Euro. Jnl of Applied Mathematics (2009), *vol.* 20, *pp.* 145–167. © 2008 Cambridge University Press
doi:10.1017/S0956792508007663 Printed in the United Kingdom

Weak solvability of a piezoelectric contact problem†

STANISŁAW MIGÓRSKI[1], ANNA OCHAL[1]
and MIRCEA SOFONEA[2]

[1]*Faculty of Mathematics and Computer Science, Institute of Computer Science, Jagiellonian University, Nawojki 11, 30072 Krakow, Poland*
email: Stanislaw.Migorski@softlab.ii.uj.edu.pl; ochal@softlab.ii.uj.edu.pl
[2]*Laboratoire de Mathématiques, Physique et Systèmes, Université de Perpignan, 52 Avenue Paul Alduy, 66860 Perpignan, France*
email: sofonea@univ-perp.fr

(*Received 13 November 2007; revised 30 May 2008; first published online 22 July 2008*)

We consider a mathematical model which describes the frictional contact between a piezoelectric body and a foundation. The material behaviour is modelled with a non-linear electro-elastic constitutive law, the contact is bilateral, the process is static and the foundation is assumed to be electrically conductive. Both the friction law and the electrical conductivity condition on the contact surface are described with subdifferential boundary conditions. We derive a variational formulation of the problem which is of the form of a system of two coupled hemi-variational inequalities for the displacement and the electric potential fields, respectively. Then we prove the existence of a weak solution to the model and, under additional assumptions, its uniqueness. The proof is based on an abstract result on operator inclusions in Banach spaces.

1 Introduction

Piezoelectric materials represent a special class of dielectric materials which exhibit significant deformations in response to an applied electric field as well as dielectric polarization in response to mechanical strains. Some materials are naturally piezoelectric, e.g. crystals, living bones, human skin, etc., others are manufactured with piezoelectric characteristics. Such materials are very important in many applications arising in biomechanics, biomedicine, structural mechanics, radioelectronics and electroacoustics. In particular, they are intensively used as sensors and actuators in various measuring equipments and in electronic clocks.

Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials. General models for electro-elastic materials can be found in [3, 9, 14] and in a more recent monograph [18]. Static frictional contact problems for electro-elastic materials were studied in [4, 10–12, 16] under the assumption that the foundation is insulated. The results in [4, 10] concern mainly the numerical simulation of

† The work of the first two authors was supported in part by the Ministry of Science and Higher Education of Poland under Grant no. N201 027 32/1449.

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

the problems while the results in [11, 12, 16] deal with the variational formulations of the problems and their unique weak solvability.

In this paper we deal with a mathematical model which describes the frictional contact between a piezoelectric body and a foundation. The body is supposed to be electro-elastic, with a non-linear elasticity operator. Unlike the models considered in [11, 12, 16], in the present paper we assume that the contact is bilateral, i.e. there is no loss of the contact during the process and, moreover, the foundation is electrically conductive. We take into account the electrical conductivity of the foundation both in the friction law and in the electrical condition on the contact surface, which are described with general subdifferential boundary conditions involving the difference between the potential on the foundation and the body surface. This leads to a new and non-standard mathematical model which is in the form of a system of two hemivariational inequalities for the displacement field and the electric potential and which can be formulated, equivalently, as an abstract inclusion of subdifferential type, with a locally Lipschitz integral functional. Details on the novelty of this model as well as on the mathematical difficulties it involves are provided in Remarks 5, 11 and 14.

The main results of this paper concern the existence and uniqueness of the weak solution of the model. To provide the existence of the solution we apply a fundamental result of Non-linear Analysis which says that a coercive pseudomonotone operator between a reflexive Banach space and its dual is surjective. The uniqueness of a solution is obtained in a case when the operator modelling the process is strongly monotone and the subdifferential of the superpotential satisfies a relaxed monotonocity condition.

Although in this paper we do not deal with real-world applications of our model, we mention that problems involving piezoelectric contact arise in smart structures and various device applications (see for instance [18] and the references therein). For instance, the relative motion of two bodies may be detected by a piezoelectric sensor in frictional contact with them, as stated in [4], and vibration of elastic plates may be obtained by the contact with a piezoelectric actuator under electric voltage (see [17] for details). Also, the contact of a read/write piezoelectric head on a hard disk is based on the mechanical deformation generated by the inverse piezoelectric effect (see for instance [2]); there, error estimates and numerical simulations in the study of the corresponding piezoelectric contact problem are provided.

The paper is structured as follows. In Section 2 we describe the model of bilateral contact between an electro-elastic body and a conductive foundation. In Section 3 we list the assumptions on the data and derive the variational formulation of the problem. In Section 4 we state and prove our main existence result, Theorem 6. The proof is carried out in several steps and are based on recent results on abstract inclusions of subdifferential type. Then, in Section 5 we state and prove a uniqueness result, Theorem 12. Some auxiliary results, needed in the proof of Theorem 6, are presented in Section 6. The paper concludes with Section 7 in which we present a short summary of the results and a discussion of open problems and future directions.

We end this section with some notation and definitions we shall use in this paper. Given a Banach space X, we denote its norm by $\|\cdot\|_X$. The dual space is denoted by X^* and $\langle\cdot,\cdot\rangle_{X^*\times X}$ is the duality pairing between X and X^*. The notation $\mathscr{L}(X,Y)$ stands for the space of linear bounded operators from a Banach space X to a Banach space Y. Let

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

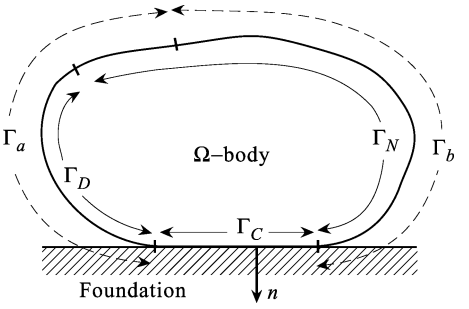


FIGURE 1. The physical setting; Γ_C is the contact surface.

$\theta : X \to \mathbb{R}$ be a locally Lipschitz function. Then, the generalized directional derivative of θ at $x \in X$ in the direction $v \in X$, denoted by $\theta^0(x; v)$, is defined by

$$\theta^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{\theta(y + \lambda v) - \theta(y)}{\lambda}.$$

The generalized gradient of θ at x, denoted by $\partial\theta(x)$, is the subset of X^* given by

$$\partial\theta(x) = \{\zeta \in X^* \ : \ \theta^0(x; v) \geqslant \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X\}.$$

A locally Lipschitz function θ is called regular (in the sense of Clarke [5]) at $x \in X$ if for all $v \in X$ the one-sided generalized directional derivative $\theta'(x; v)$ exists and satisfies $\theta^0(x; v) = \theta'(x; v)$ for all $v \in X$.

2 The contact problem

The physical setting is depicted in Figure 1 and is as follows. We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal n. The body is acted upon by body forces of density f_0 and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we consider a partition of Γ into three open disjoint parts Γ_D, Γ_N and Γ_C, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_1 and Γ_2, on the other hand. We assume that $meas\, \Gamma_D > 0$ and $meas\, \Gamma_1 > 0$. The body is clamped on Γ_D and therefore the displacement field vanishes there. Surface tractions of density f_N act on Γ_N. We also assume that the electrical potential vanishes on Γ_1 and a surface electrical charge of density q_2 is prescribed on Γ_2. The body is in frictional contact over Γ_C with a conductive obstacle, the so-called foundation. The contact is bilateral, i.e. there is no loss of the contact during the process. The process is assumed to be static, i.e. the inertial terms in the momentum balance equations are neglected, and the material behaviour is described with an electro-elastic constitutive law.

With the assumptions above, the classical model for the process is as follows.

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

Problem P. Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathcal{S}_d$, an electric potential $\varphi : \Omega \to \mathbb{R}$ and an electric displacement field $D : \Omega \to \mathbb{R}^d$ such that

$$\begin{aligned}
&\operatorname{Div}\sigma + f_0 = 0 && \text{in } \Omega, && (1)\\
&\operatorname{div} D - q_0 = 0 && \text{in } \Omega, && (2)\\
&\sigma = \mathcal{F}\varepsilon(u) - \mathcal{P}^{\top}E(\varphi) && \text{in } \Omega, && (3)\\
&D = \mathcal{P}\varepsilon(u) + \beta E(\varphi) && \text{in } \Omega, && (4)\\
&u = 0 && \text{on } \Gamma_D, && (5)\\
&\sigma n = f_N && \text{on } \Gamma_N, && (6)\\
&\varphi = 0 && \text{on } \Gamma_1, && (7)\\
&D \cdot n = q_2 && \text{on } \Gamma_2, && (8)\\
&u_n = 0 && \text{on } \Gamma_C, && (9)\\
&-\sigma_\tau \in h_\tau(\varphi - \varphi_0)\,\partial j_\tau(u_\tau) && \text{on } \Gamma_C, && (10)\\
&D \cdot n \in \partial j(\varphi - \varphi_0) && \text{on } \Gamma_C. && (11)
\end{aligned}$$

Here and below, to simplify the notation, we do not indicate the dependence of various functions on $x \in \Omega \cup \Gamma$; the indices i, j, k, l run from 1 to d and the summation over two repeated indices is applied; $\mathcal{S}_d$ represents the space of second-order symmetric tensors on $\mathbb{R}^d$; and, finally, the inner products and the Euclidean norms on $\mathbb{R}^d$ and $\mathcal{S}_d$ are denoted as follows:

$$u \cdot v = u_i v_i, \qquad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \qquad \text{for all } u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\|_{\mathcal{S}_d} = (\tau : \tau)^{1/2} \qquad \text{for all } \sigma, \tau \in \mathcal{S}_d.$$

We now describe problems (1)–(11) and provide explanation of the equations and boundary conditions.

First, equations (1) and (2) are the equilibrium equations for the stress and electric-displacement fields, respectively; here and below Div and div denote the divergence operator for tensor and vector valued functions, i.e.

$$\operatorname{Div}\sigma = (\sigma_{ij,j}) \quad \text{and} \quad \operatorname{div} D = (D_{i,i}),$$

and the index that follows a comma represents the partial derivative with respect to the corresponding component of x. We use equations (1) and (2) since the process is assumed to be static.

Next, equations (3) and (4) represent the electro-elastic constitutive law in which $\varepsilon(u)$ denotes the linearized strain tensor, $\mathcal{F}$ is the elasticity operator, assumed to be non-linear, $\mathcal{P} = (p_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{P}^{\top}$ is its transpose, $\beta = (\beta_{ij})$ denotes the electric permittivity tensor and $E(\varphi)$ is the electric field. We recall that

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E(\varphi) = -\nabla\varphi = -(\varphi_{,i}).$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

The tensors $\mathscr{P}$ and $\mathscr{P}^\top$ satisfy the equality

$$\mathscr{P}\sigma \cdot v = \sigma : \mathscr{P}^\top v \quad \text{for all } \sigma \in \mathscr{S}_d,\ v \in \mathbb{R}^d,$$

and the components of the tensor $\mathscr{P}^\top$ are given by $p^\top_{ijk} = p_{kij}$. Equation (3) indicates that the mechanical properties of the material are described by a non-linear elastic constitutive relation which takes into account the dependence of the stress field on the electric field. Relation (4) describes a linear dependence of the electric displacement field D on the strain and electric fields. Constitutive equations of the form (3) and (4) have been frequently employed in the literature (see e.g. [3, 4, 14] and the references therein).

Conditions (5) and (6) are the displacement and traction boundary conditions, whereas (7) and (8) represent the electric boundary conditions; these conditions model the fact that the displacement field and the electrical potential vanish on Γ_D and Γ_1, respectively, while the forces and free electric charges are prescribed on Γ_N and Γ_2, respectively.

Our main interest is in the boundary conditions (9), (10), (11), which describe the contact, the frictional and the electrical conductivity conditions on the contact surface Γ_C, respectively. Here and below u_n and u_τ denote the normal and the tangential displacement, respectively, given by $u_n = u \cdot n$, $u_\tau = u - u_n n$, σ_n, and σ_τ represent the normal and the tangential stress defined by $\sigma_n = (\sigma n) \cdot n$ and $\sigma_\tau = \sigma n - \sigma_n n$. The functions h_τ, j_τ and j are prescribed, the symbol ∂p denotes the Clarke subdifferential or the generalized gradient of a function p and, finally, φ_0 represents the electric potential of the surface, assumed to be given. We use condition (9) since the contact is bilateral, i.e. there is no loss of the contact during the process. Such situations can be found in many machines and in moving parts and component of mechanical equipment (see e.g. [8, 15] and the references therein).

We now present examples of frictional and electrical conductivity conditions which can be cast in a subdifferential form (10) and (11), respectively. For simplicity we restrict ourselves to the homogeneous case, i.e. we drop the dependence of various functions with respect to the spatial variable $x \in \Gamma_C$.

Example 1 We assume that the tangential traction satisfies

$$\|\sigma_\tau\|_{\mathbb{R}^d} \leqslant h_\tau(\varphi - \varphi_0), \quad \sigma_\tau = -h_\tau(\varphi - \varphi_0)\frac{u_\tau}{\|u_\tau\|_{\mathbb{R}^d}} \quad \text{if} \quad u_\tau \neq 0, \quad \text{on } \Gamma_C, \tag{12}$$

where h_τ is a prescribed non-negative function, the friction bound. Condition (12) represents a static version of Tresca's friction law in which the friction bound depends on the difference between the potential on the foundation and the body surface. This dependence models the influence of the electric variables on the friction law and it is used here since the foundation is supposed to be electrically conductive. In the case when h_τ is a constant, the friction law (12) is used in certain applications when the loads are light, or the friction is large (see [8, 15] for details). It is easy to check that (12) leads to the subdifferential condition (10) in which

$$j_\tau(\xi) = \|\xi\|_{\mathbb{R}^d} \quad \text{for all} \quad \xi \in \mathbb{R}^d. \tag{13}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

Example 2 Since the functional (13) is non-differentiable at zero, several regularization parameters of the friction law (12) can be used. A simple example is given by

$$-\sigma_\tau = h_\tau(\varphi - \varphi_0)\frac{u_\tau}{\sqrt{\|u_\tau\|^2_{\mathbb{R}^d} + \rho^2}} \quad \text{on } \Gamma_C, \tag{14}$$

where ρ represents a positive regularization parameter. It is easy to see that (14) can be cast on the form (10) with the choice

$$j_\tau(\xi) = \frac{\xi}{\sqrt{\|\xi\|^2_{\mathbb{R}^d} + \rho^2}} \quad \text{for all } \xi \in \mathbb{R}^d. \tag{15}$$

The regularization (15) of the functional j_τ in (13) was intensively used in the literature, mainly for numerical reasons (see e.g. [15]).

Example 3 Let $d = 2$ and note that in this case, at each point of the contact surface there is only a tangential direction. Denoted below by σ_τ and u_τ are the scalar components of the tangential traction and the tangential displacement, respectively. We assume that

$$-\sigma_\tau = h_\tau(\varphi - \varphi_0)\, p_\tau(u_\tau) \qquad \text{on } \Gamma_C, \tag{16}$$

where both h_τ and p_τ are prescribed as non-negative functions. Let $j_\tau : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j_\tau(\xi) = \int_0^\xi p_\tau(s)\, ds \quad \text{for all } \xi \in \mathbb{R},$$

where $p_\tau : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function. With this choice of j_τ, by using the arguments in [13, p. 17], it is easy to see that the frictional condition (16) is equivalent to the multi-valued condition (10). Note that here the function p_τ is not assumed to be increasing and, therefore, the superpotential function j_τ is not necessarily a convex function. Concrete examples of such functions can be found in [13, p. 53]. They were used in literature in order to model the contact between reinforcement and concrete in concrete structures or the partial cracking and crushing of the adhesive materials.

Example 4 During the process of contact the normal component of the electric displacement field or the free charge is assumed to depend on the difference between the potential on the foundation and the body surface. Thus,

$$D \cdot n = p(\varphi - \varphi_0) \quad \text{on } \Gamma_C, \tag{17}$$

where p is a prescribed real-valued continuous function. A possible choice of the function p is $p(r) = kr$, where k is a positive constant, the electrical conductivity coefficient. Let $j : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j(\xi) = \int_0^\xi p(s)\, ds \quad \text{for all } \xi \in \mathbb{R}. \tag{18}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

Then, using again the arguments in [13, p. 17], it is easy to see that the contact electrically condition (17) is equivalent to the multi-valued condition (11). Note that here the function p is not assumed to be increasing and, therefore, the potential function j is not necessarily a convex function. Note also that when $p \equiv 0$, then (17) leads to

$$D \cdot n = 0 \quad \text{on } \Gamma_C. \tag{19}$$

The electrical boundary condition (19), used in [4, 10, 11, 16], models the case when the obstacle is a perfect insulator.

3 Variational formulation

To present the variational formulation of Problem P we need some additional notation and preliminaries. We start by introducing the spaces

$$H = L^2(\Omega;\mathbb{R}^d), \quad \mathscr{H} = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega;\mathscr{S}_d),$$

$$H_1 = \{u \in H : \varepsilon(u) \in \mathscr{H}\} = H^1(\Omega;\mathbb{R}^d), \quad \mathscr{H}_1 = \{\tau \in \mathscr{H} : \operatorname{Div}\tau \in H\},$$

$$W = \{D \in H : \operatorname{div} D \in L^2(\Omega)\},$$

where ε, Div and div denote the deformation and the divergence operators, defined in Section 2. The spaces H, $\mathscr{H}$, H_1, $\mathscr{H}_1$ and W are Hilbert spaces equipped with the inner products

$$\langle u, v\rangle_H = \int_\Omega u \cdot v \, dx, \quad \langle \sigma, \tau\rangle_{\mathscr{H}} = \int_\Omega \sigma : \tau \, dx,$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}}, \quad \langle \sigma, \tau\rangle_{\mathscr{H}_1} = \langle \sigma, \tau\rangle_{\mathscr{H}} + \langle \operatorname{Div}\sigma, \operatorname{Div}\tau\rangle_H,$$

$$\langle D, E\rangle_W = \langle D, E\rangle_H + \langle \operatorname{div} D, \operatorname{div} E\rangle_{L^2(\Omega)}.$$

Given $v \in H^{1/2}(\Gamma;\mathbb{R}^d)$ we denote by v_n and v_τ the normal and the tangential components of v on the boundary Γ, i.e. $v_n = v \cdot n$ and $v_\tau = v - v_n n$. Similarly, for a regular tensor field $\sigma : \Omega \to \mathscr{S}_d$, we define its normal and tangential components by $\sigma_n = (\sigma n) \cdot n$ and $\sigma_\tau = \sigma n - \sigma_n n$. Recall that the following Green formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_{\mathscr{H}} + \langle \operatorname{Div}\sigma, v\rangle_H = \int_\Gamma \sigma n \cdot v \, d\Gamma \quad \text{for} \quad v \in H_1. \tag{20}$$

Also, for a regular vector field $D : \Omega \to \mathbb{R}^d$, the following Green formula holds:

$$\langle D, \nabla\psi\rangle_H + \langle \operatorname{div} D, \psi\rangle_{L^2(\Omega)} = \int_\Gamma D \cdot n \, \psi \, d\Gamma \quad \text{for} \quad \psi \in H^1(\Omega). \tag{21}$$

For the displacement and the electric potential fields we introduce the spaces

$$V = \{v \in H^1(\Omega;\mathbb{R}^d) \ : \ v = 0 \ \text{ on } \Gamma_D, \ v_n = 0 \ \text{ on } \Gamma_C\},$$

$$\Phi = \{\varphi \in H^1(\Omega) \ : \ \varphi = 0 \ \text{ on } \Gamma_1\}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

which are closed subspaces of H_1 and $H^1(\Omega)$, respectively. On V and Φ we consider the inner products and the corresponding norms given by

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}} \quad \text{for all } u, v \in V.$$

$$\langle \varphi, \psi\rangle_\Phi = \langle \nabla\varphi, \nabla\psi\rangle_H, \quad \|\psi\|_\Phi = \|\nabla\psi\|_H \quad \text{for all } \varphi, \psi \in \Phi.$$

Since $meas\,\Gamma_D$ and $meas\,\Gamma_1$ are positive, it follows from the Korn and the Friedrichs–Poincaré inequalities, respectively, that $(V, \|\cdot\|_V)$ and $(\Phi, \|\cdot\|_\Phi)$ are Hilbert spaces.

In the study of problems (1)–(11) we assume that the elasticity operator $\mathcal{F}$, the piezoelectric tensor $\mathcal{P}$ and the electric permittivity tensor β satisfy the following hypotheses:

$\underline{H(\mathcal{F})}$: $\mathcal{F} : \Omega \times \mathcal{S}_d \to \mathcal{S}_d$ is such that

(i) $\mathcal{F}(\cdot, \varepsilon)$ is measurable on Ω for all $\varepsilon \in \mathcal{S}_d$;

(ii) $\mathcal{F}(x, \cdot)$ is continuous on $\mathcal{S}_d$ for a.e. $x \in \Omega$;

(iii) $\|\mathcal{F}(x, \varepsilon)\|_{\mathcal{S}_d} \leqslant \alpha_1(b(x) + \|\varepsilon\|_{\mathcal{S}_d})$ for all $\varepsilon \in \mathcal{S}_d$, a.e. $x \in \Omega$ with $b \in L^2(\Omega)$, $\alpha_1 > 0$;

(iv) $(\mathcal{F}(x, \varepsilon_1) - \mathcal{F}(x, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geqslant 0$ for all $\varepsilon_1, \varepsilon_2 \in \mathcal{S}_d$ and a.e. $x \in \Omega$;

(v) $\mathcal{F}(x, \varepsilon) : \varepsilon \geqslant \alpha_2 \|\varepsilon\|^2_{\mathcal{S}_d}$ for all $\varepsilon \in \mathcal{S}_d$ and a.e. $x \in \Omega$ with $\alpha_2 > 0$.

$\underline{H(\mathcal{P})}$: $\mathcal{P} : \Omega \times \mathcal{S}_d \to \mathbb{R}^d$ is such that $\mathcal{P}(x, \varepsilon) = p(x)\,\varepsilon$ for all $\varepsilon \in \mathcal{S}_d$, a.e. $x \in \Omega$ and $p(x) = \{p_{ijk}(x)\}$, $p_{ijk} \in L^\infty(\Omega)$.

$\underline{H(\beta)}$: $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is such that $\beta(x, \xi) = \beta(x)\,\xi$ for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$ and

(i) $\beta(x) = \{\beta_{ij}(x)\}$ with $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$;

(ii) $\beta_{ij}(x)\xi_i\xi_j \geqslant m_\beta \|\xi\|^2_{\mathbb{R}^d}$ for all $\xi = (\xi_i) \in \mathbb{R}^d$ and a.e. $x \in \Omega$ with $m_\beta > 0$.

The functions j_τ and j satisfy

$\underline{H(j_\tau)}$: $j_\tau : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j_\tau(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $j_\tau(\cdot, 0) \in L^1(\Gamma_C)$;

(ii) $j_\tau(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_C$;

(iii) $\|\partial j_\tau(x, \xi)\|_{\mathbb{R}^d} \leqslant c_\tau(1 + \|\xi\|_{\mathbb{R}^d})$ for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Gamma_C$ with $c_\tau > 0$;

(iv) $j^0_\tau(x, \xi; -\xi) \leqslant d_\tau(1 + \|\xi\|_{\mathbb{R}^d})$ for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Gamma$ with $d_\tau \geqslant 0$;

(v) either $j_\tau(x, \cdot)$ or $-j_\tau(x, \cdot)$ is regular for a.e. $x \in \Gamma_C$.

$\underline{H(j)}$: $j : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $j(\cdot, r)$ is measurable for all $r \in \mathbb{R}$ and $j(\cdot, 0) \in L^1(\Gamma_C)$;

(ii) $j(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_C$;

(iii) $|\partial j(x, r)| \leqslant c(1 + |r|)$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $c > 0$;

(iv) $j^0(x, r; -r) \leqslant d(1 + |r|)$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $d \geqslant 0$;

(v) either $j(x, \cdot)$ or $-j(x, \cdot)$ is regular for a.e. $x \in \Gamma_C$.

Here and below the symbols ∂j_τ and ∂j denote the generalized subdifferentials of $j_\tau(x, \cdot)$ and $j(x, \cdot)$, respectively.

The function h_τ satisfies

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

$\underline{H(h_\tau)}$: $h_\tau : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $h_\tau(\cdot, r)$ is measurable for all $r \in \mathbb{R}$ and $h_\tau(x, \cdot)$ is continuous for a.e. $x \in \Gamma_C$;

(ii) $0 \leqslant h_\tau(x, r) \leqslant \overline{h}_\tau$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $\overline{h}_\tau > 0$.

Note that assumptions $H(j_\tau)$, $H(j)$ and $H(h_\tau)$ are valid for the boundary conditions described in Examples 1–4 in Section 2, provided that appropriate assumptions are imposed on the functions used to describe the corresponding contact processes. To provide an example, consider a bilateral frictional contact process with electrical potential dependent friction law (12), associated to the electrical contact condition (17). Assume that $h_\tau,\ p : \mathbb{R} \to \mathbb{R}$ are continuous functions such that

$$0 \leqslant h_\tau(r) \leqslant \overline{h}_\tau, \quad |p(r)| \leqslant c\,(1 + |r|) \quad \text{for all } r \in \mathbb{R}$$

with $c, \overline{h}_\tau > 0$. Then the functions j_τ and j defined by (13) and (18) satisfy the conditions $H(j_\tau)$ and $H(j)$, respectively, and the function h_τ satisfies the condition $H(h_\tau)$. We conclude that our results below are valid in the study of this example.

The forces, tractions, volume and surface-free charge densities satisfy

$\underline{H(f)}$: $f_0 \in L^2(\Omega; \mathbb{R}^d)$, $f_N \in L^2(\Gamma_N; \mathbb{R}^d)$,

$\underline{H(q)}$: $q_0 \in L^2(\Omega)$, $q_2 \in L^2(\Gamma_2)$,

whereas the potential of the surface satisfies

$\underline{H(\varphi_0)}$: $\varphi_0 \in L^\infty(\Gamma_C)$.

We now turn to the variational formulation of the contact problems (1)–(11). To this end, we assume in what follows that u, σ, φ, D are regular functions which solve (1)–(11). Let $v \in V$. We use the equilibrium equation (1) and the Green formula (20) to find that

$$\langle \sigma, \varepsilon(v) \rangle_{\mathscr{H}} = \int_\Omega f_0 \cdot v \, dx + \int_\Gamma \sigma n \cdot v \, d\Gamma. \tag{22}$$

We take into account the boundary conditions (5) and (6) and equality $v_n = 0$ on Γ_C to see that

$$\int_\Gamma \sigma n \cdot v \, d\Gamma = \int_{\Gamma_N} f_N \cdot v \, d\Gamma + \int_{\Gamma_C} \sigma_\tau \cdot v_\tau \, d\Gamma. \tag{23}$$

On the other hand, from the definition of the Clarke subdifferential combined with the inclusion (10), we have $-\sigma_\tau \cdot v_\tau \leqslant h_\tau(\varphi - \varphi_0) j_\tau^0(u_\tau; v_\tau)$ on Γ_C which implies that

$$\int_{\Gamma_C} \sigma_\tau \cdot v_\tau \, d\Gamma \geqslant - \int_{\Gamma_C} h_\tau(\varphi - \varphi_0)\, j_\tau^0(u_\tau; v_\tau)\, d\Gamma. \tag{24}$$

Consider the element $f \in V^*$ given by

$$\langle f, v \rangle_{V^* \times V} = \langle f_0, v \rangle_H + \langle f_N, v \rangle_{L^2(\Gamma_N; \mathbb{R}^d)} \quad \text{for all } \ v \in V. \tag{25}$$

We combine (22)–(24) and use (25) to see that

$$\langle \sigma, \varepsilon(v) \rangle_{\mathscr{H}} + \int_{\Gamma_C} h_\tau(\varphi - \varphi_0) j_\tau^0(u_\tau; v_\tau)\, d\Gamma \geqslant \langle f, v \rangle_{V^* \times V}. \tag{26}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

Similarly, for every $\psi \in \Phi$, from (2) and (21) we deduce that

$$\langle D, \nabla\psi\rangle_H + \int_\Omega q_0\psi\, dx = \int_\Gamma D\cdot n\,\psi\, d\Gamma$$

and by (7), (8), (11), we get

$$-\langle D, \nabla\psi\rangle_H + \int_{\Gamma_C} j^0(\varphi-\varphi_0;\psi)\, d\Gamma \geqslant \langle q,\psi\rangle_{\Phi^*\times\Phi} \quad \text{for all} \quad \psi\in\Phi, \tag{27}$$

where q is the element of Φ^* given by

$$\langle q,\psi\rangle_{\Phi^*\times\Phi} = \langle q_0,\psi\rangle_{L^2(\Omega)} - \langle q_2,\psi\rangle_{L^2(\Gamma_2)} \quad \text{for all} \quad \psi\in\Phi. \tag{28}$$

We substitute (3) in (26), (4) in (27) and use equality $E(\varphi) = -\nabla\varphi$ to derive the following variational formulation of Problem P, in terms of displacement and electric potential fields.

Problem P_V. Find a displacement field $u \in V$ and an electric potential $\varphi \in \Phi$ such that

$$\begin{aligned}\langle \mathscr{F}\varepsilon(u),\varepsilon(v)\rangle_{\mathscr{H}} + \langle \mathscr{P}^\top\nabla\varphi,\varepsilon(v)\rangle_{\mathscr{H}} + \int_{\Gamma_C} h_\tau(\varphi-\varphi_0)\, j_\tau^0(u_\tau;v_\tau)\, d\Gamma \\ \geqslant \langle f, v\rangle_{V^*\times V} \quad \text{for all } v\in V,\end{aligned} \tag{29}$$

$$\begin{aligned}\langle \beta\nabla\varphi,\nabla\psi\rangle_H - \langle \mathscr{P}\varepsilon(u),\nabla\psi\rangle_H + \int_{\Gamma_C} j^0(\varphi-\varphi_0;\psi)\, d\Gamma \\ \geqslant \langle q,\psi\rangle_{\Phi^*\times\Phi} \quad \text{for all } \psi\in\Phi.\end{aligned} \tag{30}$$

The existence and the uniqueness of a solution to Problem P_V will be analysed in Sections 4 and 5, respectively. We end this section with some comments on the novelty of the model presented in Problem P_V.

Remark 5 Assume now that the foundation is insulated and the friction law does not depend on the electric variables, i.e. $j \equiv 0$ and h_τ is a constant function. Then inequality (30) reduces to a linear variational equation in which the boundary term vanishes, and φ disappears from the boundary term in (29). This represents an important simplification of the model and leads to mathematical models of the form of those studied in [4, 10–16]. We conclude that taking into account the electrical conductivity of the foundation, in both the friction law and the electrical condition on the contact surface, one is led to a new mathematical model. The latter is in the form of a genuine system of two hemi-variational inequalities in which there is a strong coupling of the unknowns in the boundary terms.

4 Existence of the solution

The existence of the solution to Problem P_V is given by the following result.

Theorem 6 *Assume that $H(\mathscr{F})$, $H(\mathscr{P})$, $H(\beta)$, $H(j_\tau)$, $H(j)$, $H(h_\tau)$, $H(f)$, $H(q)$ and $H(\varphi_0)$ hold. Then Problem P_V has at least one solution.*

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

A quadruple of functions (u, σ, φ, D) which satisfies (3), (4), (29) and (30) is called a *weak solution* of the piezoelectric contact Problem P. It follows from above that, under the assumptions of Theorem 6, there exists at least one weak solution of Problem P. To describe precisely the regularity of the weak solution, we note that the constitutive relations (3) and (4), assumptions $H(\mathscr{F})$, $H(\mathscr{P})$, $H(\beta)$ and regularity $u \in V$, $\varphi \in \Phi$ show that $\sigma \in \mathscr{H}$ and $D \in H$. Moreover, using (3), (4), (29) and (30), it follows that (26) and (27) hold for all $v \in V$, $\psi \in \Phi$ which imply

$$\operatorname{Div} \sigma + f_0 = 0, \quad \operatorname{div} D - q_0 = 0 \quad \text{in } \Omega.$$

It follows now from $H(f)$ and $H(q)$ that $\operatorname{Div} \sigma \in H$ and $\operatorname{div} D \in L^2(\Omega)$, respectively. We conclude that the weak solution of the piezoelectric contact Problem P satisfies $(u, \sigma, \varphi, D) \in V \times \mathscr{H}_1 \times \Phi \times W$.

We now present the proof of Theorem 6 which is based on an abstract result for an operator inclusion in a Banach space. To this end we need some preliminary material that we introduce in what follows.

For an open bounded set Ω in $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ, assume that Y is a closed subspace of $H^1(\Omega; \mathbb{R}^\rho)$, $\rho \geqslant 1$ and $Z = H^\delta(\Omega; \mathbb{R}^\rho)$ with a fixed $\delta \in (1/2, 1)$. We know (cf. Chapter 3 of [6]) that $Y \subset Z \subset L^2(\Omega; \mathbb{R}^\rho) \subset Z^* \subset Y^*$ with continuous, dense and compact embeddings. Let $A : Y \to Y^*$ be an operator and $N : Z \to 2^{Z^*}$ be a multi-valued mapping. Consider the following inclusion:

$$\text{given } \ g \in Y^* \ \text{ find } \ y \in Y \ \text{ such that } \ g \in (A + N)y. \tag{31}$$

We say that an element $y \in Y$ is a solution to (31) if and only if there exists $z \in Z^*$ such that $g = Ay + z$ and $z \in Ny$. Recall also (cf. [6, 19]) that a single-valued operator $A : Y \to Y^*$ is pseudomonotone if it is bounded (i.e. it maps bounded subsets of Y into bounded subsets of Y^*) and

$$\langle Au, u - v \rangle_{Y^* \times Y} \leqslant \liminf \langle Au_n, u_n - v \rangle_{Y^* \times Y}$$

for all $v \in Y$ whenever the sequence $\{u_n\}$ converges weakly in Y to u with

$$\limsup \langle Au_n, u_n - u \rangle_{Y^* \times Y} \leqslant 0.$$

We recall the following result.

Lemma 7 *Under the above notation, assume that*

$\underline{H(A)}$: $A : Y \to Y^*$ *is an operator which is pseudomonotone and coercive;*

$\underline{H(N)}$: $N : Z \to 2^{Z^*}$ *is a multi-valued map such that*

(i) *N has a non-empty convex and weakly compact values;*

(ii) $\|Nz\|_{Z^*} \leqslant c_0(1 + \|z\|_Z)$ *for all $z \in Z$ with $c_0 > 0$;*

(iii) $\langle Nz, z \rangle_{Z^* \times Z} \geqslant -c_1 \|z\|_Z - c_2$ *for all $z \in Z$ with $c_1, c_2 \geqslant$;*

(iv) *the graph of N is closed in $Z \times (w\text{-}Z^*)$ topology.*

Then problem (31) has at least one solution.

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

The proof of Lemma 7 is based on a surjectivity result (cf. Theorem 1.3.70 of [6]) for multi-valued pseudomonotone maps and can be found in [12].

In what follows we associate with Problem P_V an operator inclusion and apply Lemma 7. Let us introduce the product space $Y = V \times \Phi \subset H^1(\Omega;\mathbb{R}^{d+1})$ which is a Hilbert space endowed with the inner product

$$\langle y, z\rangle_Y = \langle u, v\rangle_V + \langle \varphi, \psi\rangle_\Phi \quad \text{for all} \quad y = (u,\varphi) \in Y \ \text{and} \ z = (v,\psi) \in Y.$$

The corresponding norm is denoted by $\|\cdot\|_Y$. Let $A : Y \to Y^*$ be an operator defined by

$$\langle Ay, z\rangle_{Y^*\times Y} = \langle \mathscr{F}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} + \langle \mathscr{P}^\top \nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} + \langle \beta\nabla\varphi, \nabla\psi\rangle_H - \langle \mathscr{P}\varepsilon(u), \nabla\psi\rangle_H \tag{32}$$

for all y, $z \in Y$, $y = (u,\varphi)$ and $z = (v,\psi)$. Let $Z = H^\delta(\Omega;\mathbb{R}^{d+1})$ with some $\delta \in (1/2, 1)$. Denote by $\gamma : Z \to L^2(\Gamma_C;\mathbb{R}^{d+1})$ the trace operator and let $\gamma^* : L^2(\Gamma_C;\mathbb{R}^{d+1}) \to Z^*$ be its adjoint.

We also introduce the functional $J : L^2(\Gamma_C;\mathbb{R}^{d+1}) \times L^2(\Gamma_C;\mathbb{R}^{d+1}) \to \mathbb{R}$ given by

$$J((\overline{w},\overline{\varphi}),(w,\varphi)) = \int_{\Gamma_C} (h_\tau(\overline{\varphi} - \varphi_0) j_\tau(w_\tau) + j(\varphi - \varphi_0))\, d\Gamma \tag{33}$$

for all $(\overline{w},\overline{\varphi})$, $(w,\varphi) \in L^2(\Gamma_C;\mathbb{R}^d) \times L^2(\Gamma_C)$. Although the integrand of J is independent of $\overline{w}$, we keep the above notation to simplify the further presentation. As we will see below, under our hypotheses, $J((\overline{w},\overline{\varphi}),\cdot,\cdot)$ is locally Lipschitz on $L^2(\Gamma_C;\mathbb{R}^{d+1})$ for all $(\overline{w},\overline{\varphi}) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$. Denoting by ∂J its Clarke subdifferential, we define a multi-valued mapping $N : Z \to 2^{Z^*}$ by

$$Nz = \gamma^* \partial J(\gamma z, \gamma z) \quad \text{for all } z \in Z. \tag{34}$$

The following lemmas provide the properties of the operator A and the functional J, respectively.

Lemma 8 *If hypotheses $H(\mathscr{F})$, $H(\mathscr{P})$ and $H(\beta)$ hold, then the operator $A : Y \to Y^*$ defined by (32) satisfies $H(A)$.*

Lemma 9 *If hypotheses $H(h_\tau)$, $H(j_\tau)(i)$–(v), $H(j)(i)$–(v) and $H(\varphi_0)$ hold, then the functional J defined by (33) satisfies*

(1) *$J((\overline{w},\overline{\varphi}),\cdot,\cdot)$ is locally Lipschitz for all $(\overline{w},\overline{\varphi}) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$ (in fact Lipschitz on bounded subsets of $L^2(\Gamma_C;\mathbb{R}^{d+1})$);*

(2) *$\|\partial J((\overline{w},\overline{\varphi}),(w,\varphi))\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant \tilde{c}(1 + \|(w,\varphi)\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})})$ for all $(\overline{w},\overline{\varphi})$, $(w,\varphi) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$ with $\tilde{c} > 0$, where the subdifferential is taken with respect to (w,φ);*

(3) *for all $(\overline{w},\overline{\varphi})$, (w,φ), $(v,\psi) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$, we have*

$$J^0((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi)) = \int_{\Gamma_C} \left(h_\tau(\overline{\varphi} - \varphi_0) j_\tau^0(w_\tau; v_\tau) + j^0(\varphi - \varphi_0; \psi)\right) d\Gamma,$$

where the generalized directional derivative is taken with respect to (w,φ);

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

(4) $J^0((\overline{w},\overline{\varphi}),(w,\varphi);-(w,\varphi)) \leqslant \tilde{d}(1+\|(w,\varphi)\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})})$ *for all* $(\overline{w},\overline{\varphi}),(w,\varphi) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$ *with* $\tilde{d} > 0$;

(5) *either* $J((\overline{w},\overline{\varphi}),\cdot,\cdot)$ *or* $-J((\overline{w},\overline{\varphi}),\cdot,\cdot)$ *is regular for all* $(\overline{w},\overline{\varphi}) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$;

(6) *the graph of* $\partial J(\cdot,\cdot,\cdot,\cdot)$ *is closed in* $[L^2(\Gamma_C;\mathbb{R}^{d+1}) \times L^2(\Gamma_C;\mathbb{R}^{d+1})] \times [w\text{-}L^2(\Gamma_C;\mathbb{R}^{d+1})]$ *topology, where* ∂J *is the generalized subdifferential of* J *with respect to two last variables.*

For the proof of Lemma 8 we refer to Lemma 1 of [12]. The proof of Lemma 9 is quite technical and, for this reason, is postponed till Section 6. We conclude the list of preliminaries with a result on the chain rule for the generalized subdifferential, which will also be proved in Section 6.

Lemma 10 *Let* E *and* F *be Banach spaces,* $\psi : F \to \mathbb{R}$ *be locally Lipschitz and let* $T : E \to F$ *be given by* $Tx = Lx + y_0$ *for* $x \in E$, *where* $L \in \mathscr{L}(E,F)$ *and* $y_0 \in F$ *are fixed. Then the function* $\phi : E \to \mathbb{R}$ *defined by* $\phi(x) = \psi(Tx)$ *for* $x \in E$ *is locally Lipschitz and*

(*i*) $\phi^0(x;v) \leqslant \psi^0(Tx;Lv)$ *for all* $x, v \in E$,

(*ii*) $\partial\phi(x) \subseteq L^*\partial\psi(Tx)$ *for all* $x \in E$.

Moreover, if ψ *(or* $-\psi$*) is regular, then* ϕ *(or* $-\phi$*) is regular and in* (*i*) *and* (*ii*) *we have equalities. These equalities are also true if instead of the regularity condition, we assume that* L *is surjective.*

Now we have all the ingredients to provide the proof of Theorem 6 which is obtained in two steps, as follows.

Proof of Theorem 6 *Claim* 1. The operator N given by (34) satisfies $H(N)$.

The condition $H(N)(i)$ follows from the facts (cf. Propositions 2.1.2 and 2.1.5 of [5]) that the values of the Clarke subdifferential are non-empty, convex, weakly compact subsets of $L^2(\Gamma_C;\mathbb{R}^{d+1})$ and the closedness of the graph of $\partial J((\overline{w},\overline{\varphi}),\cdot,\cdot)$ in $L^2(\Gamma_C;\mathbb{R}^{d+1}) \times (w\text{-}L^2(\Gamma_C;\mathbb{R}^{d+1}))$ topology.

To establish $H(N)(ii)$, let $z \in Z$ and $w \in Nz$. Hence $w = \gamma^* v$ and $v \in \partial J(\gamma z, \gamma z)$. From Lemma 9(2), we know that

$$\|v\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant \tilde{c}\,\left(1 + \|\gamma z\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})}\right) \leqslant \tilde{c}\,(1 + \|\gamma\|\,\|z\|_Z).$$

Hence

$$\|w\|_{Z^*} = \|\gamma^* v\|_{Z^*} \leqslant \|\gamma^*\|\,\|v\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant c_0(1 + \|z\|_Z)$$

with some $c_0 > 0$ which proves the property $H(N)(ii)$.

For the proof of the property $H(N)(iii)$, let $z \in Y$ and $w \in Nz$. So again $w = \gamma^* v$ and $v \in \partial J(\gamma z, \gamma z)$. Using Lemma 9(4), we obtain

$$-\langle v, \gamma z\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant J^0(\gamma z, \gamma z; -\gamma z) \leqslant \tilde{d}\,(1 + \|\gamma z\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})}) \leqslant \tilde{d}(1 + \|\gamma\|\,\|z\|_Z),$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

where $\tilde{d} > 0$. Therefore, it follows that

$$\langle w, z\rangle_{Z^*\times Z} = \langle \gamma^* v, z\rangle_{Z^*\times Z} = \langle v, \gamma z\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \geqslant -\tilde{d}(1 + \|\gamma\|\,\|z\|_Z),$$

which means that $H(N)(iii)$ holds.

In order to prove $H(N)(iv)$, take $w_n \in Nz_n$ with z_n, $z \in Z$, $z_n \to z$ in Z, w_n, $w \in Z^*$ and $w_n \to w$ weakly in Z^*. We have $w_n = \gamma^* v_n$ and $v_n \in \partial J(\gamma z_n, \gamma z_n)$. Since the trace operator is linear and continuous, we get $\gamma z_n \to \gamma z$ in $L^2(\Gamma_C;\mathbb{R}^{d+1})$. From condition (2) of Lemma 9, we know that $\{v_n\}$ belongs to a bounded subset of $L^2(\Gamma_C;\mathbb{R}^{d+1})$ and we may assume that at least for a subsequence $v_n \to v$ weakly in $L^2(\Gamma_C;\mathbb{R}^{d+1})$ with some $v \in L^2(\Gamma_C;\mathbb{R}^{d+1})$. Hence, from equality $w_n = \gamma^* v_n$ we obtain $w = \gamma^* v$. Applying Lemma 9(6) we have $v \in \partial J(\gamma z, \gamma z)$. This implies that $w \in \gamma^* \partial J(\gamma z, \gamma z) = Nz$ which concludes the proof of $H(N)(iv)$ and of Claim 1.

Moreover, we observe that under the hypotheses $H(f)$ and $H(q)$ the elements f and q defined by (25) and (28), respectively, satisfy $f \in V^*$ and $q \in \Phi^*$, that is $g = (f, q) \in Y^*$.

We now apply Lemma 7 and deduce that for $g = (f, q) \in Y^*$ there exists $y \in Y$, $y = (u, \varphi) \in V \times \Phi$ such that

$$A(u, \varphi) + \gamma^* \partial J(\gamma(u, \varphi), \gamma(u, \varphi)) \ni (f, q). \tag{35}$$

Claim 2. The inclusion (35) is equivalent to the inequality

$$\begin{aligned} &\langle A(u, \varphi), (v, \psi)\rangle_{Y^*\times Y} + \int_{\Gamma_C} \left(h_\tau(\varphi - \varphi_0) j^0_\tau(u_\tau; v_\tau) + j^0(\varphi - \varphi_0; \psi)\right) d\Gamma \\ &\quad \geqslant \langle (f, q), (v, \psi)\rangle_{Y^*\times Y} \qquad \text{for all } (v, \psi) \in V \times \Phi. \end{aligned} \tag{36}$$

Indeed, let $y = (u, \varphi) \in Y$ satisfy (35). Then $A(u, \varphi) + (z, \zeta) = (f, q)$, where $(z, \zeta) = \gamma^*(w, \eta)$ and $(w, \eta) \in \partial J(\gamma(u, \varphi), \gamma(u, \varphi))$. By the definition of the Clarke subdifferential and Lemma 9(3), for all $(v, \psi) \in V \times \Phi$ we have

$$\begin{aligned} \langle (z, \zeta), (v, \psi)\rangle_{Z^*\times Z} &= \langle \gamma^*(w, \eta), (v, \psi)\rangle_{Z^*\times Z} \\ &= \langle (w, \eta), \gamma(v, \psi)\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant J^0(\gamma(u, \varphi), \gamma(u, \varphi); \gamma(v, \psi)) \end{aligned}$$

which implies that $y \in Y$ solves (36). Conversely, assume that the element $y = (u, \varphi) \in V \times \Phi$ satisfies (36). Using Lemma 9(3), for all $(v, \psi) \in V \times \Phi$ we get

$$\langle A(u, \varphi), (v, \psi)\rangle_{Y^*\times Y} + J^0(\gamma(u, \varphi), \gamma(u, \varphi); \gamma(v, \psi)) \geqslant \langle (f, q), (v, \psi)\rangle_{Y^*\times Y}.$$

Hence, from Lemma 10(i) we obtain

$$\begin{aligned} \langle (f, q) - A(u, \varphi), (v, \psi)\rangle_{Y^*\times Y} &\leqslant J^0(\gamma(u, \varphi), \gamma(u, \varphi); \gamma(v, \psi)) \\ &= (J \circ \gamma)^0((u, \varphi), (u, \varphi); (v, \psi)) \end{aligned}$$

and therefore, by the definition of the subdifferential and Lemma 10(ii), yields

$$g - Ay \in \partial(J \circ \gamma)(\gamma y, y) = \gamma^* \partial J(\gamma y, \gamma y).$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

This means that $y = (u, \varphi) \in V \times \Phi$ satisfies (35) and proves the claim.

Now using Claim 2 and choosing $(v, 0) \in V \times \Phi$ and $(0, \psi) \in V \times \Phi$ in the inequality (36), we easily obtain the inequalities (29) and (30), respectively. This means that (u, φ) solves Problem P_V and concludes the proof of the theorem. □

Remark 11 Assume now, as in Remark 5, that the foundation is insulated and the friction law does not depend on the electric variables, i.e. $j \equiv 0$ and h_τ is a constant function. This implies important simplifications in the functional (33), in the operator (34) and in the multi-valued inclusion (35), since part of the terms and variables will disappear. As a consequence, important simplifications will arise in the proof of Lemma 9 and Claims 1 and 2 above. We conclude that in order to take into account the electrical conductivity of the foundation, in both the friction law and the electrical condition on the contact surface, it requires a careful use of the mathematical arguments in a framework which is different from that used in [11, 12, 16], since now the functionals and the operators have additional terms, additional non-linearities, and involve additional variables.

5 Uniqueness of the solution

The uniqueness of a solution to Problem P_V under the general assumption $H(h_\tau)$ represents an open problem which, clearly, deserves more investigation. However, it can be obtained in the particular case when h_τ is a constant function. More precisely, we have the following result.

Theorem 12 *Assume that $H(\mathscr{P})$, $H(\beta)$, $H(j_\tau)$, $H(j)$, $H(f)$, $H(q)$ and $H(\varphi_0)$ hold and h_τ is a non-negative constant. Assume also that the elasticity operator $\mathscr{F}$ satisfies $H(\mathscr{F})(i)$, (ii), (iii), (v) and there exists $m_{\mathscr{F}} > 0$ such that*

$$(\mathscr{F}(x, \varepsilon_1) - \mathscr{F}(x, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathscr{F}} \|\varepsilon_1 - \varepsilon_2\|^2_{\mathscr{S}_d}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathscr{S}_d$, a.e. $x \in \Omega$. In addition, assume that

$H(j_\tau)(vi)$: $(\eta_1 - \eta_2, \xi_1 - \xi_2)_{\mathbb{R}^d} \geqslant -m_\tau \|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}$ *for all* $\eta_i \in \partial j_\tau(x, \xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, *a.e.* $x \in \Gamma_C$ *with* $m_\tau \geqslant 0$;

$H(j)(vi)$: $(s_1 - s_2)(r_1 - r_2) \geqslant -m|r_1 - r_2|^2$ *for all* $s_i \in \partial j(x, r_i)$, $r_i \in \mathbb{R}$, $i = 1, 2$, *a.e.* $x \in \Gamma_C$ *with* $m \geqslant 0$

and

$$\min\{m_{\mathscr{F}}, m_\beta\} > \max\{m_\tau, m\}\, c_e^2\, \|\gamma\|^2, \tag{37}$$

where $c_e > 0$ is the embedding constant of Y into Z, $\|\gamma\|$ denotes the norm of the trace operator γ in the space $\mathscr{L}(Z, L^2(\Gamma_C; \mathbb{R}^{d+1}))$ and $Z = H^\delta(\Omega; \mathbb{R}^{d+1})$ with a fixed $\delta \in (1/2, 1)$. Then Problem P_V has a unique solution.

Theorem 12 can be used in order to obtain the uniqueness of the weak solution in the study of several contact problems. To provide an example, assume that the friction is modelled with Tresca's friction law in which the friction bound is a constant. It follows from the arguments presented in Example 1 in Section 2 that this leads to a subdifferential

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

condition of the form (10) in which h_τ = constant. Using Theorem 12, it can be shown that the corresponding contact problem has a unique weak solution.

Proof of Theorem 12 Without loss of generality we assume that $h_\tau \equiv 1$ since the constant h_τ can be incorporated in the definition of superpotentials j_τ, i.e. if $h_\tau \equiv \widetilde{c}_\tau$, where $\widetilde{c}_\tau \geqslant 0$, then we take $h_\tau = 1$ and consider the function $\widetilde{c}_\tau\, j_\tau$ instead of j_τ. From the proof of Theorem 6 it follows that for any $g = (f, q) \in Y^*$ there exists $y \in Y$, $y = (u, \varphi) \in V \times \Phi$ such that

$$Ay + \gamma^* \partial J(\gamma y, \gamma y) \ni g. \tag{38}$$

In order to prove the theorem, it is enough to establish the uniqueness of a solution to the inequality (36) or equivalently (cf. Claim 2) to show the uniqueness of solutions to the inclusion (38). To this end, we state additional general conditions under which the solution of (38) is unique.

Claim 3. Assume that

(*a*) the operator $A : Y \to Y^*$ is strongly monotone, i.e.

$$\langle Ay_1 - Ay_2, y_1 - y_2\rangle_{Y^*\times Y} \geqslant m_A \|y_1 - y_2\|_Y^2 \text{ for all } y_1, y_2 \in Y \text{ with } m_A > 0,$$

(*b*) the functional $J : L^2(\Gamma_C; \mathbb{R}^{d+1})^2 \to \mathbb{R}$ satisfies the following relaxed monotonicity condition

$$\langle \zeta_1 - \zeta_2, z_1 - z_2\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \geqslant -m_J \|z_1 - z_2\|^2_{L^2(\Gamma_C;\mathbb{R}^{d+1})}$$

for all $\zeta_i \in \partial J(z_i, z_i)$, $\zeta_i,\ z_i \in L^2(\Gamma_C; \mathbb{R}^{d+1})$, $i = 1, 2$ with $m_J \geqslant 0$,

(*c*) $$m_A > m_J c_e^2 \|\gamma\|^2.$$

Then the solution of problem (38) obtained in Theorem 6 is unique.

Let $g \in Y^*$ and let $y_1,\ y_2 \in Y$ be solutions to (38). Then, there exists $\zeta_i \in \partial J(\gamma y_i, \gamma y_i)$, $\zeta_i \in L^2(\Gamma_C; \mathbb{R}^{d+1})$ such that $Ay_i + \gamma^* \zeta_i = g$ for $i = 1, 2$. Subtracting the last two equations, multiplying the result by $y_1 - y_2$ and using condition (*a*), we get

$$m_A \|y_1 - y_2\|_Y^2 + \langle \gamma^* \zeta_1 - \gamma^* \zeta_2, y_1 - y_2\rangle_{Z^*\times Z} \leqslant 0.$$

Next, by the hypothesis (*b*) we obtain

$$\begin{aligned}\langle \gamma^*(\zeta_1 - \zeta_2), y_1 - y_2\rangle_{Z^*\times Z} &= \langle \zeta_1 - \zeta_2, \gamma y_1 - \gamma y_2\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \\ &\geqslant -m_J \|\gamma y_1 - \gamma y_2\|^2_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \geqslant -m_J c_e^2 \|\gamma\|^2 \|y_1 - y_2\|_Y^2.\end{aligned}$$

The last inequality yields

$$m_A \|y_1 - y_2\|_Y^2 - m_J\, c_e^2\, \|\gamma\|^2 \|y_1 - y_2\|_Y^2 \leqslant 0$$

which, in view of the condition (*c*), implies $y_1 = y_2$. Therefore, we deduce that $\zeta_1 = \zeta_2$ which concludes the proof of Claim 3.

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

Now we verify that the operator A given by (32) satisfies condition (a) of Claim 3. To this end we note that, from the hypotheses on the operator $\mathscr{F}$ and $H(\beta)$, we have

$$\begin{aligned}
&\langle Ay_1 - Ay_2, y_1 - y_2\rangle_{Y^*\times Y}\\
&\quad = \langle \mathscr{F}\varepsilon(u_1) - \mathscr{F}\varepsilon(u_2), \varepsilon(u_1 - u_2)\rangle_{\mathscr{H}} + \langle \beta(\nabla\varphi_1 - \nabla\varphi_2), \nabla\varphi_1 - \nabla\varphi_2\rangle_H\\
&\quad \geqslant m_{\mathscr{F}} \int_\Omega \|\varepsilon(u_1 - u_2)\|^2_{\mathscr{S}_d}\, dx + m_\beta \int_\Omega \|\nabla(\varphi_1 - \varphi_2)\|^2_{\mathbb{R}^d}\, dx \geqslant m_A \|y_1 - y_2\|^2_Y
\end{aligned}$$

for all $y_i = (u_i, \varphi_i) \in Y$, $i = 1, 2$, which means that (a) holds with $m_A = \min\{m_{\mathscr{F}}, m_\beta\}$.

In what follows, we show that if $h_\tau = 1$, then the functional J defined by (33) satisfies condition (b) of Claim 3. With this choice of h_τ, this functional is independent of the first variable $(\overline{w}, \overline{\varphi})$, which is omitted, i.e.

$$J(w, \varphi) = \int_{\Gamma_C} (j_\tau(w_\tau) + j(\varphi - \varphi_0))\, d\Gamma \quad \text{for all} \quad (w, \varphi) \in L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C).$$

Let $z_i = (w_i, \varphi_i) \in L^2(\Gamma_C; \mathbb{R}^{d+1})$, $w_i \in L^2(\Gamma_C; \mathbb{R}^d)$, $\varphi_i \in L^2(\Gamma_C)$, $\zeta_i \in \partial J(z_i)$, $i = 1, 2$. From the proof of Lemma 9 (cf. (40)) for $i = 1, 2$, we have

$$\begin{aligned}
\partial J(w_i, \varphi_i) &\subseteq \int_{\Gamma_C} \partial_{(\xi, r)}(j_\tau(w_{i\tau}) + j(\varphi_i - \varphi_0))\, d\Gamma\\
&\subseteq \int_{\Gamma_C} ([\partial j_\tau(w_{i\tau})]_\tau, \partial j(\varphi_i - \varphi_0))\, d\Gamma.
\end{aligned}$$

This means that for $\zeta_i \in \partial J(w_i, \varphi_i)$ there exists $(\eta_i, s_i) \in L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C)$ such that $\eta_i(x) \in [\partial j_\tau(x, w_{i\tau}(x))]_\tau$, $s_i \in \partial j(x, \varphi_i(x) - \varphi_0(x))$ for a.e. $x \in \Gamma_C$ and

$$\langle \zeta_i, (v, \psi)\rangle_{L^2(\Gamma_C; \mathbb{R}^{d+1})} = \int_{\Gamma_C} (\eta_i(x), s_i(x)) \cdot (v(x), \psi(x))\, d\Gamma$$

for all $(v, \psi) \in L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C)$, $i = 1, 2$. Hence, we get $\eta_i(x) = [\beta_i(x)]_\tau$, $\beta_i(x) \in \partial j_\tau(x, w_{i\tau}(x))$ for a.e. $x \in \Gamma_C$ with $\beta_i \in L^2(\Gamma_C; \mathbb{R}^d)$, $i = 1, 2$. Using assumptions (a), (b) and (c) we obtain

$$\begin{aligned}
(\beta_1(x) - \beta_2(x)) \cdot (w_{1\tau}(x) - w_{2\tau}(x)) &\geqslant -m_\tau \|w_{1\tau}(x) - w_{2\tau}(x)\|^2_{\mathbb{R}^d},\\
(s_1(x) - s_2(x))(\varphi_1(x) - \varphi_2(x)) &\geqslant -m\, |\varphi_1(x) - \varphi_2(x)|^2
\end{aligned}$$

and $(\eta_1(x) - \eta_2(x)) \cdot (w_1(x) - w_2(x)) = (\beta_1(x) - \beta_2(x)) \cdot (w_{1\tau}(x) - w_{2\tau}(x))$ for a.e. $x \in \Gamma_C$. Combining the above relations, we find that

$$\begin{aligned}
&\langle \zeta_1 - \zeta_2, z_1 - z_2\rangle_{L^2(\Gamma_C; \mathbb{R}^{d+1})}\\
&\quad = \int_{\Gamma_C} ((\eta_1(x) - \eta_2(x)) \cdot (w_1(x) - w_2(x)) + (s_1(x) - s_2(x))(\varphi_1(x) - \varphi_2(x)))\, d\Gamma\\
&\quad \geqslant \int_{\Gamma_C} \left(-m_\tau \|w_{1\tau}(x) - w_{2\tau}(x)\|^2_{\mathbb{R}^d} - m\, |\varphi_1(x) - \varphi_2(x)|^2 \right) d\Gamma\\
&\quad \geqslant -m_J \int_{\Gamma_C} \left(\|w_1(x) - w_2(x)\|^2_{\mathbb{R}^d} + |\varphi_1(x) - \varphi_2(x)|^2 \right) d\Gamma = -m_J \|z_1 - z_2\|^2_{L^2(\Gamma_C; \mathbb{R}^{d+1})},
\end{aligned}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

where $m_J = \max\{m_\tau, m\}$, which implies that J satisfies condition (b). Finally we note that condition (c) of Claim 3 is a consequence of (37) which ends the proof. □

Remark 13 The hypothesis (a) in Claim 3 of the proof of Theorem 12 holds if, for instance, the operator A is bounded, hemi-continuous, coercive and strongly monotone. This follows from the fact that every strongly monotone operator is monotone and every bounded, hemi-continuous and monotone operator is pseudomonotone (cf. Proposition 27.6 of [19]). It is also well known (cf. e.g. Theorem 1.3.19 of [6]) that if J is a convex locally Lipschitz functional, then ∂J is maximal monotone; therefore, in this case the relaxed monotonicity condition (b) of Claim 3 holds with $m_J = 0$ and note also that (c) is trivially satisfied.

Remark 14 A short examination shows that assuming a contact with an insulated foundation (i.e. $j \equiv 0$) implies important simplifications in the proof of Theorem 12. Therefore, as in Remark 11, we conclude that taking into account the electric conductivity of the foundation requires the use of additional arguments and calculus in order to prove the unique solvability of the model.

6 Proofs of Lemmas 9 and 10

In this section we provide the proofs of Lemmas 9 and 10 which have been employed in the proof of Theorem 6.

Proof of Lemma 9 We start with the study of the properties of the integrand of (33). Let us fix $(\overline{w}, \overline{\varphi}) \in L^2(\Gamma_C; \mathbb{R}^{d+1})$ and define the following functions $j_1 : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$, $j_2 : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ and $p : \Gamma_C \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$
\begin{aligned}
j_1(x,\xi) &= h_\tau(x, \overline{\varphi}(x) - \varphi_0(x))\, j_\tau(x, \Upsilon\xi),\\
j_2(x,r) &= j(x, r - \varphi_0(x)),\\
p(x,\xi,r) &= j_1(x,\xi) + j_2(x,r)
\end{aligned}
$$

for $(x,\xi,r) \in \Gamma_C \times \mathbb{R}^d \times \mathbb{R}$, where $\Upsilon \in \mathscr{L}(\mathbb{R}^d, \mathbb{R}^d)$ is given by $\Upsilon\xi = \xi_\tau$ for $\xi \in \mathbb{R}^d$. Although the functions j_1, j_2 and p depend on $\overline{w}$ and $\overline{\varphi}$, for simplicity we skip this dependence in the notation.

From the hypotheses $H(j_\tau)$, $H(j)$, $H(h_\tau)$ and the fact that Υ is a linear bounded operator, we have

$$
\begin{cases}
p(\cdot,\xi,r) & \text{is measurable for all } (\xi,r) \in \mathbb{R}^d \times \mathbb{R},\\
p(x,\cdot,\cdot) & \text{is locally Lipschitz for a.e. } x \in \Gamma_C.
\end{cases}
\tag{39}
$$

From the regularity of $j_\tau(x,\cdot)$ and $j(x,\cdot)$, we infer that $p(x,\cdot,\cdot)$ is regular, so by Proposition 2.3.15 of [5], we have $\partial_{(\xi,r)} p(x,\xi,r) \subseteq \partial_\xi p(x,\xi,r) \times \partial_r p(x,\xi,r)$, where $\partial_{(\xi,r)} p$ denotes the generalized gradient of p with respect to (ξ,r), and $\partial_\xi p$ and $\partial_r p$ are the partial generalized gradients of p with respect to ξ and r, respectively. Since

$$
\begin{aligned}
\partial_\xi p(x,\xi,r) &= \partial_\xi (j_1(x,\xi) + j_2(x,r)) \subseteq \partial_\xi j_1(x,\xi),\\
\partial_r p(x,\xi,r) &= \partial_r (j_1(x,\xi) + j_2(x,r)) \subseteq \partial_r j_2(x,r)
\end{aligned}
$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

for all $(x,\xi,r)\in\Gamma_C\times\mathbb{R}^d\times\mathbb{R}$, we deduce

$$\partial_{(\xi,r)}p(x,\xi,r)\subseteq\partial_\xi j_1(x,\xi)\times\partial_r j_2(x,r). \tag{40}$$

Since $j_\tau(x,\cdot)$ is regular and $\Upsilon^*=\Upsilon$, from Lemma 10(*ii*) we get

$$\begin{aligned}\partial_\xi j_1(x,\xi)&=h_\tau(x,\overline{\varphi}(x)-\varphi_0(x))\,\partial_\xi\left(j_\tau(x,\Upsilon\xi)\right),\\&=h_\tau(x,\overline{\varphi}(x)-\varphi_0(x))\,\Upsilon^*\partial j_\tau(x,\Upsilon\xi),\\&=h_\tau(x,\overline{\varphi}(x)-\varphi_0(x))\,[\partial j_\tau(x,\xi_\tau)]_\tau,\end{aligned}$$

where ∂j_τ denotes the generalized gradient of $j_\tau(x,\cdot)$. Analogously, since $j(x,\cdot)$ is regular, we obtain $\partial_r j_2(x,r)=\partial j(x,r-\varphi_0(x))$, where ∂j is the generalized gradient of $j(x,\cdot)$.

Consider now an element $(\eta,s)\in\mathbb{R}^d\times\mathbb{R}$ such that $(\eta,s)\in\partial_{(\xi,r)}p(x,\xi,r)$. From (40) we have $\eta\in\partial_\xi j_1(x,\xi)$ and $s\in\partial_r j_2(x,r)$. Using assumptions $H(j_\tau)(iii)$ and $H(j)(iii)$ we obtain

$$\begin{aligned}&\|\eta\|_{\mathbb{R}^d}\leqslant\overline{h}_\tau\,\|[\partial j_\tau(x,\xi_\tau)]_\tau\|_{\mathbb{R}^d}\leqslant\overline{h}_\tau\,\|\partial j_\tau(x,\xi_\tau)\|_{\mathbb{R}^d}\leqslant c_\tau\,\overline{h}_\tau\,(1+\|\xi_\tau\|_{\mathbb{R}^d}),\\&\quad|s|\leqslant|\partial j(x,r-\varphi_0(x))|\leqslant c\,(1+|r-\varphi_0(x)|).\end{aligned}$$

Thus,

$$\|(\eta,s)\|_{\mathbb{R}^{d+1}}\leqslant\|\eta\|_{\mathbb{R}^d}+|s|\leqslant c_\tau\overline{h}_\tau(1+\|\xi\|_{\mathbb{R}^d})+c\,(1+|r|)+c\,|\varphi_0(x)|.$$

Using $H(\varphi_0)$, we infer

$$\|\partial_{(\xi,r)}p(x,\xi,r)\|_{\mathbb{R}^{d+1}}\leqslant\overline{c}\,(1+\|(\xi,r)\|_{\mathbb{R}^{d+1}}) \tag{41}$$

for $(x,\xi,r)\in\Gamma_C\times\mathbb{R}^d\times\mathbb{R}$ with $\overline{c}>0$. Under these notations the functional J can be written as

$$J((\overline{w},\overline{\varphi}),(w,\varphi))=\int_{\Gamma_C}p(x,w(x),\varphi(x))\,d\Gamma$$

where, as above, we omit the dependence of p on $(\overline{w},\overline{\varphi})$. Taking into account (39) and (41) and applying Theorem 2.7.5 of [5], we deduce that $J((\overline{w},\overline{\varphi}),\cdot,\cdot)$ is Lipschitz on bounded subsets of $L^2(\Gamma_C;\mathbb{R}^{d+1})$ for all $(\overline{w},\overline{\varphi})\in L^2(\Gamma_C;\mathbb{R}^{d+1})$. Therefore the conclusion (1) of the lemma holds and, moreover,

$$\partial J((\overline{w},\overline{\varphi}),(w,\varphi))\subseteq\int_{\Gamma_C}\partial_{(\xi,r)}p(x,w(x),\varphi(x))\,d\Gamma \tag{42}$$

for all $(w,\varphi)\in L^2(\Gamma_C;\mathbb{R}^{d+1})$. In addition, by the Fatou lemma we have

$$J^0((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi))\leqslant\int_{\Gamma_C}p^0(x,w(x),\varphi(x);v(x),\psi(x))\,d\Gamma \tag{43}$$

for all (w,φ), $(v,\psi)\in L^2(\Gamma_C;\mathbb{R}^{d+1})$. Condition (42) implies that for every $(u,\vartheta)\in\partial J((\overline{w},\overline{\varphi}),(w,\varphi))$ there exists $(\overline{u},\overline{\vartheta})\in L^2(\Gamma_C;\mathbb{R}^d)\times L^2(\Gamma_C)$ such that $(\overline{u}(x),\overline{\vartheta}(x))\in\partial_{(\xi,r)}p(x,w(x),\varphi(x))$ a.e. $x\in\Gamma_C$ and

$$\langle(u,\vartheta),(\overline{v},\overline{\psi})\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})}=\int_{\Gamma_C}(\overline{u}(x),\overline{\vartheta}(x))\cdot(\overline{v}(x),\overline{\psi}(x))\,d\Gamma$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

for all $(\overline{v},\overline{\varphi}) \in L^2(\Gamma_C;\mathbb{R}^d) \times L^2(\Gamma_C)$. Hence, from (41) we have

$$\begin{aligned}
\|(u,\vartheta)\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})} &= \sup_{\|(\overline{v},\overline{\varphi})\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})}\leqslant 1} |\langle (u,\vartheta),(\overline{v},\overline{\varphi})\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})}| \\
&\leqslant \Big(\int_{\Gamma_C} \|(\overline{u}(x),\overline{\vartheta}(x))\|^2_{\mathbb{R}^{d+1}}\, d\Gamma\Big)^{1/2} \\
&\leqslant \overline{c}\,\Big(\int_{\Gamma_C} (1+\|w(x)\|_{\mathbb{R}^d}+|\varphi(x)|)^2\, d\Gamma\Big)^{1/2} \\
&\leqslant \widetilde{c}\,\big(1+\|w\|_{L^2(\Gamma_C;\mathbb{R}^d)}+\|\varphi\|_{L^2(\Gamma_C)}\big) = \widetilde{c}\,\big(1+\|(w,\varphi)\|_{L^2(\Gamma_C;\mathbb{R}^{d+1})}\big)
\end{aligned}$$

with $\widetilde{c}>0$, which shows that property (2) holds.

Next, we will prove (3) and (5). We use (43) in which we estimate the integrand on the right-hand side. By the definition of the generalized directional derivative, we get

$$p^0(x,\xi,r;\eta,s) \leqslant j_1^0(x,\xi;\eta) + j_2^0(x,r;s)$$

for a.e. $x\in\Gamma_C$, all (ξ,r), $(\eta,s)\in\mathbb{R}^d\times\mathbb{R}$, and from Lemma 10($i$) we deduce that

$$\begin{aligned}
j_1^0(x,\xi;\eta) &\leqslant h_\tau(x,\overline{\varphi}(x)-\varphi_0(x))\, j_\tau^0(x,\xi_\tau;\eta_\tau), \\
j_2^0(x,r;s) &\leqslant j^0(x,r-\varphi_0(x);s)
\end{aligned}$$

for all $\xi,\ \eta\in\mathbb{R}^d$ and $r,\ s\in\mathbb{R}$. Hence, by using (43) we obtain the inequality in (3). Subsequently, we use the regularity of $j_1(x,\cdot)$, $j_2(x,\cdot)$ for a.e. $x\in\Gamma_C$ and the fact that h_τ is non-negative. We note that $p(x,\cdot,\cdot)$ is regular for a.e. $x\in\Gamma_C$. In fact, using Lemma 10(i) and the surjectivity of Υ, we have

$$\begin{aligned}
p^0(x,\xi,r;\eta,s) &\leqslant j_1^0(x,\xi;\eta) + j_2^0(x,r;s) \\
&= j_1'(x,\xi;\eta) + j_2'(x,r;s) = p'_{(\xi,r)}(x,\xi,r;\eta,s)
\end{aligned} \tag{44}$$

for all $\xi,\ \eta\in\mathbb{R}^d$ and $r,\ s\in\mathbb{R}$, a.e. $x\in\Gamma_C$, where $p'_{(\xi,r)}$ denotes the one-sided directional derivative of $p(x,\cdot,\cdot)$. Since we always have $p'_{(\xi,r)}(x,\xi,r;\eta,s)\leqslant p^0(x,\xi,r;\eta,s)$, we deduce the regularity of $p(x,\cdot,\cdot)$. From the regularity of $p(x,\cdot,\cdot)$, inequality (43) and Fatou's lemma, we obtain

$$\begin{aligned}
&J^0((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi)) \\
&\quad\geqslant \liminf_{\lambda\to 0} \frac{J((\overline{w},\overline{\varphi}),(w,\varphi)+\lambda(v,\psi)) - J((\overline{w},\overline{\varphi}),(w,\varphi))}{\lambda} \\
&\quad= \liminf_{\lambda\to 0} \int_{\Gamma_C} \frac{p(x,w(x)+\lambda v(x),\varphi(x)+\lambda\psi(x)) - p(x,w(x),\varphi(x))}{\lambda}\, d\Gamma \\
&\quad\geqslant \int_{\Gamma_C} \liminf_{\lambda\to 0} \frac{p(x,w(x)+\lambda v(x),\varphi(x)+\lambda\psi(x)) - p(x,w(x),\varphi(x))}{\lambda}\, d\Gamma \\
&\quad= \int_{\Gamma_C} p'_{(\xi,r)}(x,w(x),\varphi(x);v(x),\psi(x))\, d\Gamma \\
&\quad= \int_{\Gamma_C} p^0(x,w(x),\varphi(x);v(x),\psi(x))\, d\Gamma \ \geqslant J^0((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi))
\end{aligned}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

for all (w,φ), $(v,\psi) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$. Hence we obtain that the one-sided derivative $J'((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi))$ exists and $J'((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi)) = J^0((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi))$ for all (w,φ), (v,ψ), i.e. $J(\overline{w},\overline{\varphi},\cdot,\cdot)$ is regular and therefore (5) holds. Moreover, the last inequality together with (44) implies

$$\begin{aligned} J^0((\overline{w},\overline{\varphi}),(w,\varphi);(v,\psi)) &= \int_{\Gamma_C} p^0(x,w(x),\varphi(x);v(x),\psi(x))\,d\Gamma \\ &= \int_{\Gamma_C} \left(j_1^0(x,w(x);v(x)) + j_2^0(\varphi(x);\psi(x))\right) d\Gamma \end{aligned} \tag{45}$$

which shows that (3) holds.

The inequality (4) is a consequence of (3), combined with the Lipschitz continuity of the functions $\eta \mapsto j_\tau^0(x,\xi;\eta)$ and $s \mapsto j^0(x,r;s)$ and the assumptions $H(j_\tau)(iv)$, $H(j)(iv)$, $H(h_\tau)$ and $H(\varphi_0)$.

For the proof of the property (6) we consider sequences $\{(\overline{w}_k,\overline{\varphi}_k)\}$, $\{(w_k,\varphi_k)\}$, $\{(u_k,\vartheta_k)\} \subset L^2(\Gamma_C;\mathbb{R}^{d+1})$ such that $(u_k,\vartheta_k) \in \partial J((\overline{w}_k,\overline{\varphi}_k),(w_k,\varphi_k))$, $(\overline{w}_k,\overline{\varphi}_k) \to (\overline{w},\overline{\varphi})$, $(w_k,\varphi_k) \to (w,\varphi)$ in $L^2(\Gamma_C;\mathbb{R}^{d+1})$ and $(u_k,\vartheta_k) \to (u,\vartheta)$ weakly in $L^2(\Gamma_C;\mathbb{R}^{d+1})$ as $k \to \infty$. The latter means that

$$\langle (u_k,\vartheta_k),(v,\psi)\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant J^0((\overline{w}_k,\overline{\varphi}_k),(w_k,\varphi_k);(v,\psi))$$

for all $(v,\psi) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$. Passing to subsequences we may assume, without loss of generality, that $\overline{w}_k(x) \to \overline{w}(x)$, $w_k(x) \to w(x)$ in $\mathbb{R}^d$ for a.e. $x \in \Gamma_C$, $\overline{\varphi}_k(x) \to \overline{\varphi}(x)$, $\varphi_k(x) \to \varphi(x)$ in $\mathbb{R}$ for a.e. $x \in \Gamma_C$ and

$$\max\{\,\|\overline{w}_k(x)\|_{\mathbb{R}^d},\ \|w_k(x)\|_{\mathbb{R}^d},\ |\overline{\varphi}_k(x)|,\ |\varphi_k(x)|\,\} \leqslant l(x) \quad \text{with } l \in L^2(\Gamma_C).$$

Let $(v,\psi) \in L^2(\Gamma_C;\mathbb{R}^{d+1})$. From hypotheses $H(j_\tau)(iii)$ and $H(j)(iii)$, we have

$$\begin{aligned} j_\tau^0(x,w_{k\tau}(x);v_\tau(x)) &= \max\{\zeta\cdot v_\tau(x) \mid \zeta \in \partial j_\tau(x,w_{k\tau}(x))\} \\ &\leqslant c_\tau\,\|v(x)\|_{\mathbb{R}^d}\,(1+\|w_k(x)\|_{\mathbb{R}^d}), \\ j^0(x,\varphi_k(x)-\varphi_0(x);\psi(x)) &= \max\{\zeta\,\psi(x) \mid \zeta \in \partial j(x,\varphi_k(x)-\varphi_0(x))\} \\ &\leqslant \widehat{c}\,|\psi(x)|\,(1+|\varphi_k(x)|) \end{aligned}$$

with $\widehat{c} > 0$. Taking into account the continuity of $h_\tau(x,\cdot)$, $h(x,\cdot)$, the equality (3), Fatou's lemma and the upper semi-continuity of $j_\tau^0(x,\cdot;\eta)$ and $j^0(x,\cdot;s)$, we have

$$\begin{aligned} &\limsup_{k\to\infty} J^0((\overline{w}_k,\overline{\varphi}_k),(w_k,\varphi_k);(v,\psi)) \\ &\quad\leqslant \limsup_{k\to\infty} \int_{\Gamma_C} \left(h_\tau(\overline{\varphi}_k-\varphi_0)j_\tau^0(w_{k\tau};v_\tau) + j^0(\varphi_k-\varphi_0;\psi)\right) d\Gamma \end{aligned}$$

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

$$\leqslant \int_{\Gamma_C} \limsup_{k\to\infty} \left(|h_\tau(\overline{\varphi}_k - \varphi_0) - h_\tau(\overline{\varphi} - \varphi_0)|\, c_\tau \,\|v(x)\|_{\mathbb{R}^d} (1 + \|w_k(x)\|_{\mathbb{R}^d})\right) d\Gamma$$
$$+ \int_{\Gamma_C} \left(h_\tau(\overline{\varphi} - \varphi_0)\, j_\tau^0(w_\tau; v_\tau) + j^0(\varphi - \varphi_0; \psi)\right) d\Gamma$$
$$\leqslant \int_{\Gamma_C} \left(h_\tau(\overline{\varphi} - \varphi_0) j_\tau^0(w_\tau; v_\tau) + j^0(\varphi - \varphi_0; \psi)\right) d\Gamma$$
$$= J^0((\overline{w}, \overline{\varphi}), (w, \varphi); (v, \psi)).$$

The last equality is a consequence of (45). Hence

$$\langle (u, \vartheta), (v, \psi)\rangle_{L^2(\Gamma_C;\mathbb{R}^{d+1})} \leqslant J^0((\overline{w}, \overline{\varphi}), (w, \varphi); (v, \psi))$$

for all $(v, \psi) \in L^2(\Gamma_C; \mathbb{R}^{d+1})$. This implies $(u, \vartheta) \in \partial J((\overline{w}, \overline{\varphi}), (w, \varphi))$ which finishes the proof that the graph of ∂J is closed in a suitable topology and completes the proof of the lemma. □

Proof of Lemma 10 We note that T is a Fréchet differentiable operator, therefore, it is also Gâteaux differentiable. Moreover, for any x, $h \in E$, we have $T_F'(x)h = Lh$, where $T_F'(x)h$ denotes the Fréchet derivative of T at $x \in E$ in the direction $h \in E$. From Proposition 6.7 of [1], we have

$$\phi^0(x; v) \leqslant \psi^0(Tx; T_F'(x)v) \text{ for all } x, v \in E,$$

which implies (*i*). Since $T_F'(x)^* \in \mathscr{L}(F^*, E^*)$, by Proposition 5.6.26 of [7], we obtain

$$\partial\phi(x) \subseteq T_F'(x)^* \partial\psi(Tx) = L^* \partial\psi(Tx) \text{ for all } x \in E.$$

The conclusions on the equalities in (*i*) and (*ii*) of the lemma follow now from Proposition 5.6.26 and Remark 5.6.27 of [7]. □

7 Conclusion

This paper represents a continuation of [11, 12, 16]. There, existence results in the study of static contact problems with electro-elastic materials were presented under the assumption that the foundation is insulated. The aim of the present paper is to extend the study of [11, 12, 16] to a model in which the contact is bilateral and the foundation is assumed to be electrically conductive. Our main results are presented in Theorems 6 and 12 which state the existence and the uniqueness of the weak solution to the model, respectively. The proofs of the theorems are based on arguments on multi-valued inclusions in Banach spaces with pseudomonotone operators. Taking into account the conductivity of the foundation in the model, it leads to some mathematical difficulties, as explained in Remarks 5, 11 and 14. Although this paper provides the weak solvability of the contact model, some open problems remain to be investigated and resolved in the future. For instance, whether the assumption $h_\tau =$ constant and the smallness assumption (37) represent a limitation of the mathematical approach or an intrinsic feature of the model is an open question. On the other hand, the extension of the results to the quasi-static or dynamic case as well as to models

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press

involving the Signorini unilateral contact condition would be of great interest in the future. Also, the analysis of numerical schemes including the error estimates and the optimal error estimates, as well as the implementation issue of the numerical schemes, would be, in our opinion, of major importance in solving models for piezoelectric contact with a conductive foundation and will open avenues for new advances and applications.

References

[1] Aubin, J.-P. (1984) *L'analyse non linéaire et ses motivations économiques*, Masson, Paris.

[2] Barboteu, M., Fernandez, J. R. & Raffat, T. Numerical analysis of a dynamic piezoelectric contact problem arising in viscoelasticity, *Comput. Methods Appl. Mech. Eng.* to appear.

[3] Batra, R. C. & Yang, J. S. (1995) Saint-Venant's principle in linear piezoelectricity. *J. Elasticity* **38**, 209–218.

[4] Bisegna, P., Lebon, F. & Maceri, F. (2002) The unilateral frictional contact of a piezoelectric body with a rigid support. In: J. A. C. Martins & Manuel D. P. Monteiro Marques (editors), *Contact Mechanics*, Kluwer, Dordrecht, pp. 347–354.

[5] Clarke, F. H. (1983) *Optimization and Nonsmooth Analysis*, Wiley, Interscience, New York.

[6] Denkowski, Z., Migórski, S. & Papageorgiou, N. S. (2003) *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York.

[7] Denkowski, Z., Migórski, S. & Papageorgiou, N. S. (2003) *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston.

[8] Han, W. & Sofonea, M. (2002) *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*. In: *Studies in Advanced Mathematics* 30, American Mathematical Society, Providence, RI.

[9] Ikeda, T. (1990) *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford.

[10] Maceri, F. & Bisegna, P. (1998) The unilateral frictionless contact of a piezoelectric body with a rigid support. *Math. Comput. Model.* **28**, 19–28.

[11] Migórski, S. (2006) Hemivariational inequality for a frictional contact problem in elasto-piezoelectricity. *Discrete Continuous Dynam. Syst. – Ser. B* **6**, 1339–1356.

[12] Migórski, S. (2008) A class of hemivariational inequality for electroelastic contact problems with slip dependent friction. *Discrete Continuous Dynam. Syst. – Ser. S* **1**, 117–126.

[13] Panagiotopoulos, P. D. (1993) *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin.

[14] Patron, V. Z. & Kudryavtsev, B. A. (1988) *Electromagnetoelasticity, Piezoelectrics and Electrically Conductive Solids*, Gordon & Breach, London.

[15] Shillor, M., Sofonea, M. & Telega, J. J. (2004) *Models and Analysis of Quasistatic Contact*, Springer, Berlin.

[16] Sofonea, M. & Essoufi, El H. (2004) A piezoelectric contact problem with slip dependent coefficient of friction. *Math. Model. Anal.* **9**, 229–242.

[17] Yang, J. S., Batra, R. C. & Liang, X. Q. (1994) The cylindrical bending vibration of a laminated elastic plate due to piezoelectric actuators. *Smart Mat. Struct.* **3**, 485–493.

[18] Yang, J. & Yang, J. S. (2005) *An Introduction to the Theory of Piezoelectricity*, Springer, New York.

[19] Zeidler, E. (1990) *Nonlinear Functional Analysis and Applications II A/B*, Springer, New York.

https://doi.org/10.1017/S0956792508007663 Published online by Cambridge University Press