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Vanishing transverse entropy in smooth ergodic theory

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Abstract. For a measure-preserving transformation, the entropy being zero means that there is no increasing σ-algebra. In this note, we prove that a similar phenomenon occurs for C^2 diffeomorphisms when considering the increment between the partial entropies associated with different exponents.

1. *Introduction*

In this note we consider a $C^{1+\text{Hölder}}$ diffeomorphism preserving a Borel probability measure μ on a compact manifold M. Smooth ergodic theory provides geometric objects and numbers which we shall quickly review; see [**1, 10**] for details. Firstly, Oseledets' theorem yields a μ-almost everywhere decomposition of the tangent space

$$T_x M = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$$

and invariant functions $\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$ such that

$$0 \neq v \in E_i(x) \Longleftrightarrow \lim_{n \to \pm\infty} \frac{1}{n} \ln \|D_x f^n v\| = \lambda_i(x),$$

for some Riemannian metric on M. Then, Pesin theory says that for μ-almost every (a.e.) x, for all $\lambda > 0$ the set

$$W^\lambda(x) := \left\{ y \in M : \limsup_{n \to \infty} \frac{1}{n} \ln d(f^{-n}x,\ f^{-n}y) \leq -\lambda \right\}$$

is an embedded d_λ-dimensional disk, where $d_\lambda(x) = \sum_{i:\lambda_i(x) \geq \lambda} \text{Dim}\, E_i(x)$. For all $\lambda > 0$, the $W^\lambda(x)$ form a measurable lamination $\mathcal{W}^\lambda$. To each foliation $\mathcal{W}^\lambda$ is associated a family of conditional measures μ^λ, such that for μ-a.e. x, μ_x^λ is a measure on $W^\lambda(x)$, up to a multiplicative constant (see, e.g., [**7**]). Moreover, by the work of Ledrappier and Young, conditional measures μ_x^λ associated to $\mathcal{W}^\lambda$ are exact-dimensional: there exists an invariant

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function $\delta_\lambda(x)$ such that for μ-a.e. x,

$$\lim_{\varepsilon\to 0}\frac{\ln \mu_x^\lambda B^\lambda(x,\varepsilon)}{\ln \varepsilon}=\delta_\lambda(x),$$

where $B^\lambda(x,\varepsilon)$ is the ball of radius ε in $W^\lambda(x)$ for the induced metric. Finally, define $\gamma_i(x)$ by

$$\delta_\lambda(x)=\sum_{i:\lambda_i(x)\geq\lambda}\gamma_i(x).$$

The number $\gamma_1(x)$ is the dimension of the conditional measures $\mu_x^{\lambda_1}$, i.e. $\gamma_1(x)=\delta_{\lambda_1}(x)$. In particular, if $\mu_x^{\lambda_1}$ is a Dirac measure, then $\gamma_1(x)=0$. This property is no longer true for $\gamma_i(x)$ when $i\neq 1$. In §3 we describe a system in dimension three where the middle direction ($i=2$) is integrable, the conditional measures associated with that one-dimensional foliation are Dirac measures, and $\gamma_2(x)=1$. Our main result is that the vanishing of some $\gamma_i(x)$ corresponds to a stronger property: loosely speaking, the conditional measures associated to $\mathcal{W}^{\lambda_i}$ are carried by lower-dimensional leaves.

THEOREM 1. *Let f be a C^2 diffeomorphism preserving a Borel probability measure μ on a compact manifold M. With the above notation, for $\lambda<\lambda'$ and an invariant set A of positive measure, the following properties are equivalent:*

(1) $\delta_\lambda(x)=\delta_{\lambda'}(x)$ *for μ-a.e. $x\in A$;*
(2) $\sum_{i:\lambda\leq\lambda_i(x)<\lambda'}\gamma_i(x)=0$ *for μ-a.e. $x\in A$;*
(3) *for μ-a.e. $x\in A$, $\mu_x^\lambda=\mu_x^{\lambda'}$;*
(4) *for μ-a.e. $x\in A$, μ_x^λ is carried by a single $W^{\lambda'}$ leaf.*

Theorem 1 answers a question posed by Katok and Rodriguez Hertz in [**6**]. Indeed, one expects that for the action of a higher-rank abelian group which is truly of higher rank, either $\gamma_i(x)=0$ or μ^{λ_i} is equivalent to the product of μ^{λ_i-1} and a transversal Lebesgue class (see [**3–7**] for surveys and results). Theorem 1 is relevant since it relates some global entropy property (the vanishing of some γ_i) to the geometry of the invariant measure. The proof of Theorem 1 is given in §2.

2. *Proof of Theorem 1*

Step 1. Observe first that the equivalence between property (1) and property (2) is immediate from the definition of $\gamma_i(x)$. The equivalence between the geometric properties (3) and (4) is immediate as well. Moreover, it is also immediate that if the geometric property (3) is satisfied, then property (1) holds.

It remains to show that property (2) implies property (4). We may assume that the measure μ is ergodic. Indeed, invariant sets are $\mathcal{W}^\lambda$-saturated for any $\lambda>0$, and the decomposition of μ into ergodic measures can be seen as a decomposition into conditional measures associated with the σ-algebra of invariant sets. So, for μ-a.e. x, μ_x^λ is also the conditional measure associated to the ergodic component μ_x and (2) $\Rightarrow$ (4) is a property of the conditional measures μ_x^λ. Since μ is ergodic, the functions r, λ_i, Dim E_i, δ_λ and γ_i are almost everywhere constant, and from here on we will not write out the dependence on x of these functions.

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We assume that $0 \le i < j$ and $\lambda_i > \lambda_j > 0$, and we write W^i, W^j, $\mathcal{W}^i$ and $\mathcal{W}^j$ for W^{λ_i}, W^{λ_j}, $\mathcal{W}^{\lambda_i}$ and $\mathcal{W}^{\lambda_j}$, respectively. For $i = 0$, we take W^0_x to mean $\{x\}$. By the above discussion, Theorem 1 follows from the below proposition.

PROPOSITION 2. *With the above notation, assume that $\gamma_k = 0$ for $i < k \le j$. Then, for μ-a.e. $x \in A$, $\mu^{\lambda_j}_x$ is carried by a single W^i leaf.*

Step 2. Subordinate partitions and leafwise entropy. Recall that a measurable partition η is said to be *subordinate* to the lamination $\mathcal{W}$ if:

(1) η is increasing, i.e. $f^{-1}\eta$ refines η;
(2) η generates, i.e. $f^{-n}\eta$ tends to the partition into points as $n \to \infty$;
(3) for μ-a.e. x, $\eta(x)$ is a neighborhood of x inside the leaf $W(x)$; and
(4) for μ-a.e. x, $\bigcup_{n\ge 0} f^n(\eta(f^{-n}x))$ is the whole leaf $W(x)$.

Our proof relies on the following lemma.

LEMMA 3. *With the above notation, there exist measurable partitions η_i subordinate to $\mathcal{W}^i$ and η_j subordinate to $\mathcal{W}^j$ such that*

$$\eta_i > \eta_j \quad \text{and} \quad f^{-1}\eta_i = \eta_i \vee f^{-1}\eta_j. \tag{2.1}$$

The subordinate partitions η_i and η_j are constructed along the lines presented in [**10**] (where the corresponding constructions are for the random diffeomorphisms case) or of the original construction of Ledrappier and Young [**8, 9**]. As observed in [**12**, Proposition IX.2.7] (where the corresponding constructions are for the endomorphisms case), a careful simultaneous construction of η_i and η_j ensures property (2.1). Take for η_0 the partition of M into points.

In the rest of the proof, we fix the η_i and η_j given by Lemma 3 as well as two families μ^i_x and μ^j_x of conditional probability measures associated with η_i and η_j, respectively. Following [**9**] (see [**2**, Appendix] for the extension from C^2 diffeomorphisms to $C^{1+\text{Hölder}}$ diffeomorphisms), we introduce the leafwise entropies

$$h_i = H_\mu(f^{-1}\eta_i|\eta_i) = \int -\ln \mu^i_x(f^{-1}\eta_i(x))\, d\mu,$$
$$h_j = H_\mu(f^{-1}\eta_j|\eta_j) = \int -\ln \mu^j_x(f^{-1}\eta_j(x))\, d\mu$$

and recall the entropy formula $h_\ell = \sum_{k\le\ell} \gamma_k\lambda_k$ for $\ell = i,\ j$, so that

$$h_j - h_i = \sum_{k:i<k\le j} \gamma_k\lambda_k = \int \ln \frac{\mu^i_x(f^{-1}\eta_i(x))}{\mu^j_x(f^{-1}\eta_j(x))}\, d\mu.$$

Using property (2.1), we obtain

$$\int \ln \frac{\mu^i_x(f^{-1}\eta_j(x))}{\mu^j_x(f^{-1}\eta_j(x))}\, d\mu = \sum_{k:i<k\le j} \gamma_k\lambda_k. \tag{2.2}$$

Remark 1. In the limit case where $i = 0$, Proposition 2 is well known: μ^0_x is the Dirac measure at x and equation (2.2) reads $-\int \ln \mu^j_x(f^{-1}\eta_j(x))\, d\mu = 0$. This means that μ^j_x is carried by a single element of $f^{-1}\eta_j$. Since $f^{-n}\eta$ generates, the conditional measures μ^j_x are Dirac measures. Since $\bigcup_{n\ge 0} f^n(\eta_j(f^{-n}x))$ is the whole leaf $W^j(x)$, μ-a.e. conditional

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measure $\mu_x^{\lambda_j}$ is the Dirac measure μ_x^0. Our assertion is that this property still holds for intermediate foliations (where $i > 0$).

Step 3. Proof of Proposition 2. For any element A of the partition η_j, write $\mu_A^j := \mu_x^j$ for any $x \in A$ and define

$$I(A) := \int_A \ln \frac{\mu_y^i(f^{-1}\eta_j(y))}{\mu_y^j(f^{-1}\eta_j(y))}\, d\mu_A^j(y).$$

Then $h_j - h_i = \int_{M/\eta_j} I(A)\, d\mu_{\eta_j}(A)$, where μ_{η_j} is the quotient measure of μ on M/η_j.

Since $h_j = H_\mu(f^{-1}\eta_j|\eta_j) < +\infty$, there exists a countable partition $\{B_k : k \geq 1\}$ with finite entropy such that $f^{-1}\eta_j = \eta_j \vee \{B_k : k \geq 1\}$ (see [**13**, Lemma 10.2]). So, for μ_{η_j}-a.e. A, we partition A by $A_k := A \cap B_{k'}$ in such a way that $\mu_A^j(A_k) > 0$ and the partition $\{A_k\}$ is the trace of $f^{-1}\eta_j$ on A (in the sense of μ_A^j-mod 0).

For two probability measures ν_1 and ν_2 with $\nu_2 \ll \nu_1$ and $\rho := d\nu_2/d\nu_1$, we consider the relative entropy $H(\nu_2, \nu_1) = H(\rho \cdot \nu_1, \nu_1)$ given by

$$H(\rho \cdot \nu_1, \nu_1) = \int \rho \ln \rho\, d\nu_1 \geq 0.$$

The relative entropy $H(\rho \cdot \nu_1, \nu_1)$ is zero if and only if $\rho \equiv 1$ holds ν_1-almost everywhere (or, equivalently, $\nu_1 = \nu_2$). Now, upon noting that

$$\mu_A^j(A_k) = \int_{A/\eta_i} \mu_B^i(A_k)\, d(\mu_A^j)_{\eta_i}(B), \tag{2.3}$$

where $(\mu_A^j)_{\eta_i}$ is the measure on the quotient space A/η_i associated with the measure μ_A^j on A, we obtain the following (with the convention that $0 \cdot \ln 0 = 0$):

$$\begin{aligned}
I(A) &= \sum_k \int_{A_k} \ln \frac{\mu_x^i(f^{-1}\eta_j(x))}{\mu_x^j(f^{-1}\eta_j(x))}\, d\mu_A^j(x) \\
&= \sum_k \int_{A_k} \ln \frac{\mu_x^i(A_k)}{\mu_A^j(A_k)}\, d\mu_A^j(x) \\
&= \sum_k \mu_A^j(A_k) \cdot \int_{A/\eta_i} \frac{\mu_B^i(A_k)}{\mu_A^j(A_k)} \cdot \ln \frac{\mu_B^i(A_k)}{\mu_A^j(A_k)}\, d(\mu_A^j)_{\eta_i}(B) \\
&= \sum_k \mu_A^j(A_k) \cdot \int_{A/\eta_i} \rho_k(B) \ln \rho_k(B)\, d(\mu_A^j)_{\eta_i}(B) \quad \left(\text{with } \rho_k(B) := \frac{\mu_B^i(A_k)}{\mu_A^j(A_k)}\right) \\
&= \sum_k \mu_A^j(A_k) \cdot H(\rho_k \cdot (\mu_A^j)_{\eta_i}, (\mu_A^j)_{\eta_i}).
\end{aligned}$$

Observe that $\nu_1 := (\mu_A^j)_{\eta_i}$ is a probability measure, since it is a quotient of the probability measure μ_A, and that $\nu_2 := \rho_k \cdot \nu_1$ is also a probability measure by equation (2.3). Therefore, $I(A) \geq 0$ for μ_{η_j}-a.e. A.

Under the hypothesis that $\gamma_k = 0$ for $i < k \leq j$, we have $h_i = h_j$ and $I(A) = 0$ for μ_{η_j}-a.e. A. This implies that for μ_{η_j}-a.e. A, $\rho_k(B) = 1$ for $(\mu_A^j)_{\eta_i}$-a.e. $B \in A/\eta_i$.

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Hence $\mu_A^j(A_k) = \mu_B^i(A_k)$ holds true for any $k \geq 1$, μ_{η_j}-a.e. A and $(\mu_A^j)_{\eta_i}$-a.e. B. Equivalently,

$$\mu_A^j = \mu_B^i \text{ when restricted to the } \sigma\text{-algebra } \mathcal{F}_1 := \mathcal{B}(f^{-1}\eta_j)$$

for μ_{η_j}-a.e. A and $(\mu_A^j)_{\eta_i}$-a.e. B; here, $\mathcal{B}(\xi)$ denotes the σ-algebra generated by a partition ξ.

Using the fact that $f^{-n}\eta_i = \eta_i \vee f^{-n}\eta_j$ for any $n \geq 1$ (which can be deduced inductively from property (2.1)) together with the fact that $nh_j - nh_i = H_\mu(f^{-n}\eta_j|\eta_j) - H_\mu(f^{-n}\eta_i|\eta_i)$, we can deduce that

$$\mu_A^j = \mu_B^i \text{ when restricted to all the } \sigma\text{-algebras } \mathcal{F}_n := \mathcal{B}(f^{-n}\eta_j),\ n \geq 1.$$

Saying that η_j generates means that for any $X \in \mathcal{B}(M)$, one can find for all $n \geq 1$ a set $X_n \in \mathcal{F}_n$ such that $\mu(X \Delta X_n) \to 0$ as $n \to \infty$. For μ_{η_j}-a.e. A, we have $\mu_A(X \Delta X_n) \to 0$ as $n \to \infty$. Since $\mathcal{B}(M)$ is countably generated, for μ_{η_j}-a.e. A the union $\bigcup_n \mathcal{F}_n$ generates $\mathcal{B}(M)$ (in the sense of μ_A-mod 0). In the same way, for μ_{η_i}-a.e. B the union $\bigcup_n \mathcal{F}_n$ also generates $\mathcal{B}(M)$ (in the sense of μ_B-mod 0). It follows that for μ_{η_j}-a.e. A fixed,

$$\mu_A^j = \mu_B^i \tag{2.4}$$

holds true for $(\mu_A^j)_{\eta_i}$-a.e. B.

Equation (2.4) means that for μ-a.e. x, the measure μ_x^j is carried by a single element of η_i. By invariance, it follows that the conditional measure associated with $f^n\eta_j$ is carried by a single element of $f^n\eta_i$. Since $\bigcup_{n\geq 0} f^n(\eta_j(f^{-n}x))$ is the whole leaf $W^j(x)$, μ-a.e. conditional measure $\mu_x^{\lambda_j}$ is carried by a single $\bigcup_{n\geq 0} f^n(\eta_i(f^{-n}x))$, which is $W^i(x)$. □

3. *An example*

In this section, we give an example in which the weak stable foliation E_j is uniquely integrable and defines a foliation $\mathcal{W}^j$, the associated conditional measures are Dirac, but the corresponding γ_j is Dim E_j.

Choose $0 > \lambda_2 > \ln(1/2) > \lambda_3$. Consider a smooth mapping f from $\mathbb{R}^3$ into itself which sends the cube $\Omega := [0, 1]^3$ into a horseshoe $f(\Omega)$ with the direction e_1 expanding and such that the intersection $f(\Omega) \cap \Omega$ is $[0, 1] \times \Lambda$, where $\Lambda \subset [0, 1]^2$ is made up of two disjoint rectangles R_0 and R_1 whose sides are parallel to the axes, with height e^{λ_3} and width e^{λ_2}. The intersection of R_0 with the plane $x_2 = 0$ and the intersection of R_1 with the plane $x_2 = 1$ are non-empty.

There is a unique f-invariant measure μ on $\bigcap_n f^n\Omega$ such that

$$\mu\left(\bigcap_{n=0}^{N-1} f^{-n}([0, 1] \times R_{i_n})\right) = 2^{-N}$$

for all sequences $\{i_n\} \in \{0, 1\}^N$. The Oseledets directions are $E_1(x)$ and the two other coordinate axes. The Lyapunov exponents are some $\lambda_1 > 0$ and the negative numbers λ_2 and λ_3 that we chose. The direction E_2 is integrable: the local leaves of $\mathcal{W}^2$ are the lines with constant x_1 and x_3. By construction, there is at most one point of $\bigcap_n f^n\Omega$ in each such local leaf.

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On the other hand, the local stable manifolds are the planes with constant x_1. The conditional measures on these planes are all the same, namely the uniform measure m on Λ. The numbers γ_2 and γ_3 can be read from the uniform measure m: γ_3 is the dimension of the conditional measures on the strong stable manifolds $x_2 = \text{constant}$, while γ_2 is the dimension of the projection of m along the strong stable foliation on the transversal $x_3 = 0$. Finally, observe that the projection of m along the strong stable foliation is (up to affine reparametrization) the infinite Bernoulli convolution with parameter e^{λ_2} (see [**11**]). By a famous result of Solomyak, for almost every choice of λ_2 with $0 > \lambda_2 > \ln(1/2)$, the infinite Bernoulli convolution of parameter e^{λ_2} is absolutely continuous and, in particular, $\gamma_2 = 1$.

Question. One might wonder whether a companion of Theorem 1 holds for the case where the transverse dimensions are maximum. The analog of Remark 1 holds true: when $i = 0$ and $\gamma_\ell = \text{Dim } E_\ell$ for $\ell = 1, \ldots, j$, the measures μ^j are absolutely continuous (with respect to the Lebesgue measures on $\mathcal{W}^j$); this can be proved in a similar way to [**8**], where Ledrappier and Young proved the equivalence between Pesin's entropy formula and the SBR property. The generalized statement would be the following.

STATEMENT 1. *Let f be a C^2 diffeomorphism preserving a Borel probability measure μ on a compact manifold M. With the notation of Theorem 1, for $\lambda < \lambda'$ and an invariant set A of positive measure, the following properties are equivalent:*

(1) *$\gamma_i(x) = \text{Dim } E_i(x)$ for i such that $\lambda \le \lambda_i(x) < \lambda'$ and for μ-a.e. $x \in A$;*

(2) *for μ-a.e. $x \in A$, the measure μ_x^λ is obtained by integrating the measures $\mu_y^{\lambda'}$ on $W^{\lambda'}$ leaves with respect to some transversal measure in the Lebesgue class.*

In the particular case of the example, which is not atypical, Statement 1 would imply that an infinite Bernoulli convolution of dimension one is absolutely continuous. This, while compatible with known conjectures about infinite Bernoulli convolutions, is far from being understood at present.

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