+1}$$
$$\times\left[2^{2h}z\middle|\begin{matrix}\left(\frac{2+u-w-r}{2},h\right),(a_1,e_1),\ldots,(a_p,e_p),\left(\frac{2+u+w+r}{2},h\right),\\ \left(\frac{2+u-w+r}{2},h\right),\left(\frac{2+u+w-r}{2},h\right);\\ (u+1,2h),(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right]J_r(x),\qquad(3\cdot1)$$

where h is a positive number and

$$\sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_j\leqslant 0,\quad \sum_{j=1}^{n}e_j-\sum_{j=n+1}^{p}e_j+\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_i\equiv k>0,\quad |\arg z|<\tfrac{1}{2}k\pi,$$

$$\Re(w+v-u+2hb_i/f_i)>0\quad(i=1,2,\ldots,m),$$

$$\Re(2ha_j/e_j-u)<2h+1\quad(j=1,2,\ldots,n),\quad r\equiv\mu+2s+1.$$

Proof. Let

$$f(x)=x^{-u}J_w(x)\,H_{p,q}^{m,n}\left[zx^{2h}\middle|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right]$$
$$=\sum_{s=0}^{\infty}C_s\,J_{\mu+2s+1}(x).\qquad(3\cdot2)$$

Equation (3·2) is valid, since $f(x)$ is continuous and of bounded variation in the open interval $(0,\infty)$ when $u\geqslant 0$.

Multiplying both sides of (3·2) by $x^{-1}J_{\mu+2l+1}(x)$ and integrating with respect to x from 0 to ∞, we have

$$\int_0^\infty x^{-u-1}J_{\mu+2l+1}(x)\,J_w(x)\,H_{p,q}^{m,n}\left[zx^{2h}\middle|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right]dx$$
$$=\sum_{s=0}^{\infty}C_s\int_0^\infty x^{-1}J_{\mu+2l+1}(x)\,J_{\mu+2s+1}(x)\,dx.$$

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Now using (2·1) and the orthogonality property of Bessel functions ((5), p. 291, (6)), we get

$$C_t = 2^{-u}vH^{m+1,n+1}_{p+4,q+1}\left[2^{2h}z\left|\begin{matrix}\left(\frac{2+u-w-v}{2},h\right),(a_1,e_1),\ldots,(a_p,e_p),\left(\frac{2+u+w+v}{2},h\right),\\ \left(\frac{2+u-w+v}{2},h\right),\left(\frac{2+u+w-v}{2},h\right);\\ (u+1,2h),(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right], \tag{3·3}$$

where $v \equiv \mu + 2t + 1$. From (3·2) and (3·3), the formula (3·1) follows.

4. *Particular cases.* On specializing the parameters the H-function can be reduced to Meijer's G-function, MacRobert's E-function and other higher transcendental functions. Therefore, the results (2·1) and (3·1) lead to the generalization of many results. Only one interesting particular case is given below:

In (3·1), assuming h as a positive integer, putting $e_j = f_i = 1$ $(j = 1, 2, \ldots, p;\ i = 1, 2, \ldots, q)$, using the formula

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,1),\ldots,(a_p,1)\\(b_1,1),\ldots,(b_q,1)\end{matrix}\right.\right] = G^{m,n}_{p,q}\left[z\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right]$$

and simplifying with the help of (1·1), ((2), p. 4, (11)) and ((2), p. 207, (1)), we obtain

$$x^{-u}J_w(x)\,G^{m,n}_{p,q}\left[zx^{2h}\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right] = \frac{h^{-u-\frac{1}{2}}}{\sqrt{(\pi)}}$$

$$\times\sum_{s=0}^{\infty} rG^{m+2h,n+h}_{p+4h,q+2h}\left[2^{2h}z\left|\begin{matrix}\Delta\left(h,\frac{2+u-w-r}{2}\right),a_1,\ldots,a_p,\Delta\left(h,\frac{2+u+w+r}{2}\right),\\ \Delta\left(h,\frac{2+u-w+r}{2}\right),\Delta\left(h,\frac{2+u+w-r}{2}\right);\\ \Delta(2h,u+1),b_1,\ldots,b_q\end{matrix}\right.\right]J_r(x), \tag{4·1}$$

where the symbol $\Delta(h,\alpha)$ represents the set of parameters

$$\frac{\alpha}{h},\frac{\alpha+1}{h},\ldots,\frac{\alpha+h-1}{h} \quad \text{and} \quad 2(m+n) > p+q, \quad |\arg z| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$

$$\Re(w-u+2hb_j) > 0 \quad (j=1,2,\ldots,m),$$

$$\Re(2ha_j-u) < 2h+1 \quad (j=1,2,\ldots,n), \quad r \equiv \mu+2s+1.$$

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