Proc. Camb. Phil. Soc. (1970), **68**, 27
PCPS 68–4
Printed in Great Britain

Order-continuous functions and order-connected spaces

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(*Received* 25 *March* 1968)

We introduce and compare four procedures for defining the order-continuity of a function from one topological ordered space into another, where each reduces to the usual conception when the orderings of the two spaces are trivial. Chiefly for the purposes of this comparison, we use the idea of an 'order-connected' space, and in the course of investigating under which types of order-continuous functions this property is preserved, we are helped in assessing their relative importance.

Preliminaries. Let X be an arbitrary partially ordered set.

Let $A \subseteq X$; we write

$$i_X(A) \equiv U\{[x, \rightarrow] : x \in A\}, d_X(A) \equiv U\{[\leftarrow, x] : x \in A\}$$

(where $[x, \rightarrow]$, $[\leftarrow, x]$ are the (order)-closed intervals $\{y \in X : x \leqslant y\}, \{y \in X : y \leqslant x\}$ respectively in X) and $c_X(A) \equiv i_X(A) \cap d_X(A)$. A subset B of X is said to be *increasing* (*decreasing*) *in* X if and only if $B = i_X(B)$ ($B = d_X(B)$). B is *convex in* X if and only if $B = c_X(B)$. If $A \subseteq Y \subseteq X$, we write $i_Y(A) = i_X(A) \cap Y$, $d_Y(A) = d_X(A) \cap Y$ and $c_Y(A) = c_X(A) \cap Y = i_Y(A) \cap d_Y(A)$, and if $B \subseteq Y \subseteq X$, we say that B is *increasing* (*decreasing, convex*) *in* Y if and only if $B = i_Y(B)$ ($B = d_Y(B), B = c_Y(B)$).

A topological ordered space $(X, \mathscr{T}, \leqslant)$ is a set X endowed with both a topology $\mathscr{T}$ and a partial order $\leqslant$. Since the symbol $\leqslant$ is used to denote any partial order, we usually refer to the ordered space as $(X, \mathscr{T})$. Let $Y \subseteq X$, where $(X, \mathscr{T})$ is an arbitrary topological ordered space, then $(Y, \mathscr{T}_Y)$ with the induced order (where $\mathscr{T}_Y$ is the topology for Y induced by $\mathscr{T}$), is a *topological ordered subspace* of $(X, \mathscr{T})$. An ordered subspace $(Y, \mathscr{T}_Y)$ of an ordered space $(X, \mathscr{T})$ is said to be *$\mathscr{T}$-compatibly ordered* if and only if for each $\mathscr{T}_Y$-closed set F, increasing (decreasing) in Y, there exists a $\mathscr{T}$-closed set F^*, increasing (decreasing) in X, such that $F = F^* \cap Y$.

Let X, X' be partially ordered sets; a function $f: X \rightarrow X'$ is said to be *order-preserving* if and only if $x \leqslant y$ in X implies $f(x) \leqslant f(y)$ in X'.

Lemma 1. *Let X, X' be partially ordered sets and let f be a function of X into X', then the following conditions are equivalent*:

(i) *f is order-preserving.*

(ii) *For each increasing (decreasing) set A' in X', $f^{-1}(A')$ is increasing (decreasing) in X.*

(iii) *For each $x' \in X'$, $f^{-1}([x', \rightarrow])$ ($f^{-1}([\leftarrow, x'])$) is increasing (decreasing) in X.*

A topological ordered space $(X, \mathscr{T})$ is said to be *order-separated* if and only if there exist disjoint $\mathscr{T}$-closed subsets A, B of X with A increasing and B decreasing in X, such that $X = A \cup B$. $(X, \mathscr{T})$ is said to be *order-connected* if and only if it is not order

https://doi.org/10.1017/S0305004100001018 Published online by Cambridge University Press

separated. If $Y \subseteq X$, then Y is said to be *$\mathscr{T}$-order-connected* (or just *order-connected* in X if no ambiguity can arise) if and only if the ordered subspace $(Y,\mathscr{T}_Y)$ is order-connected.

Let $(X,\mathscr{T})$ and $(X',\mathscr{T}')$ be ordered spaces and let f be a function of X into X'. Then f is *weakly $(\mathscr{T},\mathscr{T}')$-order-continuous* if and only if for each increasing (decreasing) $\mathscr{T}'$-open, $\mathscr{T}'$-closed set $U' \subseteq X'$, $f^{-1}(U')$ is increasing (decreasing) $\mathscr{T}$-open, $\mathscr{T}$-closed respectively, in X. The function f is *normally $(\mathscr{T},\mathscr{T}')$-order-continuous* if and only if for each $\mathscr{T}'_Y$-open, $\mathscr{T}'_Y$-closed set U', increasing (decreasing) in $Y = f(X)$, $f^{-1}(U')$ is increasing (decreasing) $\mathscr{T}$-open, $\mathscr{T}$-closed respectively, in X (that is, f, considered as a function of X into Y, is weakly $(\mathscr{T},\mathscr{T}'_Y)$-order-continuous). The function f is *compatibly $(\mathscr{T},\mathscr{T}')$-order-continuous* if and only if f is weakly $(\mathscr{T},\mathscr{T}')$-order-continuous and $f(X)$ is a $\mathscr{T}'$-compatibly ordered subspace of $(X',\mathscr{T}')$. Finally, f is *strongly $(\mathscr{T},\mathscr{T}')$-order-continuous* if and only if f is $(\mathscr{T},\mathscr{T}')$-continuous (in the usual abstract sense) and f is order-preserving.

Remark 1. Let $(X,\mathscr{T})$ and $(X',\mathscr{T}')$ be ordered spaces and let f be a function of X into X'. Then $(Y,\mathscr{T}'_Y)$, where $Y = f(X)$, is a $\mathscr{T}'$-compatibly ordered subspace of $(X',\mathscr{T}')$ if and only if for each $\mathscr{T}'_Y$-closed set A', increasing (decreasing) in Y, there exists a $\mathscr{T}'$-closed set A, increasing (decreasing) in X', such that $f^{-1}(A') = f^{-1}(A)$.

Proof. This follows from the fact that A' is the trace on Y of A if and only if $f^{-1}(A') = f^{-1}(A)$.

Remark 2. Let $(X,\mathscr{T})$ and $(X',\mathscr{T}')$ be ordered spaces and let f be a function of X into X'.

(i) If f is strong $(\mathscr{T},\mathscr{T}')$-order-continuous, then f is normally $(\mathscr{T},\mathscr{T}')$-order-continuous.

(ii) If f is compatibly $(\mathscr{T},\mathscr{T}')$-order-continuous then f is normally $(\mathscr{T},\mathscr{T}')$-order-continuous.

(iii) If f is normally $(\mathscr{T},\mathscr{T}')$-order-continuous, then f is weakly $(\mathscr{T},\mathscr{T}')$-order-continuous.

Proof. (i) Let $Y = f(X)$, then the result follows, using Lemma 1, if we note that f is here both order-preserving and continuous also when regarded as a function from X into Y.

(ii) This is immediate by Remark 1.

(iii) Let $Y = f(X)$ and let U' be increasing $\mathscr{T}'$-open in X', then $U' \cap Y$ is $\mathscr{T}'_Y$-open and increasing in Y so that, by hypothesis, $f^{-1}(U') = f^{-1}(U' \cap Y)$ is $\mathscr{T}$-open and increasing in X. The obvious dual argument suffices when U' is decreasing in X'.

LEMMA 2. *Let $(X,\mathscr{T})$ and $(X',\mathscr{T}')$ be topological ordered spaces and let f be a function of X into X'; if $(X,\mathscr{T})$ is order-connected and if f is normally $(\mathscr{T},\mathscr{T}')$-order-continuous, then $f(X)$ is order-connected in X'.*

Proof. Let $Y = f(X)$ and suppose Y is not order-connected in X'. Then, there exist disjoint $\mathscr{T}'_Y$-closed sets A, B, with A increasing in Y and B decreasing in Y, such that $Y = A \cup B$. By hypothesis, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\mathscr{T}$-closed, increasing and decreasing, respectively in X, and $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$, so that $(X,\mathscr{T})$ is order-separated.

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Some remarks on the motivation for the various definitions of order-continuity are relevant at this point. The proof of Lemma 2 is clearly a mere variation on the classical argument, but for this very reason it is included here in order to clarify why it may break down for a *weakly*-order-continuous function (see Example 1 below); this is bound up with the fact that, unlike the classical case, a loss of generality may be entailed here by assuming that the function is onto X' (that is, if $f: X \to X'$ is weakly $(\mathscr{T},\mathscr{T}')$-order-continuous, then it does not necessarily follow that f, considered as a function from X into $Y = f(X)$, is weakly $(\mathscr{T},\mathscr{T}'_Y)$-order-continuous). It was this difficulty that suggested the need for some strengthening of the original (weak) conception of 'order-continuity', resulting in the introduction of 'normally' and 'compatibly' order-continuous functions. On the other hand, it is seen in Example 3 that even a compatibly order-continuous function need not be continuous in the abstract sense; we remedy this deficiency by the further notion of 'strong' order-continuity.

The following examples show that there exist no logical relationships between the four types of order-continuity other than those contained in, or implied by, Remark 2:

Example 1. Let $X = A \cup \{u, v\}$, where $A = \{x_n : n \in J^+\}$ and where x_n is the element $(1/(n+1), 1-1/(n+1))$, u is the element $(1, 1)$ and v is the element $(0, 0)$ of the real Cartesian plane. With the cardinal ordering ($(a, b) \leqslant (c, d)$ in X if and only if $a \leqslant c$ and $b \leqslant d$ in R, the real line), X is a partially ordered set. Let the subset $\{(2, y): 0 \leqslant y \leqslant 1 \text{ in } R\}$ of the real plane be denoted by X'. Let $\mathscr{C}$, $\mathscr{C}'$ denote the cofinite topologies for X, X' respectively. (The cofinite topology on a set has, for its proper closed sets, the family of finite subsets.) If $c' \equiv (2, \frac{1}{2})$ let f be the function from X into X' defined by writing $f(u) = a' \equiv (2, 1)$, $f(v) = b' = (2, 0)$ and $f(x_n) = c'$. Then, if $Y = f(X)$, $(Y, \mathscr{C}'_Y)$ is a discrete space.

We note that the only $\mathscr{C}'$-open increasing (decreasing) sets in X' are X', ϕ (the empty set), and $X' - \{b'\}$ (X', ϕ, and $X' - \{a'\}$), each of which has an inverse under f easily verified to be both $\mathscr{C}$-open and increasing (decreasing) in X. Consequently, f is weakly $(\mathscr{C},\mathscr{C}')$-order-continuous. However, $Y \equiv f(X)$ is not $\mathscr{C}'$-order-connected, for $Y = \{a'\} \cup \{b', c'\}$ and $\{a'\}$, $\{b', c'\}$ are $\mathscr{C}'_Y$-closed, increasing and decreasing, respectively, in Y. It follows, by Lemma 2, that f is not normally $(\mathscr{C},\mathscr{C}')$-order-continuous since $(X, \mathscr{C})$ is clearly order-connected. It is of interest to note that the topologies used can be defined intrinsically in terms of the ordering relations on X, X' respectively, $\mathscr{C}$ being identical with the well-known interval topology I for X ((1), p. 60), and $\mathscr{C}'$ with the diverse topology (see below, Ex. 2) for X'.

Example 2. Let $X = C \cup \{u\}$, $X' = A \cup B \cup \{v\}$, where $C \equiv \{c_n : n \in J^+\}$, $A \equiv \{a_n : n \in J^+\}$, $B \equiv \{b_n : n \in J^+\}$ and c_n, a_n, b_n, u, v, are the elements $(\frac{1}{2} - 1/(n+1))$, $(-\frac{1}{2} + 1/(n+1))$, $(1, \frac{1}{2} + 1/(n+1))$, $(\frac{1}{2} - 1/(n+1), \frac{1}{2} + 1/(n+1))$, $(0, -1)$ and $(0, \frac{1}{2})$ respectively of the real plane. With the usual cardinal ordering of the plane in each case, X and X' are partially ordered sets. Let $\mathscr{F}$ be the diverse topology for X (that is, $\mathscr{F}$ denotes the topology for X which has, as a sub-basis for its closed sets, the family of diverse (or totally unordered) subsets of X), and let $\mathscr{D}'$ be the Dedekind topology for X' (that is, $\mathscr{D}'$ denotes the topology for X' which has, as a sub-basis for its closed sets, the family of Dedekind—complete subsets of X', where $K' \subseteq X'$ is Dedekind—

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complete if and only if K contains the infima (suprema) of its down (up)-directed subsets, when these elements exist (see (3)). Since X is the union of two diverse subsets, C and $\{u\}$, $(X, \mathscr{F})$ is a discrete space (see (2); Theorem 3, section 2).

We define a function $f: X \to X'$ by agreeing that $f(c_n) = b_n$, for each $n \in J^+$, and $f(u) = v$. Since $(X, \mathscr{F})$ is a discrete space, f is $(\mathscr{F}, \mathscr{D}')$-continuous and it is clear that f is order-preserving so that f is strongly $(\mathscr{F}, \mathscr{D}')$-order-continuous.

However, if $Y = B \cup \{v\}$ in X', then, since B is clearly a diverse subset of X' it follows, by ((2); Lemma 5, section 2) that B is $\mathscr{D}'$-closed and thus $\mathscr{D}'_Y$-closed. Also, it is easily seen that B is increasing in Y and that $i_{X'}(B) = A \cup B$. But $A \cup B$ is not $\mathscr{D}'$-closed in X', for A is down-directed and has v as its infimum in X'. It follows that X' is the only $\mathscr{D}'$-closed subset of X', containing B, which is also increasing in X'. Consequently $(Y, \mathscr{D}'_Y)$, is not $\mathscr{D}'$-compatibly ordered and f is not compatibly $(\mathscr{F}, \mathscr{D}')$-order-continuous.

Example 3. Let $X = A \cup B \cup \{c\}$, where $A \equiv \{a_n : n \in J^+\}, B \equiv \{b_n : n \in J^+\}$ and where a_n, b_n, c are the elements $(1/n, 1 - 1/n)$, $(-1/n, 1/n - 1)$ and $(0, 0)$ of the real plane. Let $X' \equiv \{x', y', z'\}$, where x', y', z' are the elements $(2,1)$, $(2, 0)$ and $(2, 1)$ of the real plane. With the cardinal ordering of the plane in each case, X and X' are partially ordered sets. We shall consider the ordered spaces (X, I) and $(X', \mathscr{T}')$, where $\mathscr{T}'$ is the topology for X' whose open sets are $\{y'\}$, ϕ, and X'.

A function f from X into X' is defined by agreeing that $f(a_n) = z'$, $f(b_n) = x'$ for each $n \in J^+$, and $f(c) = y'$. Since $x' < y' < z'$ in X' the only increasing or decreasing $\mathscr{T}'$-open sets in X' are ϕ and X'. Hence, f is weakly $(I, \mathscr{T}')$-order-continuous. As $f(X) = X'$, f is onto X' so that, in particular, $f(X)$ is a $\mathscr{T}'$-compatibly ordered subspace of $(X', \mathscr{T}')$ and f is compatibly $(I, \mathscr{T}')$-order-continuous.

Each sub-basic I-open subset of X containing c is of the form $\{u : u \not\leqslant b_i\}$ or $\{v : a_j \not\leqslant v\}$ for some $i \in J^+$, $j \in J^+$ and it follows that the sequence $\{a_n\}$ I-converges to c in X. Thus f is not $(I, \mathscr{T}')$-continuous since $f^{-1}(\{y'\}) = \{c\}$ is not I-open in X. Consequently, f is not strongly $(I, \mathscr{T}')$-order-continuous.

THEOREM 1. *The following conditions are equivalent for an ordered space* $(X, \mathscr{T})$:

(i) $(X, \mathscr{T})$ *is order-connected.*

(ii) *The only simultaneously* $\mathscr{T}$*-closed and* $\mathscr{T}$*-open, increasing or decreasing, subsets of* X *are* ϕ *and* X.

(iii) *Given any two distinct elements in* X, *there exists an order-connected subset in* X *containing both.*

(iv) *There exists no compatibly-*$(I, \mathscr{T})$*-order-continuous function* f *from* X *into* R *(where* I *denotes the interval or natural topology for the real line* R*) such that* $f(X)$ *consists of exactly two elements.*

Proof. To see, for example, that (iv) implies (i), suppose $(X, \mathscr{T})$ is not order-connected, then there exist disjoint, $\mathscr{T}$-closed sets A, B, with A increasing, B decreasing in X, such that $X = A \cup B$. We define $f: X \to R$ by writing $f(x) = 0$ if $x \in B$, $f(x) = 1$ if $x \in A$.

It is obvious that f is weakly $(\mathscr{T}, I)$-order-continuous and that $f(X)$ is an I-compatibly ordered subset of R.

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THEOREM 2. *Let* $(X,\mathscr{T})$ *be an ordered space; if* M *is order-connected in* X *and* $M \subseteq N \subseteq c_X(M)$ *then* N *is order-connected in* X.

Proof. Suppose N is not order-connected in X, then $N = A \cup B$, where A, B are disjoint, $\mathscr{T}_N$-closed with A increasing, B decreasing in N. Then $A \cap M \neq \phi$, for otherwise $M \subseteq B$ and consequently $A = A \cap (N \cap c_X(B)) = A \cap c_N(B) = A \cap B = \phi$. Similarly $B \cap M \neq \phi$. Thus, $A \cap M$, $B \cap M$ are non-empty, disjoint, $\mathscr{T}_M$-closed, with $A \cap M$ increasing, $B \cap M$ decreasing in M, so that $M = (A \cap M) \cup (B \cap M)$ is not order-connected in X.

THEOREM 3. *Each order-component of an ordered space* $(X,\mathscr{T})$ *is* $\mathscr{T}$*-closed and convex in* X.

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