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TESTING FOR LONG MEMORY

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This paper introduces a new test statistic for the null hypothesis of short memory against long memory alternatives. The novelty of our statistic is that it is based on only high-order sample autocovariances and by construction eliminates the effects of nuisance parameters typically induced by short memory autocorrelation. For practically relevant situations where the short memory process is not directly observed, but instead appears as the disturbance term in a deterministic linear regression model, we are able to demonstrate that our residual-based statistic has an asymptotic standard normal distribution under the null hypothesis. We also establish consistency of the statistic under long memory alternatives. The finite-sample properties of our procedure are compared to other well-known tests in the literature via Monte Carlo simulations. These show that the empirical size properties of the new statistic can be very robust compared to existing tests and also that it competes well in terms of power.

1. INTRODUCTION

One of the major problems for statistical inference is the elimination of nuisance parameters. It is relatively straightforward to test the null of white noise against memory (short memory, long memory, or integrated behavior). However, the white noise null is, in most situations, rather unrealistic. It often makes more sense to test for the null of short memory against long memory or integrated behavior. If the short memory can be modeled parametrically (e.g., with an autoregressive moving average (ARMA) process) it is usually possible to fit such a model and whiten the data prior to testing. The main difficulty, however, is that the parametric model for short memory is usually unknown and any fitted model may, therefore, not include the "true" parametric data generating process (DGP). In this situation, any subsequent test will typically suffer from size distortions, even asymptotically. In contrast to prewhitening the data,

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we propose a nonparametric method of dealing with nuisance parameters for a class of short memory linear processes that may be observed or be the disturbances of a deterministic linear regression model. So, provided that the class of linear processes is appropriate there is no need to postulate and fit a parametric model for any short memory behavior.

The statistic we suggest is a simple variant of the test for $I(d)$ long memory proposed by Robinson (1994) and Tanaka (1999). It is constructed from appropriately chosen and weighted high-order sample autocovariances. By not including low-order sample autocovariances, we are effectively able to eliminate the effects of the nuisance parameters induced by the presence of $I(0)$ short memory autocorrelation under the null. We demonstrate that our statistic is asymptotically normal under the short memory null when it is calculated using residuals from a quite general deterministic linear regression model. This property renders our statistic convenient for practical implementation, as in many empirical applications the short memory process is not directly observed but instead appears as the disturbance term in a deterministic linear regression model that may cater, e.g., for constants, time trends, or structural breaks. We also establish consistency of our statistic under both stationary and nonstationary long memory alternatives, as characterized by the $I(d)$, $d > 0$ model.

We perform Monte Carlo simulation experiments that demonstrate that our new statistic can exhibit very good size control across a range of stationary short memory DGPs and sample sizes, to an extent not matched by the other tests used in the comparison. This size control is not achieved at the expense of reasonable power, however, with the new test displaying particularly good power for nonstationary $I(d)$ alternatives. Somewhat interestingly, our preferred variant of the new statistic has superior empirical size and power properties to the well-known Kwiatkowski, Phillips, Schmidt, and Shin (KPSS) (1992) test for a short memory null hypothesis against an $I(1)$ alternative.

The structure of the paper is as follows. Section 2 provides some simple motivation for our new test. In Section 3 we demonstrate the asymptotic standard normality of the statistic under a general short memory linear process assumption and also establish its consistency against long memory processes. Some finite-sample issues associated with regression effects are also discussed here. Finally, in Section 4 we report the results of the Monte Carlo simulation experiments. Proofs appear in the Appendix.

2. MOTIVATION

Consider the linear regression model

$$y_t = x_t'\beta + z_t, \tag{1}$$

where the disturbances satisfy the $I(d)$ process

$$(1 - L)^d z_t = u_t \tag{2}$$

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and where u_t is a zero mean stationary short memory process. We are concerned with the hypothesis testing problem $H_0: d = 0$, $H_1: d > 0$. That is, we wish to test the null hypothesis that z_t has $I(0)$ short memory against the alternative that it has $I(d)$ long memory. The autocovariance function of z_t is denoted $\gamma_j = E(z_t z_{t-j})$, which is absolutely summable under the null hypothesis but not under the alternative.

Abstracting from regression effects and supposing for the moment that z_t is observed, our starting point is the Lagrange multiplier (LM) statistic proposed by Robinson (1994) and Tanaka (1999) under the assumption that u_t is Gaussian white noise, i.e.,

$$\tilde{N} = T^{1/2} \sum_{j=1}^{T-1} \frac{\tilde{\gamma}_j}{j},$$

where $\tilde{\gamma}_j = T^{-1} \sum_{t=j+1}^{T} z_t z_{t-j}$. Under the null hypothesis and when u_t is white noise these authors demonstrate that $\tilde{N}$ (when appropriately studentized) has a standard normal limiting distribution. Crucially, it is the fact that the expectation of each $\tilde{\gamma}_j$ is zero that implies that the asymptotic distribution is centered at zero. However, in the more general case when u_t is autocorrelated, we find that

$$|E(\tilde{N})| = T^{1/2} \left| \sum_{j=1}^{T-1} \frac{T-j}{jT} \gamma_j \right| \to \infty$$

because

$$\sum_{j=1}^{T-1} \frac{T-j}{jT} \gamma_j \to \sum_{j=1}^{\infty} \frac{\gamma_j}{j} \neq 0.$$

This implies that when u_t is autocorrelated the statistic is incorrectly centered under the null and the divergence of $|E(\tilde{N})|$ shows that a one-tailed test using $\tilde{N}$ will have an asymptotic size of either zero or one, if standard normal critical values are referenced. To overcome this difficulty, Robinson (1994) provides a modified statistic, based on the periodogram, whose asymptotic null distribution is centered at zero. Tanaka (1999) approaches the problem by suggesting that the test is calculated not from the z_t but instead via the residuals from an ARMA model estimated for z_t. In the case where z_t *is* generated by a stationary ARMA model, and assuming the correct model is fitted (or overfitted), this approach asymptotically recenters the statistic at zero.

In this paper we pursue a third approach to eliminating the mean effect induced by the dependence in u_t under H_0. We modify the statistic $\tilde{N}$ directly by including only the sample autocovariances at long lags. Specifically, we consider the truncated statistic

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$$\tilde{N}_k = (T-k)^{1/2} \sum_{j=k}^{T-1} \frac{\tilde{\gamma}_j}{j-k+1},$$

which is basically $\tilde{N}$ calculated only from sample autocovariances at lag k and above. Essential to our approach here is the fact that long and short memory can be distinguished by the eventual rate of decay of the autocovariance function of z_t, i.e., the rate at which $\gamma_j \to 0$ as $j \to \infty$. Now,

$$|E(\tilde{N}_k)| \leq (T-k)^{1/2} \sum_{j=k}^{T-1} \frac{1}{j-k+1} |\gamma_j|,$$

and it is shown subsequently that under H_0, this converges to zero for suitable choice of k where $k \to \infty$ as $T \to \infty$, giving a correctly centered asymptotic null distribution. Furthermore we show that $\tilde{N}_k$, when suitably studentized, has an asymptotic standard normal null distribution and is divergent under the alternative—hence giving a consistent test for long memory. In essence, $\tilde{N}_k$ disregards any short-run information under H_0 contained in the lower order sample autocovariances up to lag $k-1$. Long memory effects arising under H_1 are not eliminated, however, because long-run information is contained in all the higher order sample autocovariances from lag k onward.

3. DISTRIBUTION THEORY

Let $\hat{z}_t$ be the ordinary least squares (OLS) residuals from (1) with sample autocovariances[1]

$$\hat{\gamma}_j = T^{-1} \sum_{t=(j\vee 0)+1}^{T-(j\wedge 0)} \hat{z}_t \hat{z}_{t-j}.$$

Our studentized test statistic is given by

$$\hat{S}_k^* = \frac{\hat{N}_k}{\hat{\omega}_l},$$

where

$$\hat{N}_k = (T-k)^{1/2} \sum_{j=k}^{T-1} \frac{1}{j-k+1} \hat{\gamma}_j, \qquad \hat{\omega}_l^2 = \sum_{j=-l}^{l} h_j \sum_{i=-l}^{l} \hat{\gamma}_i \hat{\gamma}_{i+j}$$

and $h_0 = \pi^2/6$ and $h_j = H_{|j|}/|j|$ for $j = \pm 1, \pm 2, \ldots$ where H_j are the harmonic numbers $\sum_{i=1}^{j} i^{-1}$.

The derivation of the asymptotic null distribution of $\hat{S}_k^*$ makes use of the following linear process assumption on u_t in (2).

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Assumption LP. Let $u_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ where

(a) $\sum_{j=0}^{\infty} |c_j| < \infty$,
(b) $\varepsilon_t \sim$ i.i.d.$(0,1)$,
(c) $E(\varepsilon_t^4) = m_4 < \infty$.

The following theorem shows that under the null hypothesis the distribution of $\hat{S}_k^*$ is asymptotically standard normal where x_t is a vector of deterministic regressors.

THEOREM 1. *If we assume that*

(i) Assumption LP holds,
(ii) $k = (cT)^{1/2}$ *for some constant* $c > 0$,
(iii) $l \to \infty$ *as* $T \to \infty$ *such that* $l^2/T \to 0$,
(iv) x_t *is a p-dimensional vector of deterministic regressors such that*

$$D_T^{-1} x_{[T\tau]} \to X(\tau) < \infty$$

uniformly in τ *for some matrix* D_T, *and*

$$T^{-1} \sum_{t=1}^{T} D_T^{-1} x_t x_t' D_T^{-1\prime} \to \int_0^1 X(\tau) X(\tau)' \, d\tau > 0,$$

(v) $\sum_{j=1}^{\infty} \ln j \cdot |\gamma_j| < \infty$,

then under $H_0: d = 0$, $\hat{S}_k^* \xrightarrow{d} N(0,1)$.

Note that $\sum_{j=1}^{\infty} \ln j \cdot |\gamma_j| < \infty$ is only a slightly stronger condition than $\sum_{j=1}^{\infty} |\gamma_j| < \infty$, which would follow directly from Assumption LP(i). Notice also that although the numerator of the statistic $\hat{S}_k^*$ only uses information on high-order sample autocovariances, for the correct studentization we do employ short-run information in the construction of $\hat{\omega}_l^2$. The next theorem establishes consistency of $\hat{S}_k^*$ under the alternative hypothesis of long memory.

THEOREM 2. *Under* H_1 *with* $d > 0$, *conditions (i)–(iv) of Theorem 1, and*

(i) $\sum_{j=0}^{\infty} c_j \neq 0$,
(ii) $\sum_{j=1}^{\infty} j |c_j| < \infty$,

the statistic $\hat{S}_k^*$ *diverges to* $+\infty$ *at a rate at least* $T^d \ln T / \ln l$ *for* $0 < d < \frac{1}{4}$, $T^d \ln T / (l^{2d-1/2} (\ln l))$ *for* $\frac{1}{4} < d < \frac{1}{2}$, *and* $(T^{1/2} \ln T)/(l^{1/2} \ln l)$ *for* $d > \frac{1}{2}$.

This shows that a consistent test for long memory is obtained by rejecting H_0 for values of $\hat{S}_k^*$ greater than the upper tail standard normal critical value. Condition (i) of this theorem rules out moving average (MA) unit roots in u_t so

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that z_t is $I(d)$. Condition (ii) is an additional relatively weak technical condition from equation (13) of Tanaka (1999).

The proof of Theorem 1 proceeds by establishing that $\hat{S}_k^* - \tilde{S}_k^* \xrightarrow{p} 0$, where $\tilde{S}_k^*$ is the infeasible statistic calculated from the true z_t rather than OLS estimates $\hat{z}_t$. That is, the effect of estimating the regression (1) has an asymptotically negligible effect on the null distribution of $\hat{S}_k^*$. However, this estimation can have an important effect in finite samples. In particular, a little algebra shows that the numerator of $\hat{S}_k^*$ can be written as

$$\hat{N}_k = \tilde{N}_k - b + o_p(1),$$

where

$$b = q(T-k)^{1/2} \sum_{j=k}^{T-1} \frac{1}{j-k+1}, \qquad q = (T-k)^{-1} \sum_{t=1}^{T} z_t x_t' \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \sum_{t=1}^{T} x_t z_t.$$

The term b approximates the effect of the regression estimation on the numerator $\hat{N}_k$. Even though b is itself asymptotically negligible, it is a term that is unambiguously positive and can therefore contribute a substantial negative finite-sample bias to the statistic $\hat{S}_k^*$, with the effect of reducing the size of the test in finite samples. It turns out that the quality of the asymptotic approximation provided by Theorem 1 can be improved if we define the estimate of the bias induced by b, i.e., $E(b)$, by

$$\hat{b} = \hat{q}(T-k)^{1/2} \sum_{j=k}^{T-1} \frac{1}{j-k+1}, \qquad \hat{q} = (T-k)^{-1} \operatorname{tr}\left[\left(\frac{1}{T} \sum_{t=1}^{T} x_t x_t' \right)^{-1} \hat{\Omega}(x_t \hat{z}_t) \right],$$

where $\hat{\Omega}(.)$ is any standard long-run variance matrix estimator. A "bias corrected" statistic is then defined to be

$$\hat{S}_k = \frac{\hat{N}_k + \hat{b}}{\hat{\omega}_l}.$$

We then have the following corollary.

COROLLARY 1. *Under the conditions of Theorem 1,* $\hat{S}_k \xrightarrow{d} N(0,1)$*; under the conditions of Theorem 2,* $\hat{S}_k$ *diverges to* $+\infty$ *at the same rates as* $\hat{S}_k^*$.

Corollary 1 (proof omitted) follows because $\hat{b}$ is easily shown to be of a smaller order in probability than $\hat{N}_k$ under both H_0 and H_1. It is the statistic $\hat{S}_k$ that we recommend for use in practice.

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4. FINITE-SAMPLE RESULTS

Although the choice of the scaling parameter c, which controls the truncation k, has no asymptotic effect on the properties of $\hat{S}_k$, the finite-sample performance of $\hat{S}_k$ will inherently depend on the specific value selected by the user. Here we evaluate, via Monte Carlo simulation, the properties of $\hat{S}_k$ for various choices of c in the context of an ARMA(1,1) DGP: $(1 - \rho L)u_t = (1 - \theta L)\varepsilon_t$, $\varepsilon_t \sim$ i.i.d.$N(0,1)$. The sample sizes selected are $T = 200$, 500, 1,000, and for c we consider the values $c = 0.5, 1.0, 2.0$, setting $k = \lfloor (cT)^{1/2} \rfloor$ ($\lfloor . \rfloor$ denoting the integer part of the argument). Size calculations are performed under the null hypothesis that $d = 0$ in (2), and power calculations are carried out for $d = 0.2, 0.4, 0.6, 0.8, 1.0$. As regards the supplementary user-supplied items: in $\hat{\omega}_l^2$ we use a bandwidth of $l = \lfloor (\frac{2}{3})T^{12/25} \rfloor$, and for $\hat{\Omega}(.)$ in the bias correction we employ the quadratic spectral (QS) kernel with Newey and West (1994) automatic bandwidth selection, using a nonstochastic prior bandwidth of $\lfloor 4(T/100)^{2/25} \rfloor$.[2]

For comparison, three other well-known tests of long memory are included in the simulations. The first is the KPSS test, shown by Lee and Schmidt (1996) to provide a consistent test of short memory against long memory. Here, we implement the KPSS test using a long-run variance estimator computed using the QS kernel/Newey–West automatic bandwidth selection, with prior bandwidth set at $\lfloor 8(T/100)^{2/25} \rfloor$. This combination is promoted by Hobijn, Franses, and Ooms (2004) as a means of improving the finite-sample size properties of the KPSS test under the $I(0)$ null, while retaining decent levels of power against $I(1)$ alternatives.[3]

Two versions of the time domain test first suggested by Robinson (1994), and further analyzed by Tanaka (1999), are also considered. The test requires the parameterization of a model for u_t, and so we calculate versions of the test assuming that u_t follows an AR(1) process and an ARMA(1,1) process, estimating each model using an exact maximum likelihood procedure. For these experiments, the assumption of an ARMA(1,1) model is always correct, and so the test is theoretically valid. However, the assumption of an AR(1) model is incorrect (except when $\theta = 0$), and so this version of the test will give some idea of the consequences of misspecifying the estimated model for u_t. We implemented the statistic in a form denoted S'_{T2} following Tanaka (1999, Thm. 3.3), which provides the necessary formulas. The tests are carried out using upper tail critical values to test against long memory.

We also include two versions of the test proposed by Lobato and Robinson (1998), which is an LM test based on the Whittle log-likelihood constructed for z_t. We carry out the test in its one-sided form (see Lobato and Robinson, 1998, eqn. (2.2)) and note that the test for long memory is carried out using a lower tail critical value. The statistic is a function of the periodogram of $\hat{z}_t$ and requires the choice of a bandwidth parameter m. We consider two choices for this bandwidth. The first is the formula m^* suggested by Lobato and Robinson (1998,

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p. 481), which is based on an AR(1) approximation to the spectrum of u_t. The second version of the Lobato and Robinson (1998) test is carried out using an ARMA(1,1) approximation to the spectrum of u_t, with the resulting formula for m^* following from equation (3.8) of Delgado and Robinson (1996).

In Tables 1 and 2, the tests are constructed assuming a constant term in (1). Entries refer to empirical rejection frequencies at asymptotic critical values yielding a nominal size of 0.05. Standard normal critical values apply in each case apart from the KPSS test. All results are based on 10,000 replications.

The empirical sizes for the tests are presented in Table 1. When $c = 0.5$, the $\hat{S}_k$ test can be seen to control size very well across the ARMA parameter space with the exception of when the autoregressive (AR) coefficient takes its largest value of $\rho = 0.9$. In this case its size generally exceeds 0.20 when $T = 200$. Increasing the sample size does, however, offset this effect reasonably quickly. When $c = 1.0$, $\hat{S}_k$ is only ever modestly oversized for $\rho = 0.9$ when $T = 200$ (though never above 0.10), and this problem is rapidly corrected as the sample size increases. For $c = 2.0$, there are no notable size distortions anywhere. This is entirely consistent with what we would expect in finite samples; other things being equal, the larger the AR root, the larger the value of c that will be required to correctly center the statistic.

The KPSS test has good size properties for small amounts of autocorrelation in u_t but becomes oversized across all of the sample sizes for strong autocorrelation of an AR form, or large positive MA in small samples. The Robinson (1994)/Tanaka (1999) test assuming an AR(1) model for u_t provides conservative sizes overall when the AR(1) model is correct but can be either badly oversized or undersized if the AR(1) model is incorrect and ARMA(1,1) is the true DGP. Those cases where the AR(1) model is misspecified are included to demonstrate the consequences of model misspecification in the parametric test. The Robinson (1994)/Tanaka (1999) test assuming an ARMA(1,1) model for u_t is always based on a correctly specified model in these particular simulations, and therefore they provide reasonable size control, albeit being somewhat undersized for weak AR autocorrelation when $T = 200$. Of course, if the DGP for u_t were of a higher order than ARMA(1,1) then the size properties of this test would be expected to be rather poor. The Lobato and Robinson (1998) tests can be seen to provide reasonable size control for negative values of θ, although with some undersizing for large values of ρ for $T = 200$. However, the AR(1) version has poor size properties for positive values of θ, with sizes approaching both zero and one in various areas of the parameter space. The ARMA(1,1) version does not suffer from this problem, although there is evidence of undersizing when $T = 200$ and there are also some sizes over 0.10 for $\rho = 0$ in the larger sample sizes. Despite this being a nonparametric test, the obvious implication of these results is that it is important to have a good approximating model from which to compute the bandwidth parameter for this test.

The corresponding empirical powers for the tests are reported in Tables 2A–E. Recall that these are not size adjusted powers, and so comparisons across the

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TABLE 1. Empirical size of tests when $d = 0.0$

	$T = 200$					$T = 500$					$T = 1{,}000$				
ρ/θ	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8
							$\hat{S}_k$, $c = 0.5\|1.0\|2.0$								
0.0	0.04|0.04|0.04	0.04|0.04|0.03	0.04|0.04|0.04	0.04|0.04|0.04	0.04|0.04|0.04	0.05|0.04|0.05	0.05|0.04|0.04	0.05|0.04|0.04	0.04|0.05|0.05	0.05|0.05|0.05	0.05|0.05|0.05	0.05|0.04|0.04	0.04|0.05|0.04	0.05|0.05|0.05	0.05|0.05|0.04
0.5	0.03|0.03|0.03	0.03|0.03|0.03	0.03|0.04|0.04	0.04|0.04|0.04	0.05|0.05|0.05	0.04|0.04|0.04	0.04|0.04|0.04	0.04|0.04|0.04	0.04|0.05|0.04	0.05|0.05|0.05	0.05|0.05|0.04	0.04|0.05|0.05	0.05|0.04|0.05	0.05|0.05|0.04	0.05|0.05|0.04
0.7	0.04|0.03|0.03	0.04|0.03|0.03	0.03|0.03|0.03	0.03|0.03|0.02	0.04|0.05|0.05	0.04|0.04|0.03	0.04|0.04|0.03	0.04|0.04|0.03	0.04|0.04|0.04	0.05|0.05|0.05	0.04|0.05|0.04	0.04|0.04|0.04	0.04|0.04|0.04	0.04|0.04|0.04	0.05|0.05|0.04
0.9	0.22|0.10|0.04	0.21|0.09|0.04	0.23|0.09|0.04	0.20|0.09|0.04	0.12|0.07|0.04	0.14|0.07|0.04	0.15|0.06|0.04	0.14|0.07|0.04	0.14|0.08|0.04	0.11|0.06|0.04	0.10|0.05|0.04	0.10|0.05|0.04	0.11|0.05|0.04	0.11|0.05|0.04	0.08|0.05|0.04
							KPSS								
0.0	0.05	0.05	0.05	0.05	0.16	0.05	0.05	0.05	0.05	0.07	0.06	0.06	0.06	0.05	0.05
0.5	0.05	0.05	0.05	0.05	0.07	0.06	0.06	0.06	0.05	0.04	0.06	0.06	0.06	0.05	0.03
0.7	0.06	0.06	0.06	0.06	0.04	0.07	0.07	0.07	0.07	0.04	0.07	0.07	0.07	0.07	0.04
0.9	0.14	0.14	0.14	0.14	0.10	0.12	0.12	0.12	0.12	0.11	0.12	0.12	0.12	0.11	0.11
							Robinson (1994)/Tanaka (1999) AR(1)								
0.0	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.01	0.00	0.03	0.00	0.00
0.5	0.98	0.39	0.01	0.05	0.00	1.0	0.85	0.02	0.11	0.00	1.0	0.99	0.02	0.20	0.00
0.7	1.0	0.95	0.01	0.17	0.00	1.0	1.0	0.02	0.50	0.00	1.0	1.0	0.02	0.83	0.00
0.9	1.0	0.99	0.06	0.02	0.50	1.0	1.0	0.05	0.02	0.91	1.0	1.0	0.05	0.01	1.0
							Robinson (1994)/Tanaka (1999) ARMA(1,1)								
0.0	0.02	0.01	0.02	0.01	0.02	0.02	0.02	0.02	0.02	0.01	0.03	0.03	0.03	0.03	0.02
0.5	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.01	0.01	0.02	0.02	0.02	0.02	0.02
0.7	0.01	0.01	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.03
0.9	0.06	0.06	0.04	0.01	0.04	0.05	0.04	0.03	0.01	0.02	0.05	0.04	0.03	0.01	0.02
							Lobato and Robinson (1998) AR(1)								
0.0	0.03	0.04	0.03	0.00	0.00	0.04	0.05	0.04	0.00	0.00	0.05	0.06	0.05	0.00	0.00
0.5	0.01	0.02	0.06	0.15	0.00	0.03	0.04	0.08	0.27	0.00	0.04	0.05	0.10	0.37	0.00
0.7	0.00	0.01	0.03	0.34	0.00	0.02	0.03	0.06	0.50	0.00	0.03	0.04	0.09	0.61	0.00
0.9	0.00	0.00	0.00	0.25	0.66	0.03	0.04	0.04	0.46	0.96	0.07	0.07	0.07	0.61	1.0
							Lobato and Robinson (1998) ARMA(1,1)								
0.0	0.09	0.09	0.01	0.01	0.00	0.11	0.11	0.02	0.01	0.01	0.11	0.11	0.02	0.02	0.01
0.5	0.06	0.06	0.05	0.02	0.00	0.08	0.08	0.08	0.03	0.01	0.10	0.10	0.10	0.05	0.01
0.7	0.03	0.04	0.03	0.04	0.01	0.06	0.06	0.06	0.05	0.01	0.08	0.08	0.09	0.08	0.01
0.9	0.00	0.00	0.00	0.00	0.05	0.04	0.04	0.04	0.04	0.06	0.07	0.07	0.07	0.07	0.07

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TABLE 2A. Empirical power of tests when $d = 0.2$

	$T = 200$					$T = 500$					$T = 1{,}000$				
ρ/θ	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8
								$\hat{S}_k$, $c = 0.5\|1.0\|2.0$							
0.0	0.11|0.08|0.05	0.11|0.07|0.05	0.10|0.07|0.05	0.10|0.07|0.05	0.05|0.05|0.04	0.18|0.13|0.09	0.18|0.14|0.09	0.17|0.14|0.09	0.15|0.11|0.08	0.06|0.05|0.05	0.29|0.20|0.14	0.29|0.19|0.14	0.28|0.20|0.13	0.20|0.15|0.11	0.06|0.06|0.05
0.5	0.10|0.07|0.04	0.11|0.08|0.05	0.11|0.08|0.05	0.10|0.07|0.05	0.07|0.06|0.05	0.20|0.15|0.09	0.20|0.14|0.10	0.19|0.14|0.09	0.19|0.14|0.09	0.09|0.08|0.07	0.28|0.21|0.15	0.28|0.20|0.15	0.28|0.20|0.15	0.28|0.20|0.14	0.13|0.10|0.08
0.7	0.14|0.08|0.05	0.13|0.08|0.05	0.13|0.08|0.05	0.13|0.08|0.04	0.08|0.07|0.05	0.20|0.15|0.10	0.21|0.15|0.10	0.21|0.15|0.10	0.20|0.14|0.10	0.14|0.12|0.08	0.30|0.22|0.16	0.30|0.21|0.15	0.30|0.22|0.16	0.31|0.22|0.15	0.22|0.16|0.12
0.9	0.45|0.24|0.10	0.47|0.23|0.10	0.46|0.23|0.10	0.44|0.24|0.09	0.32|0.18|0.08	0.51|0.28|0.16	0.52|0.28|0.15	0.51|0.29|0.15	0.51|0.28|0.15	0.39|0.24|0.14	0.56|0.32|0.21	0.56|0.32|0.20	0.56|0.32|0.19	0.56|0.31|0.19	0.48|0.28|0.18
								KPSS							
0.0	0.14	0.14	0.14	0.14	0.14	0.20	0.20	0.20	0.21	0.15	0.25	0.25	0.25	0.25	0.19
0.5	0.14	0.14	0.14	0.14	0.09	0.20	0.20	0.20	0.20	0.15	0.25	0.25	0.25	0.25	0.21
0.7	0.15	0.15	0.15	0.15	0.10	0.20	0.20	0.20	0.20	0.17	0.25	0.25	0.25	0.25	0.24
0.9	0.25	0.25	0.25	0.25	0.22	0.28	0.28	0.28	0.28	0.26	0.31	0.31	0.31	0.31	0.30
								Robinson (1994)/Tanaka (1999) AR(1)							
0.0	0.38	0.08	0.33	0.15	0.00	0.91	0.42	0.80	0.42	0.00	1.0	0.79	0.98	0.73	0.00
0.5	1.0	0.95	0.03	0.31	0.00	1.0	1.0	0.16	0.81	0.00	1.0	1.0	0.42	0.99	0.00
0.7	1.0	1.0	0.20	0.06	0.05	1.0	1.0	0.38	0.23	0.19	1.0	1.0	0.63	0.49	0.41
0.9	1.0	1.0	0.78	0.00	0.74	1.0	1.0	0.96	0.00	1.0	1.0	1.0	1.0	0.00	1.0
								Robinson (1994)/Tanaka (1999) ARMA(1,1)							
0.0	0.28	0.16	0.02	0.22	0.05	0.75	0.60	0.14	0.61	0.15	0.97	0.91	0.42	0.90	0.35
0.5	0.03	0.03	0.03	0.03	0.03	0.18	0.18	0.19	0.18	0.19	0.45	0.47	0.49	0.47	0.46
0.7	0.16	0.07	0.01	0.01	0.01	0.30	0.16	0.07	0.08	0.07	0.54	0.32	0.22	0.28	0.23
0.9	0.72	0.57	0.35	0.13	0.03	0.94	0.81	0.51	0.13	0.15	1.0	0.96	0.72	0.16	0.42
								Lobato and Robinson (1998) AR(1)							
0.0	0.13	0.22	0.55	0.10	0.00	0.38	0.54	0.88	0.41	0.00	0.66	0.81	0.99	0.79	0.00
0.5	0.01	0.03	0.12	0.51	0.00	0.12	0.17	0.35	0.84	0.00	0.31	0.37	0.62	0.97	0.00
0.7	0.00	0.00	0.01	0.29	0.13	0.08	0.08	0.14	0.61	0.32	0.24	0.24	0.32	0.84	0.55
0.9	0.00	0.00	0.00	0.01	0.89	0.15	0.15	0.15	0.16	1.0	0.37	0.38	0.38	0.38	1.0
								Lobato and Robinson (1998) ARMA(1,1)							
0.0	0.48	0.36	0.08	0.10	0.02	0.81	0.74	0.25	0.42	0.10	0.96	0.93	0.49	0.81	0.26
0.5	0.12	0.12	0.12	0.09	0.02	0.36	0.37	0.35	0.31	0.11	0.63	0.63	0.62	0.58	0.29
0.7	0.02	0.02	0.03	0.04	0.03	0.14	0.16	0.17	0.19	0.13	0.34	0.35	0.37	0.41	0.34
0.9	0.00	0.00	0.00	0.00	0.03	0.15	0.15	0.15	0.15	0.15	0.37	0.38	0.38	0.38	0.35

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TABLE 2B. Empirical power of tests when $d = 0.4$

	$T = 200$					$T = 500$					$T = 1{,}000$				
ρ/θ	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8
								$\hat{S}_k$, $c = 0.5\|1.0\|2.0$							
0.0	0.28|0.19|0.11	0.29|0.19|0.10	0.29|0.18|0.10	0.26|0.18|0.10	0.10|0.08|0.06	0.53|0.40|0.26	0.53|0.39|0.26	0.53|0.39|0.25	0.53|0.39|0.25	0.18|0.14|0.11	0.75|0.59|0.43	0.76|0.58|0.43	0.75|0.58|0.43	0.73|0.58|0.43	0.30|0.24|0.19
0.5	0.32|0.20|0.10	0.31|0.19|0.10	0.30|0.19|0.11	0.31|0.19|0.11	0.17|0.15|0.09	0.55|0.40|0.26	0.55|0.40|0.25	0.54|0.41|0.26	0.54|0.40|0.26	0.37|0.30|0.21	0.76|0.59|0.43	0.76|0.59|0.43	0.76|0.59|0.43	0.76|0.58|0.43	0.62|0.49|0.37
0.7	0.36|0.21|0.11	0.35|0.22|0.11	0.35|0.22|0.11	0.34|0.21|0.11	0.24|0.16|0.09	0.57|0.42|0.27	0.57|0.42|0.26	0.57|0.41|0.26	0.57|0.42|0.26	0.49|0.37|0.25	0.77|0.60|0.44	0.78|0.60|0.44	0.78|0.60|0.43	0.77|0.59|0.43	0.71|0.56|0.40
0.9	0.69|0.43|0.20	0.69|0.43|0.20	0.69|0.42|0.20	0.68|0.43|0.20	0.56|0.37|0.19	0.81|0.61|0.35	0.81|0.60|0.35	0.80|0.60|0.35	0.80|0.60|0.36	0.76|0.55|0.35	0.91|0.71|0.52	0.91|0.71|0.51	0.91|0.71|0.51	0.91|0.71|0.51	0.88|0.69|0.49
								KPSS							
0.0	0.26	0.26	0.26	0.26	0.24	0.37	0.37	0.37	0.37	0.38	0.50	0.50	0.49	0.49	0.52
0.5	0.26	0.26	0.26	0.26	0.24	0.37	0.37	0.37	0.37	0.38	0.50	0.50	0.50	0.49	0.50
0.7	0.27	0.27	0.27	0.27	0.24	0.36	0.37	0.36	0.37	0.36	0.50	0.50	0.50	0.50	0.49
0.9	0.37	0.37	0.37	0.37	0.34	0.44	0.43	0.43	0.44	0.42	0.54	0.54	0.54	0.54	0.53
								Robinson (1994)/Tanaka (1999) AR(1)							
0.0	0.84	0.16	0.45	0.82	0.00	1.0	0.63	0.95	1.0	0.01	1.0	0.94	1.0	1.0	0.02
0.5	1.0	0.99	0.15	0.14	0.13	1.0	1.0	0.27	0.55	0.47	1.0	1.0	0.44	0.88	0.82
0.7	1.0	1.0	0.80	0.00	0.41	1.0	1.0	0.97	0.00	0.91	1.0	1.0	1.0	0.00	1.0
0.9	1.0	1.0	1.0	0.06	0.07	1.0	1.0	1.0	0.03	0.06	1.0	1.0	1.0	0.01	0.04
								Robinson (1994)/Tanaka (1999) ARMA(1,1)							
0.0	0.37	0.16	0.06	0.27	0.17	0.90	0.68	0.39	0.22	0.61	1.0	0.96	0.84	0.07	0.92
0.5	0.12	0.04	0.02	0.03	0.04	0.21	0.11	0.10	0.31	0.23	0.35	0.23	0.28	0.76	0.51
0.7	0.70	0.45	0.16	0.01	0.08	0.92	0.67	0.23	0.04	0.52	0.99	0.86	0.33	0.12	0.91
0.9	0.99	0.98	0.88	0.48	0.05	1.0	1.0	0.99	0.64	0.10	1.0	1.0	1.0	0.80	0.23
								Lobato and Robinson (1998) AR(1)							
0.0	0.10	0.22	0.70	0.78	0.00	0.53	0.70	0.97	1.0	0.04	0.85	0.94	1.0	1.0	0.20
0.5	0.00	0.00	0.03	0.51	0.19	0.25	0.25	0.34	0.91	0.57	0.63	0.63	0.71	0.99	0.87
0.7	0.00	0.00	0.00	0.08	0.59	0.26	0.26	0.26	0.44	0.94	0.63	0.63	0.63	0.79	1.0
0.9	0.00	0.00	0.00	0.00	0.54	0.37	0.37	0.36	0.35	0.92	0.74	0.75	0.75	0.75	0.99
								Lobato and Robinson (1998) ARMA(1,1)							
0.0	0.61	0.39	0.09	0.22	0.11	0.94	0.84	0.43	0.43	0.50	1.0	0.98	0.76	0.66	0.88
0.5	0.03	0.04	0.06	0.08	0.06	0.36	0.39	0.41	0.43	0.36	0.72	0.75	0.77	0.78	0.75
0.7	0.00	0.00	0.002	0.01	0.18	0.26	0.26	0.26	0.27	0.59	0.63	0.63	0.63	0.63	0.89
0.9	0.00	0.00	0.00	0.00	0.01	0.37	0.37	0.36	0.35	0.33	0.74	0.75	0.75	0.75	0.71

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TABLE 2C. Empirical power of tests when $d = 0.6$

	$T = 200$					$T = 500$					$T = 1{,}000$				
ρ/θ	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8
							$\hat{S}_k$, $c = 0.5\|1.0\|2.0$								
0.0	0.56|0.39|0.22	0.56|0.39|0.22	0.55|0.39|0.22	0.55|0.39|0.22	0.32|0.24|0.16	0.84|0.70|0.51	0.83|0.70|0.51	0.83|0.70|0.51	0.83|0.70|0.51	0.66|0.56|0.40	0.96|0.88|0.74	0.96|0.88|0.74	0.97|0.88|0.74	0.96|0.88|0.74	0.90|0.81|0.68
0.5	0.58|0.41|0.24	0.58|0.40|0.23	0.58|0.40|0.23	0.57|0.41|0.23	0.44|0.32|0.20	0.86|0.71|0.53	0.85|0.70|0.52	0.85|0.71|0.52	0.84|0.71|0.51	0.77|0.66|0.49	0.97|0.88|0.74	0.96|0.88|0.74	0.96|0.88|0.74	0.96|0.88|0.74	0.95|0.86|0.73
0.7	0.63|0.42|0.24	0.63|0.42|0.24	0.62|0.43|0.23	0.62|0.42|0.24	0.46|0.36|0.21	0.85|0.71|0.52	0.86|0.72|0.52	0.86|0.72|0.52	0.85|0.72|0.52	0.82|0.68|0.51	0.97|0.89|0.76	0.97|0.89|0.75	0.97|0.89|0.76	0.96|0.88|0.75	0.95|0.87|0.74
0.9	0.85|0.63|0.37	0.85|0.63|0.36	0.85|0.63|0.36	0.85|0.63|0.36	0.77|0.56|0.32	0.95|0.84|0.61	0.95|0.83|0.61	0.95|0.84|0.60	0.94|0.83|0.60	0.93|0.81|0.59	0.99|0.93|0.80	0.99|0.93|0.80	0.99|0.93|0.79	0.99|0.93|0.79	0.98|0.92|0.78
							KPSS								
0.0	0.41	0.41	0.41	0.41	0.40	0.55	0.55	0.55	0.55	0.57	0.72	0.72	0.72	0.72	0.73
0.5	0.41	0.41	0.42	0.41	0.39	0.55	0.55	0.55	0.55	0.55	0.72	0.72	0.72	0.72	0.72
0.7	0.42	0.42	0.42	0.42	0.40	0.55	0.55	0.55	0.55	0.54	0.72	0.72	0.72	0.72	0.72
0.9	0.50	0.50	0.49	0.49	0.47	0.58	0.58	0.58	0.59	0.58	0.75	0.75	0.75	0.75	0.75
							Robinson (1994)/Tanaka (1999) AR(1)								
0.0	0.83	0.18	0.01	0.95	0.32	0.94	0.18	0.01	0.99	0.81	0.98	0.17	0.00	0.99	0.98
0.5	1.0	1.0	0.60	0.00	0.60	1.0	1.0	0.78	0.00	0.98	1.0	1.0	0.90	1.0	1.0
0.7	1.0	1.0	1.0	0.02	0.23	1.0	1.0	1.0	0.01	0.67	1.0	1.0	1.0	0.00	0.83
0.9	1.0	1.0	1.0	0.92	0.00	1.0	1.0	1.0	1.0	0.00	1.0	1.0	1.0	1.0	0.00
							Robinson (1994)/Tanaka (1999) ARMA(1,1)								
0.0	0.01	0.01	0.02	0.01	0.50	0.01	0.01	0.04	0.01	0.94	0.00	0.01	0.07	0.01	0.96
0.5	0.48	0.22	0.05	0.01	0.20	0.62	0.22	0.03	0.01	0.83	0.73	0.22	0.01	0.01	1.0
0.7	0.98	0.91	0.58	0.09	0.07	1.0	0.99	0.76	0.05	0.49	1.0	1.0	0.88	0.02	0.87
0.9	0.98	1.0	1.0	0.90	0.05	1.0	1.0	1.0	1.0	0.01	1.0	1.0	1.0	1.0	0.00
							Lobato and Robinson (1998) AR(1)								
0.0	0.00	0.02	0.33	0.99	0.23	0.49	0.51	0.86	1.0	0.79	0.88	0.89	0.99	1.0	0.99
0.5	0.00	0.00	0.00	0.10	0.75	0.50	0.49	0.48	0.65	0.99	0.89	0.89	0.89	0.94	1.0
0.7	0.00	0.00	0.00	0.00	0.67	0.50	0.50	0.51	0.49	0.99	0.89	0.89	0.89	0.89	1.0
0.9	0.00	0.00	0.00	0.00	0.02	0.60	0.60	0.60	0.59	0.57	0.93	0.93	0.93	0.93	0.92
							Lobato and Robinson (1998) ARMA(1,1)								
0.0	0.24	0.11	0.02	0.00	0.30	0.80	0.63	0.51	0.48	0.88	0.98	0.93	0.89	0.88	1.0
0.5	0.00	0.00	0.00	0.01	0.25	0.50	0.49	0.49	0.51	0.71	0.89	0.89	0.89	0.89	0.94
0.7	0.00	0.00	0.00	0.00	0.13	0.50	0.50	0.51	0.49	0.63	0.89	0.89	0.89	0.89	0.93
0.9	0.00	0.00	0.00	0.00	0.00	0.59	0.60	0.60	0.59	0.57	0.93	0.93	0.93	0.93	0.92

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TABLE 2D. Empirical power of tests when $d = 0.8$

	$T = 200$					$T = 500$					$T = 1{,}000$				
ρ/θ	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8
							$\hat{S}_k$, $c = 0.5\|1.0\|2.0$								
0.0	0.79|0.62|0.40	0.79|0.62|0.40	0.78|0.61|0.40	0.78|0.61|0.39	0.66|0.53|0.35	0.96|0.89|0.75	0.96|0.90|0.75	0.96|0.90|0.75	0.96|0.90|0.75	0.93|0.87|0.72	1.0 |0.98|0.93	1.0 |0.98|0.93	1.0 |0.98|0.92	1.0 |0.98|0.92	1.0 |0.97|0.91
0.5	0.80|0.63|0.41	0.80|0.62|0.41	0.79|0.62|0.40	0.79|0.62|0.40	0.70|0.54|0.37	0.96|0.89|0.75	0.96|0.90|0.74	0.96|0.90|0.74	0.96|0.90|0.73	0.95|0.88|0.71	1.0 |0.98|0.92	1.0 |0.98|0.92	1.0 |0.98|0.92	1.0 |0.98|0.93	1.0 |0.98|0.91
0.7	0.84|0.66|0.43	0.84|0.65|0.42	0.83|0.65|0.42	0.83|0.65|0.42	0.73|0.58|0.37	0.97|0.90|0.74	0.97|0.90|0.75	0.97|0.90|0.75	0.96|0.90|0.75	0.95|0.88|0.73	1.0 |0.98|0.92	1.0 |0.98|0.92	1.0 |0.98|0.92	1.0 |0.98|0.92	1.0 |0.98|0.92
0.9	0.94|0.80|0.53	0.93|0.80|0.53	0.93|0.79|0.53	0.93|0.79|0.53	0.89|0.74|0.49	0.99|0.95|0.81	0.99|0.95|0.81	0.99|0.95|0.82	0.99|0.95|0.81	0.98|0.94|0.79	1.0 |0.99|0.94	1.0 |0.99|0.94	1.0 |0.99|0.94	1.0 |0.99|0.94	1.0 |0.99|0.94
							KPSS								
0.0	0.55	0.55	0.55	0.54	0.54	0.72	0.72	0.72	0.72	0.72	0.84	0.84	0.84	0.85	0.85
0.5	0.55	0.55	0.55	0.55	0.53	0.72	0.72	0.72	0.72	0.72	0.84	0.84	0.84	0.84	0.85
0.7	0.56	0.56	0.56	0.56	0.53	0.72	0.72	0.72	0.72	0.71	0.85	0.85	0.85	0.85	0.85
0.9	0.61	0.61	0.61	0.60	0.60	0.75	0.75	0.75	0.75	0.74	0.86	0.86	0.86	0.86	0.85
							Robinson (1994)/Tanaka (1999) AR(1)								
0.0	0.96	0.52	0.00	0.10	0.89	1.0	0.65	0.00	0.03	1.0	1.0	0.78	0.00	0.00	1.0
0.5	1.0	1.0	0.99	0.01	0.29	1.0	1.0	1.0	0.00	0.41	1.0	1.0	1.0	0.00	0.30
0.7	1.0	1.0	1.0	0.71	0.00	1.0	1.0	1.0	0.90	0.00	1.0	1.0	1.0	0.99	0.00
0.9	1.0	1.0	1.0	1.0	0.02	1.0	1.0	1.0	1.0	0.00	1.0	1.0	1.0	1.0	0.00
							Robinson (1994)/Tanaka (1999) ARMA(1,1)								
0.0	0.00	0.00	0.01	0.05	0.23	0.00	0.00	0.00	0.03	0.07	0.00	0.00	0.00	0.01	0.00
0.5	0.96	0.78	0.33	0.03	0.13	1.0	0.94	0.36	0.00	0.39	1.0	0.99	0.39	0.00	0.48
0.7	0.98	1.0	0.97	0.45	0.01	1.0	1.0	1.0	0.55	0.00	1.0	1.0	1.0	0.68	0.00
0.9	0.94	1.0	1.0	1.0	0.24	1.0	1.0	1.0	1.0	0.24	1.0	1.0	1.0	1.0	0.25
							Lobato and Robinson (1998) AR(1)								
0.0	0.00	0.00	0.01	0.58	0.85	0.71	0.70	0.70	0.96	1.0	0.97	0.97	0.97	1.0	1.0
0.5	0.00	0.00	0.00	0.00	0.78	0.71	0.71	0.71	0.71	1.0	0.98	0.98	0.98	0.98	1.0
0.7	0.00	0.00	0.00	0.00	0.12	0.71	0.71	0.71	0.72	0.78	0.98	0.98	0.98	0.98	0.99
0.9	0.00	0.00	0.00	0.00	0.00	0.77	0.77	0.76	0.77	0.75	0.99	0.99	0.99	0.99	0.98
							Lobato and Robinson (1998) ARMA(1,1)								
0.0	0.00	0.00	0.00	0.00	0.19	0.71	0.70	0.70	0.69	0.73	0.97	0.97	0.97	0.98	0.97
0.5	0.00	0.00	0.00	0.00	0.14	0.71	0.71	0.71	0.71	0.76	0.98	0.98	0.98	0.98	0.98
0.7	0.00	0.00	0.00	0.00	0.02	0.71	0.71	0.71	0.721	0.70	0.98	0.98	0.98	0.98	0.97
0.9	0.00	0.00	0.00	0.00	0.00	0.77	0.77	0.76	0.77	0.75	0.99	0.99	0.99	0.99	0.98

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TABLE 2E. Empirical power of tests when $d = 1.0$

	$T = 200$					$T = 500$					$T = 1{,}000$				
ρ/θ	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8	−0.8	−0.4	0.0	0.4	0.8
								$\hat{S}_k$, $c = 0.5\|1.0\|2.0$							
0.0	0.92|0.80|0.61	0.92|0.79|0.60	0.92|0.79|0.60	0.91|0.79|0.60	0.86|.75|54	1.0 |0.97|0.89	1.0 |0.97|0.89	0.99|0.97|0.89	0.99|0.97|0.89	0.99|0.97|0.88	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.97
0.5	0.92|0.81|0.60	0.92|0.80|0.60	0.91|0.80|0.60	0.91|0.79|0.59	0.85|0.76|0.57	1.0 |0.98|0.89	0.99|0.98|0.89	1.0 |0.98|0.89	1.0 |0.97|0.89	0.99|0.97|0.88	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98
0.7	0.93|0.81|0.61	0.93|0.81|0.60	0.92|0.81|0.60	0.92|0.81|0.60	0.89|0.77|0.57	1.0 |0.97|0.90	1.0 |0.97|0.90	1.0 |0.97|0.89	0.99|0.97|0.89	0.99|0.97|0.88	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98	1.0 |1.0|0.98
0.9	0.97|0.89|0.69	0.97|0.89|0.68	0.97|0.89|0.68	0.96|0.89|0.68	0.96|0.86|0.66	1.0 |0.99|0.92	1.0 |0.99|0.92	1.0 |0.99|0.92	1.0 |0.98|0.91	0.99|0.98|0.91	1.0 |1.0|0.99	1.0 |1.0|0.99	1.0 |1.0|0.99	1.0 |1.0|0.98	1.0 |1.0|0.98
								KPSS							
0.0	0.68	0.68	0.68	0.68	0.66	0.84	0.84	0.84	0.84	0.84	0.94	0.94	0.94	0.94	0.94
0.5	0.68	0.68	0.68	0.68	0.67	0.84	0.84	0.84	0.84	0.84	0.95	0.95	0.94	0.94	0.94
0.7	0.69	0.68	0.68	0.69	0.67	0.84	0.84	0.84	0.84	0.84	0.95	0.95	0.94	0.94	0.94
0.9	0.71	0.71	0.71	0.71	0.70	0.86	0.85	0.86	0.85	0.88	0.95	0.95	0.95	0.95	0.95
								Robinson (1994)/Tanaka (1999) AR(1)							
0.0	1.0	1.0	0.13	0.00	0.75	1.0	1.0	0.11	0.00	0.48	1.0	1.0	0.10	0.00	0.24
0.5	1.0	1.0	1.0	0.64	0.00	1.0	1.0	1.0	0.85	0.00	1.0	1.0	1.0	0.97	0.00
0.7	1.0	1.0	1.0	1.0	0.00	1.0	1.0	1.0	1.0	0.00	1.0	1.0	1.0	1.0	0.00
0.9	1.0	1.0	1.0	1.0	0.90	1.0	1.0	1.0	1.0	0.99	1.0	1.0	1.0	1.0	1.0
								Robinson (1994)/Tanaka (1999) ARMA(1,1)							
0.0	0.14	0.14	0.17	0.23	0.04	0.11	0.13	0.13	0.19	0.08	0.10	0.10	0.12	0.17	0.09
0.5	0.98	1.0	0.95	0.37	0.01	1.0	1.0	1.0	0.47	0.00	1.0	1.0	1.0	0.61	0.00
0.7	0.95	1.0	1.0	0.96	0.01	1.0	1.0	1.0	1.0	0.00	1.0	1.0	1.0	1.0	0.00
0.9	0.79	1.0	1.0	1.0	0.86	0.98	1.0	1.0	1.0	0.99	1.0	1.0	1.0	1.0	1.0
								Lobato and Robinson (1998) AR(1)							
0.0	0.00	0.00	0.00	0.01	0.95	0.84	0.85	0.84	0.85	1.0	1.0	1.0	1.0	0.99	1.0
0.5	0.00	0.00	0.00	0.00	0.09	0.84	0.83	0.84	0.84	0.86	1.0	0.99	1.0	0.99	1.0
0.7	0.00	0.00	0.00	0.00	0.00	0.85	0.84	0.85	0.85	0.84	1.0	1.0	0.99	0.99	1.0
0.9	0.00	0.00	0.00	0.00	0.00	0.88	0.88	0.86	0.88	0.87	1.0	1.0	1.0	1.0	1.0
								Lobato and Robinson (1998) ARMA(1,1)							
0.0	0.00	0.00	0.00	0.00	0.00	0.84	0.83	0.85	0.84	0.84	1.0	1.0	1.0	1.0	0.99
0.5	0.00	0.00	0.00	0.00	0.02	0.84	0.85	0.83	0.84	0.83	1.0	1.0	1.0	1.0	1.0
0.7	0.00	0.00	0.00	0.00	0.00	0.86	0.85	0.85	0.85	0.83	0.99	1.0	1.0	1.0	1.0
0.9	0.03	0.00	0.00	0.00	0.00	0.88	0.88	0.88	0.88	0.87	1.0	1.0	1.0	1.0	1.0

tests need to be made keeping their relative size properties firmly in mind. It is clear that, for all c, the $\hat{S}_k$ test has power that increases with both d and T. For a given d and T, these powers also decline monotonically with c because the positive offset of the statistic declines as c increases. Our opinion is that the potential problems of controlling size when $c = 0.5$ are probably sufficient to outweigh its power advantages over $c = 1.0$, whereas we feel that the reverse is true on comparing $c = 1.0$ and $c = 2.0$. This leaves $c = 1.0$ as our compromise choice on which we base subsequent comparisons with the other tests.

The power of $\hat{S}_k$ ($c = 1.0$) is greater than that of KPSS almost everywhere when d is in the nonstationary region (for stationary values of d KPSS has the greater power unless the sample size is large). Hence, bearing in mind its superior control of size, $\hat{S}_k$ ($c = 1.0$) would appear to provide a very worthy competitor to KPSS as a test of short memory against nonstationary long memory and particularly in the $I(1)$ case.

As might be expected, the powers of the Robinson (1994)/Tanaka (1999) tests are usually very good when the model for u_t is specified correctly, although there are some curious exceptions in some parts of the ARMA parameter space for nonstationary values of d where the powers are extremely low. In fact, this test does not appear to be consistent for nonstationary d when u_t is noise. We conjecture that, in such situations, prewhitening via AR/ARMA estimation is largely extracting the effects of nonstationary d (this certainly explains the results for $d = 1$). The Lobato and Robinson (1998) test using the ARMA(1,1) approximating model exhibits very good power properties for the largest sample size, $T = 1{,}000$, but is generally less powerful than $\hat{S}_k$ ($c = 1.0$) for the other sample sizes and is profoundly so when $T = 200$ and d is nonstationary.

In the case where (1) contains a linear trend term, simulations (not reported but available from the authors) show that the good size control properties of the $\hat{S}_k$ ($c = 1.0$) test continue to hold but with the absence of the modest oversizing seen previously. The size properties of the KPSS test also generally mimic those of the constant only case with the exception that a large positive MA component leads to much more serious oversizing in smaller samples. For the other four tests, their patterns of size largely carry over from the constant case but with almost every size being slightly smaller than before. The test powers are, overall, lower than in the constant case except for KPSS, which appears less affected by the presence of the trend.

In summary, then, we conclude from these simulations that $\hat{S}_k$ ($c = 1.0$) exhibits very good size control across a range of DGPs and sample sizes, to an extent not matched by the other tests in the comparison. For tests that rely on prewhitening, knowledge of the correct parametric model appears essential to avoid potentially severe size distortions. Similarly, bandwidth selection in the periodogram-based tests requires a good approximating parametric model to avoid finite-sample size distortion. Significantly, the size control of the new test does not appear to be achieved at the expense of being competitive in terms

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of power, particularly when nonstationary long memory alternatives are under consideration.

NOTES

1. We use the notation $j \vee 0 = \max(j,0)$ and $j \wedge 0 = \min(j,0)$.
2. Our results did not appear particularly sensitive to the specific choices made for these supplementary parameters.
3. We also considered estimating the long-run variance using the recently proposed prewhitening-based method of Sul, Phillips, and Choi (2005). Although this procedure arguably displayed slightly better size control for a pure AR(1) DGP, we found that its degree of size control given an additional MA(1) component generally fell well below that obtainable from the Hobijn et al. (2004) procedure, in terms of both under- and oversizing.

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APPENDIX: Proofs

A.1. Proof of Theorem 1. Similar to the definition of $\hat{S}_k^*$, we define a statistic $\tilde{S}_k^*$ based on the true z_t, rather than $\hat{z}_t$, as

$$\tilde{S}_k^* = \frac{T-k}{T} \frac{\tilde{N}_k}{\tilde{\omega}_l},$$

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where

$$\tilde{N}_k = T(T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k}\tilde{\gamma}_j, \qquad \tilde{\omega}_l^2 = \sum_{j=-l}^{l} h_j \sum_{i=-l}^{l} \tilde{\gamma}_i \tilde{\gamma}_{i+j}, \qquad \tilde{\gamma}_j = T^{-1} \sum_{t=(j\vee 0)+1}^{T-(j\wedge 0)} z_t z_{t-j},$$

and $w_j = 1/(j+1)$ is defined for notational convenience. The proof uses $(T-k)/T \to 1$ and the next four steps, which are proved in the following sequence of lemmas.

L1. $\tilde{N}_k - E(\tilde{N}_k) \xrightarrow{d} N(0,\omega^2)$ where $\omega^2 = \sum_{i=-\infty}^{\infty}(\sum_{h=0}^{\infty} w_h \gamma_{h+i})^2 < \infty$,
L2. $E(\tilde{N}_k) \to 0$,
L3. $\tilde{\omega}_l^2 \xrightarrow{p} \omega^2$,
L4. $\hat{N}_k - \tilde{N}_k \xrightarrow{p} 0$ and $\hat{\omega}_l^2 - \tilde{\omega}_l^2 \xrightarrow{p} 0$.

The result of Theorem 1 then follows by the continuous mapping theorem (CMT).

LEMMA 1. *Under the conditions of Theorem 1,* $\tilde{N}_k - E(\tilde{N}_k) \xrightarrow{d} N(0,\omega^2)$.

Proof. Define the artificial autocovariances $\bar{\gamma}_j = \sum_{i=j\vee 0}^{\infty} |c_i||c_{i-j}|$, which clearly exist and are absolutely summable under the condition $\sum_{j=0}^{\infty}|c_j| < \infty$. These $\bar{\gamma}_j$ provide useful bounds throughout the proof. Lemma 5 shows that the claimed expression for ω^2 exists.

First note that $\tilde{N}_k$ can be rewritten as

$$\tilde{N}_k = (T-k)^{-1/2} \sum_{t=k+1}^{T} z_t v_{t-k},$$

where $v_t = \sum_{i=0}^{\infty} f_{t,i}\varepsilon_{t-i}$ and $f_{t,i} = \sum_{h=0}^{i\wedge(t-1)} w_h c_{i-h}$. In this notation

$$E(\tilde{N}_k) = (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k}(T-j)\gamma_j$$

$$= (T-k)^{-1/2} \sum_{t=k+1}^{T} \sum_{i=k}^{\infty} c_i f_{t-k,i-k}.$$

Substituting these expressions and rearranging gives

$$\tilde{N}_k - E(\tilde{N}_k) = (T-k)^{-1/2} \sum_{t=k+1}^{T} (s_{k,t} + r_{k,t-k}),$$

where

$$s_{k,t} = \sum_{i=1}^{\infty} g_{k,t,k-i}(L)\varepsilon_t \varepsilon_{t-i}, \qquad r_{k,t} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{j+k} f_{t,i}(\varepsilon_{t-i}\varepsilon_{t-j} - \delta_{i,j}), \tag{A.1}$$

$$g_{k,t,k-i}(L) = \sum_{j=(k-i)\vee 0}^{k-1} c_j f_{t-k,i-k+j} L^j,$$

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and $\delta_{i,j} = 1$ for $i = j$ and is zero otherwise. Applying the Beveridge–Nelson decomposition to $g_{k,t,k-i}(L)$ gives the martingale difference

$$m_{k,t} = \varepsilon_t \zeta_{k,t-1}, \qquad \zeta_{k,t-1} = \sum_{i=1}^{\infty} g_{k,t,k-i}\, \varepsilon_{t-i},$$

where $g_{k,t,k-i} = g_{k,t,k-i}(1)$. The existence of the $\zeta_{k,t-1}$ process as a mean square limit follows from Lemma 6(i) with $s = 0$. The central limit theorem (CLT) is proved by showing

(a) $(T-k)^{-1/2} \sum_{t=k+1}^{T} m_{k,t} \xrightarrow{d} N(0, \omega^2)$,
(b) $(T-k)^{-1/2} \sum_{t=k+1}^{T} (s_{k,t} - m_{k,t}) \xrightarrow{p} 0$,
(c) $(T-k)^{-1/2} \sum_{t=k+1}^{T} r_{k,t-k} \xrightarrow{p} 0$.

These results combine to give Lemma 1.

For (a) the martingale difference CLT is proved by showing

(i) $E[(T-k)^{-1} \sum_{t=k+1}^{T} m_{k,t}^2] \to \omega^2$, $(T-k)^{-1} \sum_{t=k+1}^{T} m_{k,t}^2 \xrightarrow{p} \omega^2$,
(ii) $\max_{1 \le t \le T} |T^{-1/2} m_{k,t}| \xrightarrow{p} 0$.

Part (i) is shown by Lemma 6(iii) in Appendix A.3 (Technical Appendix),

$$E\left[(T-k)^{-1} \sum_{t=k+1}^{T} m_{k,t}^2\right] = (T-k)^{-1} \sum_{t=k+1}^{T} \left(\sum_{i=1}^{\infty} g_{k,t,k-i}^2\right) \to \omega^2.$$

To also show $(T-k)^{-1} \sum_{t=k+1}^{T} m_{k,t}^2 \xrightarrow{p} \omega^2$, use $m_{k,t} = \varepsilon_t \zeta_{k,t-1}$ to write

$$(T-k)^{-1} \sum_{t=k+1}^{T} m_{k,t}^2 = (T-k)^{-1} \sum_{t=k+1}^{T} (\varepsilon_t^2 - 1)\zeta_{k,t-1}^2 + (T-k)^{-1} \sum_{t=k+1}^{T} \zeta_{k,t-1}^2. \qquad \textbf{(A.2)}$$

For any $s \ge 0$

$$\begin{aligned} \operatorname{cov}(\zeta_{k,t}^2, \zeta_{k,t-s}^2) &= 2E(\zeta_{k,t}\zeta_{k,t-s})^2 + \kappa_4(\zeta_{k,t}, \zeta_{k,t}, \zeta_{k,t-s}, \zeta_{k,t-s}) \\ &= 2\left(\sum_{i=s+1}^{\infty} g_{k,t,k-i}\, g_{k,t-s,k-i+s}\right)^2 + \kappa_4(\varepsilon_t) \sum_{i=s+1}^{\infty} g_{k,t,k-i}^2 g_{k,t-s,k-i-s}^2, \qquad \textbf{(A.3)} \end{aligned}$$

where $\kappa_4(.)$ denotes the (finite) fourth-order cumulant (see Hannan, 1970, Ch. 1, eqn. (5.1)), and Lemma 6(i) and (ii) provide the convergence of these sums. The martingale difference array $(\varepsilon_t^2 - 1)\zeta_{k,t-1}^2$ therefore has uniformly bounded variance ($s = 0$ in (A.3)), and hence the first term of (A.2) converges to zero in probability. Using (A.3), the variance of the second term of (A.2) can be written

$$\begin{aligned} &\operatorname{var}\left[(T-k)^{-1} \sum_{t=k+1}^{T} \zeta_{k,t-1}^2\right] \\ &\quad \le 2(T-k)^{-2} \sum_{s=0}^{T-k-1} \sum_{t=k+s+1}^{T} \operatorname{cov}(\zeta_{k,t}^2, \zeta_{k,t-s}^2) \\ &\quad = 2(T-k)^{-2} \sum_{s=0}^{T-k-1} \sum_{t=k+s+1}^{T} \left(\sum_{i=s+1}^{\infty} g_{k,t,k-i}\, g_{k,t-s,k-i+s}\right)^2 \qquad \textbf{(A.4)} \\ &\qquad + 2\kappa_4(\varepsilon_t)(T-k)^{-2} \sum_{s=0}^{T-k-1} \sum_{t=k+s+1}^{T} \sum_{i=s+1}^{\infty} g_{k,t,k-i}^2 g_{k,t-s,k-i-s}^2. \qquad \textbf{(A.5)} \end{aligned}$$

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Using the Cauchy–Schwarz (CS) inequality in (A.4) and then applying Lemma 6(i) and (ii) shows that (A.4) is bounded by

$$2\bar{\omega}^2\left[(T-k)^{-1}\cdot k\bar{\omega}^2+(T-k)^{-1}\sum_{s=1}^{T-k}\sum_{i=s}^{\infty}\left(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h-i}\right)^2\right].$$

Now $(T-k)^{-1}k\to 0$, and the Cesaro sum disappears because $\sum_{i=s}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h-i})^2\to 0$ as $s\to\infty$ by Lemma 5, and so it follows that (A.4) disappears as $T\to\infty$. Rearranging sums in (A.5) gives

$$(T-k)^{-2}\sum_{t=1}^{T-k}\sum_{i=1}^{\infty}\left(\sum_{s=i}^{t+i-1}g_{k,t+k,k-s}^2\right)g_{k,t,k-i}^2$$

$$\le (T-k)^{-2}\sum_{t=1}^{T-k}\left(\sum_{s=1}^{\infty}g_{k,t+k,k-s}^2\right)\left(\sum_{i=1}^{\infty}g_{k,t,k-i}^2\right),$$

and this disappears by Lemma 6(i). This shows that $T^{-1}\sum_{t=k+1}^{T}m_{k,t}^2\xrightarrow{p}\omega^2$ as required.

For part (ii) it can be shown that for any $\varepsilon>0$

$$\Pr\left(\max_{1\le t\le T}|T^{-1/2}m_{k,t}|>\varepsilon\right)=\Pr\left(T^{-1}\max_{1\le t\le T}m_{k,t}^2>\varepsilon^2\right)$$

$$\le\sum_{t=1}^{T}\Pr(T^{-1}m_{k,t}^2>\varepsilon^2)$$

$$\le\frac{T^{-1}E(m_{k,t}^4)}{\varepsilon^4},$$

and the preceding arguments for (i) show that $E(m_{k,t}^4)$ is uniformly bounded for all k and t. Thus $\max_{1\le t\le T}|T^{-1/2}m_{k,t}|\xrightarrow{p}0$ as required.

We prove (b) by using

$$s_{k,t}-m_{k,t}=\sum_{i=1}^{\infty}\sum_{j=(k-i)\vee 1}^{k-1}c_jf_{t-k,i-k+j}(\varepsilon_{t-j}\varepsilon_{t-i-j}-\varepsilon_t\varepsilon_{t-i}),$$

which gives

$$(T-k)^{-1/2}\sum_{t=k+1}^{T}(s_{k,t}-m_{k,t})$$

$$=(T-k)^{-1/2}\sum_{i=1}^{\infty}\sum_{j=(k-i)\vee 1}^{k-1}\sum_{t=k-j+1}^{k}c_jf_{t+j-k,i-k+j}\varepsilon_t\varepsilon_{t-i} \tag{A.6}$$

$$+(T-k)^{-1/2}\sum_{i=1}^{\infty}\sum_{j=(k-i)\vee 1}^{k-1}\sum_{t=k+1}^{T-j}c_j(f_{t-k+j,i-k+j}-f_{t-k,i-k+j})\varepsilon_t\varepsilon_{t-i} \tag{A.7}$$

$$+(T-k)^{1/2}\sum_{i=1}^{\infty}\sum_{j=(k-i)\vee 1}^{k-1}\sum_{t=T-j+1}^{T}c_jf_{t-k,i-k+j}\varepsilon_t\varepsilon_{t-i}. \tag{A.8}$$

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Clearly all terms have zero means, and so it is sufficient to show that their variances disappear. First (A.6) can be written

$$(T-k)^{-1/2}\sum_{i=1}^{\infty}\sum_{t=2}^{k}\left(\sum_{j=(k-i)\vee(k+1-t)}^{k-1}c_j f_{t+j-k,i-k+j}\right)\varepsilon_t\varepsilon_{t-i},$$

which has variance

$$\begin{aligned}
&(T-k)^{-1}\sum_{i=1}^{\infty}\sum_{t=2}^{k}\left(\sum_{j=(k-i)\vee(k+1-t)}^{k-1}c_j f_{t+j-k,i-k+j}\right)^2\\
&\quad=(T-k)^{-1}\sum_{i=1}^{\infty}\sum_{t=2}^{k}\left(\sum_{j=(k-i)\vee(k+1-t)}^{k-1}c_j\sum_{h=0}^{(t+j-k-1)\wedge(i-k+j)}w_h c_{i-k+j-h}\right)^2\\
&\quad\le\frac{k}{T-k}\sum_{i=1}^{\infty}\left(\sum_{j=(k-i)\vee 1}^{k-1}\sum_{h=0}^{j\wedge(i-k+j)}|w_h||c_j||c_{j-k+i-h}|\right)^2\\
&\quad=\frac{k}{T-k}\sum_{i=-\infty}^{k-1}\left(\sum_{h=0}^{k-1-(i\vee 0)}|w_h|\sum_{j=h+(i\vee 0)}^{k-1}|c_j||c_{j-i-h}|\right)^2\\
&\quad\le\frac{k}{T-k}\sum_{i=-\infty}^{\infty}\left(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h+i}\right)^2,
\end{aligned}$$

and this disappears using $k/T\to 0$ and Lemma 5. Similarly (A.8) can be written

$$(T-k)^{-1/2}\sum_{i=1}^{\infty}\sum_{t=T-k+2}^{T}\left(\sum_{j=(k-i)\vee 1}^{T-t+1}c_j f_{t-k,i-k+j}\right)\varepsilon_t\varepsilon_{t-i},$$

which disappears in exactly the same way. Rearranging sums in (A.7) gives

$$(T-k)^{-1/2}\sum_{i=0}^{\infty}\sum_{t=1}^{T-k-1}\sum_{j=1}^{(k-1)\wedge(T-k-t)}c_j(f_{t+j,i}-f_{t,i})\varepsilon_{t+k}\varepsilon_{t-i+j},$$

which has variance

$$(T-k)^{-1}\sum_{i=0}^{\infty}\sum_{t=1}^{T-k-1}\left(\sum_{j=1}^{(k-1)\wedge(T-k-t)}c_j(f_{t+j,i}-f_{t,i})\right)^2. \tag{A.9}$$

Using

$$f_{t+j,i}-f_{t,i}=\begin{cases}\sum_{h=t}^{t+j-1}w_h c_{i-h}, & t\le i-j,\\ \sum_{h=t}^{i}w_h c_{i-h}, & i-j+1\le t\le i,\\ 0, & t>i,\end{cases}$$

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(A.9) can be bounded by

$$(T-k)^{-1}\sum_{t=1}^{T-k-1}\sum_{i=t+1}^{\infty}\left(\sum_{j=1}^{i-t}|c_j|\sum_{h=t}^{t+j-1}|w_h||c_{i-h}|\right)^2$$
$$+(T-k)^{-1}\sum_{t=1}^{T-k-1}\sum_{i=t+1}^{t+k-2}\left(\sum_{j=i-t+1}^{k-1}|c_j|\sum_{h=t}^{i}|w_h||c_{i-h}|\right)^2$$
$$\le 2\left(\sum_{j=1}^{\infty}|c_j|\right)^2(T-k)^{-1}\sum_{t=1}^{T-k-1}\sum_{i=t+1}^{\infty}\left(\sum_{h=t}^{i}|w_h||c_{i-h}|\right)^2,$$

which disappears by Cesaro summation because

$$\sum_{i=t+1}^{\infty}\left(\sum_{h=t}^{i}|w_h||c_{i-h}|\right)^2=\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}|w_{h+t}||w_{j+t}|\bar{\gamma}_{j-h}$$
$$\le 2\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}|w_{h+t}||w_{h+j+t}|\bar{\gamma}_j$$
$$\le 2\left(\sum_{h=t}^{\infty}w_h^2\right)\sum_{j=0}^{\infty}\bar{\gamma}_j\to 0$$

as $t\to\infty$.

To prove (c) we consider

$$E\left[\left((T-k)^{-1/2}\sum_{t=1}^{T-k}r_{k,t}\right)^2\right]\le 2(T-k)^{-1}\sum_{s=0}^{T-k-1}\sum_{t=s+1}^{T-k}|E(r_{k,t}r_{k,t-s})|, \tag{A.10}$$

where for $s\ge 0$

$$|E(r_{k,t}r_{k,t-s})|\le a\sum_{j=s}^{\infty}\sum_{i=s}^{\infty}(|c_{j+k}||c_{j-s+k}||f_{t,i}||f_{t-s,i-s}|+|c_{j+k}||c_{i-s+k}||f_{t,i}||f_{t-s,j-s}|)$$
$$\le a\sum_{j=s}^{\infty}|c_{j+k}||c_{j-s+k}|\cdot\sum_{i=s}^{\infty}\left(\sum_{h=0}^{i}|w_h||c_{i-h}|\right)\left(\sum_{g=0}^{i-s}|w_h||c_{i-s-g}|\right)$$
$$+a\sum_{j=s}^{\infty}|c_{j+k}|\left(\sum_{h=0}^{j-s}w_hc_{j-s-h}\right)\cdot\sum_{i=s}^{\infty}|c_{i-s+k}|\left(\sum_{g=0}^{i}|w_g||c_{i-g}|\right)$$
$$\le a\sum_{j=0}^{\infty}|c_{j+k}||c_{j+s+k}|\cdot\sum_{g=s}^{\infty}\sum_{h=0}^{\infty}|w_h||w_{g-s}|\bar{\gamma}_{h-g}$$
$$+a\left(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h+k+s}\right)\left(\sum_{g=0}^{\infty}|w_g|\bar{\gamma}_{g+k-s}\right)$$
$$\le a\sum_{j=0}^{\infty}|c_{j+k}||c_{j+s+k}|\cdot\sum_{h=-\infty}^{\infty}\sum_{g=0\vee(-h)}^{\infty}|w_g||w_{g+h}|\bar{\gamma}_{h-s} \tag{A.11}$$
$$+a\sum_{h=0}^{\infty}\sum_{g=h}^{\infty}|w_h||w_{g-h}|\bar{\gamma}_{s+h+k}\bar{\gamma}_{s+h-g-k}, \tag{A.12}$$

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where $a = \max(1, m_4)$ and using $|f_{t,i}| \leq \sum_{h=0}^{i} |w_h||c_{i-h}|$ for all t. Substituting (A.11) and (A.12) into (A.10) and rearranging summations gives a bound for (A.10):

$$a \sum_{j=k}^{\infty} |c_j| \sum_{s=j}^{\infty} |c_s| \cdot \left(\sum_{g=0}^{\infty} w_g^2 \right) \left(\sum_{h=-\infty}^{\infty} \bar{\gamma}_h \right) + a \sum_{g=0}^{\infty} \sum_{h=0}^{g} |w_h||w_{g-h}| \sum_{s=0}^{\infty} \bar{\gamma}_{s+k} \bar{\gamma}_{s-g-k}$$

$$\leq a \left(\sum_{j=k}^{\infty} |c_j| \right)^2 \left(\sum_{g=0}^{\infty} w_g^2 \right) \left(\sum_{h=-\infty}^{\infty} \bar{\gamma}_h \right) + a \left(\sum_{h=0}^{\infty} w_h^2 \right) \left(\sum_{s=k}^{\infty} \bar{\gamma}_s \right) \left(\sum_{g=-\infty}^{\infty} \bar{\gamma}_g \right),$$

which disappears as $T \to \infty$, and hence $k \to \infty$, by the summability of $|c_j|$ and $\bar{\gamma}_s$.

LEMMA 2. *Under the conditions of Theorem 1,* $E(\tilde{N}_k) \to 0$.

Proof. We have

$$E(\tilde{N}_k) = (T-k)^{1/2} \sum_{j=k}^{T-1} w_{j-k} \left(1 - \frac{j}{T} \right) \gamma_j.$$

Under $H_0 \colon \sum_{j=0}^{\infty} |\gamma_j| < \infty$ we want to choose k so that $E(\tilde{N}_k) \to 0$. Now

$$|E(\tilde{N}_k)| \leq (T-k)^{1/2} \sum_{j=k}^{T-1} \frac{1}{j-k+1} |\gamma_j|$$

and

$$\sum_{k=1}^{T-1} \sum_{j=k}^{T-1} \frac{1}{j-k+1} |\gamma_j| = \sum_{j=1}^{T-1} \left(\sum_{k=1}^{j} \frac{1}{k} \right) |\gamma_j| \leq \sum_{j=1}^{T-1} (\ln j + 2\gamma) |\gamma_j|,$$

where γ is Euler's constant. Because $\sum_{j=1}^{\infty} \ln j \cdot |\gamma_j| < \infty$ we thus have $\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} (1/(j-k+1)) |\gamma_j| < \infty$ and hence $k \sum_{j=k}^{\infty} (1/(j-k+1)) |\gamma_j| \to 0$, so that k must grow such that $k/(T-k)^{1/2} \not\to 0$, i.e., k is at least $O(T^{1/2})$. ■

LEMMA 3. *Under the conditions of Theorem 1,* $\tilde{\omega}_l^2 \xrightarrow{p} \omega^2$.

Proof. The limiting variance has the form

$$\omega^2 = \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{i=j \vee h}^{\infty} w_{i-h} w_{i-j} \gamma_h \gamma_j$$

$$= \left(\sum_{i=0}^{\infty} w_i^2 \right) \sum_{h=-\infty}^{\infty} \gamma_h^2 + 2 \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} w_i w_{i-j} \right) \sum_{h=-\infty}^{\infty} \gamma_h \gamma_{j+h}$$

$$= \frac{\pi^2}{6} \sum_{h=-\infty}^{\infty} \gamma_h^2 + 2 \sum_{j=1}^{\infty} \frac{H_j}{j} \sum_{h=-\infty}^{\infty} \gamma_h \gamma_{j+h}$$

because for $j = 1, 2, \ldots$, $\sum_{i=j}^{\infty} w_i w_{i-j} = \sum_{i=1}^{\infty} i^{-1} (i+j)^{-1} = j^{-1} \sum_{i=1}^{j} i^{-1} = H_j / j$ where H_j is the jth harmonic number. So

$$\omega^2 = \sum_{j=-\infty}^{\infty} h_j \sum_{h=-\infty}^{\infty} \gamma_h \gamma_{j+h},$$

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where $h_0 = \pi^2/6$ and $h_j = |j|^{-1}H_{|j|}$ for $j = \pm 1, \pm 2, \ldots$. Note that $\sum_{j=0}^{\infty} h_j^2 < \infty$ and $\sum_{j=0}^{l} h_j = O((\ln l)^2)$. The estimator based on z_t is

$$\tilde{\omega}_l^2 = \sum_{j=-l}^{l} h_j \sum_{h=-l}^{l} \tilde{\gamma}_h \tilde{\gamma}_{h+j},$$

and we define

$$\omega_l^2 = \sum_{j=-l}^{l} h_j \sum_{h=-l}^{l} \gamma_h \gamma_{h+j}.$$

We note that

$$E|\tilde{\omega}_l^2 - \omega^2| \le E|\tilde{\omega}_l^2 - \omega_l^2| + |\omega_l^2 - \omega^2|,$$

and we show that each of these terms disappears asymptotically. First

$$\begin{aligned} E|\tilde{\omega}_l^2 - \omega_l^2| &= E\left| \sum_{j=-l}^{l} h_j \sum_{h=-l}^{l} ((\tilde{\gamma}_{h+j} - \gamma_{h+j})(\tilde{\gamma}_h - \gamma_h) \right. \\ &\qquad \left. + \gamma_{h+j}(\tilde{\gamma}_h - \gamma_h) + \gamma_h(\tilde{\gamma}_{h+j} - \gamma_{h+j})) \right| \\ &\le \sum_{j=-l}^{l} h_j \left(\sum_{h=-2l}^{2l} E(\tilde{\gamma}_h - \gamma_h)^2 \right) \\ &\quad + 2 \sum_{j=-l}^{l} h_j \left(\sum_{h=\infty}^{\infty} |\gamma_{h+j}| \right) \left(\max_{h=0,1,\ldots,2l} E|\tilde{\gamma}_h - \gamma_h| \right) \\ &\le \sum_{j=-l}^{l} h_j \cdot laT^{-1} + 2 \sum_{j=-l}^{l} h_j \left(\sum_{h=\infty}^{\infty} |\gamma_{h+j}| \right) a^{1/2} T^{-1/2} \to 0, \end{aligned}$$

for some $a > 0$. The bounds in the second inequality follow because

$$\begin{aligned} E[(\tilde{\gamma}_h - \gamma_h)^2] &= T^{-2} \sum_{t=h+1}^{T} \sum_{s=h+1}^{T} (E(z_t z_{t-h} z_s z_{s-h}) - \gamma_h^2) + \frac{h^2}{T^2} \gamma_h^2 \\ &= T^{-2} \sum_{t=h+1}^{T} \sum_{s=h+1}^{T} (\gamma_{t-s}^2 + \gamma_{t-s+h}\gamma_{t-s-h} + \kappa_4(z_t z_{t-h} z_s z_{s-h})) + \frac{h^2}{T^2} \gamma_h^2 \end{aligned}$$

and

$$\begin{aligned} T^{-2} \sum_{t=h+1}^{T} \sum_{s=h+1}^{T} \gamma_{t-s}^2 &\le T^{-2} \sum_{s=1}^{T} \sum_{t=1-s}^{T-s} \gamma_t^2 \\ &= T^{-2} \sum_{t=1-T}^{T-1} \sum_{s=1\vee(1-t)}^{T\wedge(T-t)} \gamma_t^2 \le T^{-1} \sum_{t=-\infty}^{\infty} \gamma_t^2, \end{aligned}$$

$$T^{-2} \sum_{t=h+1}^{T} \sum_{s=h+1}^{T} \gamma_{t-s+h}\gamma_{t-s-h} = T^{-2} \sum_{s=h+1}^{T} \sum_{t=h-s+1}^{T-s} \gamma_{t+h}\gamma_{t-h} \le T^{-1} \sum_{t=-\infty}^{\infty} \gamma_t^2,$$

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and the term involving the fourth cumulant is handled similarly, showing that $E[(\tilde{\gamma}_h - \gamma_h)^2] \leq T^{-1}a$ for some a uniformly in $h \leq 2l$. Second

$$
\begin{aligned}
|\omega^2 - \omega_l^2| &= \left| \sum_{j=-\infty}^{\infty} h_j \sum_{h=-\infty}^{\infty} \gamma_h \gamma_{j+h} - \sum_{j=-l}^{l} h_j \sum_{h=-l}^{l} \gamma_h \gamma_{h+j} \right| \\
&\leq \sum_{h=-\infty}^{\infty} |\gamma_h| \sum_{j=l+1}^{\infty} h_j |\gamma_{j-h}| + \sum_{h=-\infty}^{\infty} |\gamma_h| \sum_{j=l+1}^{\infty} h_j |\gamma_{j+h}| \\
&\quad + \sum_{h=l+1}^{\infty} |\gamma_h| \sum_{j=-l}^{l} h_j |\gamma_{j-h}| + \sum_{h=l+1}^{\infty} |\gamma_h| \sum_{j=-l}^{l} h_j |\gamma_{h+j}| \\
&\leq 2 \left(\sum_{j=l+1}^{\infty} h_j^2 \right)^{1/2} \left(\sum_{h=-\infty}^{\infty} |\gamma_h| \right) \left(\sum_{j=-\infty}^{\infty} \gamma_j^2 \right)^{1/2} \\
&\quad + 2 \left(\sum_{h=l+1}^{\infty} |\gamma_h| \right) \left(\sum_{j=-\infty}^{\infty} h_j^2 \right)^{1/2} \left(\sum_{j=-\infty}^{\infty} \gamma_j^2 \right)^{1/2} \\
&\to 0
\end{aligned}
$$

as $l \to \infty$. Thus $E|\tilde{\omega}_l^2 - \omega^2| \to 0$ as required. ■

LEMMA 4. *Under the conditions of Theorem 1,*

(i) $\hat{N}_k - \tilde{N}_k \xrightarrow{p} 0$,
(ii) $\hat{\omega}_l^2 - \tilde{\omega}_l^2 \xrightarrow{p} 0$.

Proof. (i) The numerator of the statistic can be written

$$
\begin{aligned}
\hat{N}_k = \tilde{N}_k &- (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k} \sum_{t=j+1}^{T} z_t X'_{T,t-j} \sum_{t=1}^{T} X_{T,t} z_t \\
&- \sum_{t=1}^{T} z_t X'_{T,t} (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k} \sum_{t=j+1}^{T} X_{T,t} z_{t-j} \\
&+ \sum_{t=1}^{T} z_t X'_{T,t} (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k} \sum_{t=j+1}^{T} X_{T,t} X'_{T,t-j} \sum_{t=1}^{T} X_{T,t} z_t, \qquad \textbf{(A.13)}
\end{aligned}
$$

where $X_{T,t} = (\sum_{t=1}^{T} x_t x_t')^{-1/2} x_t$. We show that the last three terms disappear asymptotically. Note that $\sum_{t=1}^{T} X'_{T,t} X_{T,t} = p$, and so

$$
\begin{aligned}
E\left(\left\| \sum_{t=1}^{T} X_{T,t} z_t \right\|^2 \right) &= \sum_{t=1}^{T} \sum_{s=1}^{T} X'_{T,t} X_{T,s} \gamma_{t-s} \\
&\leq \sum_{t=1-T}^{T-1} |\gamma_t| \left| \sum_{s=1 \vee (1-t)}^{T \wedge (T-t)} X'_{T,s} X_{T,s+t} \right| \\
&\leq p \sum_{t=1-T}^{T-1} |\gamma_t| \\
&\leq p \sum_{t=-\infty}^{\infty} |\gamma_t|. \qquad \textbf{(A.14)}
\end{aligned}
$$

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The second inequality follows by the vector CS inequality $(\sum_{i=1}^{n} a_i' b_i)^2 \le (\sum_{i=1}^{n} a_i' a_i)(\sum_{i=1}^{n} b_i' b_i)$. This bound also applies to $E(\|\sum_{t=j+1}^{T} X_{T,t-j} z_t\|^2)$ and $E(\|\sum_{t=j+1}^{T} X_{T,t} z_{t-j}\|^2)$. Thus the second term on the right-hand side of (A.13) disappears as follows:

$$
\begin{aligned}
E\left\| (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k} \sum_{t=j+1}^{T} z_t X_{T,t-j}' \sum_{t=1}^{T} X_{T,t} z_t \right\| & \\
\le (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k} \cdot E\left(\left\| \sum_{t=j+1}^{T} X_{T,t-j} z_t \right\|^2 \right)^{1/2} & \cdot E\left(\left\| \sum_{t=1}^{T} X_{T,t} z_t \right\|^2 \right)^{1/2} \\
\le (T-k)^{-1/2} \sum_{j=k}^{T-1} w_{j-k} \cdot p \sum_{t=-\infty}^{\infty} |\gamma_t| = O\left(\frac{\ln T}{T^{1/2}} \right). &
\end{aligned}
$$

The third and fourth terms disappear similarly, which proves (i).

(ii) To show $\hat{\omega}_l^2 - \tilde{\omega}_l^2 \xrightarrow{p} 0$, we write

$$
\begin{aligned}
E|\hat{\omega}_l^2 - \tilde{\omega}_l^2| = E\Bigg| \sum_{j=-l}^{l} h_j \sum_{h=-l}^{l} & ((\hat{\gamma}_{h+j} - \tilde{\gamma}_{h+j})(\hat{\gamma}_h - \tilde{\gamma}_h) + \tilde{\gamma}_{h+j}(\hat{\gamma}_h - \tilde{\gamma}_h) \\
& + \tilde{\gamma}_h(\hat{\gamma}_{h+j} - \tilde{\gamma}_{h+j})) \Bigg|,
\end{aligned}
$$

in which

$$
\begin{aligned}
E|\hat{\gamma}_h - \tilde{\gamma}_h| &= E\left| T^{-1} \sum_{t=h+1}^{T} (\hat{z}_t \hat{z}_{t-h} - z_t z_{t-h}) \right| \\
&\le E\left| T^{-1} \sum_{t=h+1}^{T} z_t X_{T,t-h}' \left(\sum_{t=1}^{T} X_{T,t} z_t \right) \right| \\
&\quad + E\left| \left(\sum_{t=1}^{T} z_t X_{T,t}' \right) T^{-1} \sum_{t=h+1}^{T} X_{T,t} z_{t-h} \right| \\
&\quad + E\left| \left(\sum_{t=1}^{T} z_t X_{T,t}' \right) T^{-1} \sum_{t=h+1}^{T} X_{T,t} X_{T,t-h}' \left(\sum_{t=1}^{T} X_{T,t} z_t \right) \right| \\
&\le T^{-1} \left(E\left\| \sum_{t=h+1}^{T} z_t X_{T,t-h}' \right\|^2 \right)^{1/2} \left(E\left\| \sum_{t=1}^{T} X_{T,t} z_t \right\|^2 \right)^{1/2} \\
&\quad + T^{-1} \left(E\left\| \sum_{t=h+1}^{T} X_{T,t} z_{t-h} \right\|^2 \right)^{1/2} \left(E\left\| \sum_{t=1}^{T} X_{T,t} z_t \right\|^2 \right)^{1/2} \\
&\quad + T^{-1} \left\| \sum_{t=h+1}^{T} X_{T,t} X_{T,t-h}' \right\| E\left\| \sum_{t=1}^{T} X_{T,t} z_t \right\|^2 \\
&\le 3T^{-1} p \sum_{t=1-T}^{T-1} |\gamma_t|, \qquad \textbf{(A.15)}
\end{aligned}
$$

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by application of (A.14). It thus follows that

$$E|\hat{\omega}_l^2 - \tilde{\omega}_l^2| \leq \sum_{j=-l}^{l} h_j \cdot lT^{-1}a = O(l(\ln l)^2/T) \to 0$$

for some $a > 0$. ■

A.2. Proof of Theorem 2. For $d < 0.5$ the statistic can be written as

$$\hat{S}_k^* = \frac{(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}(\hat{\gamma}_j - \gamma_j) + (T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j}{\omega_l + (\hat{\omega}_l - \omega_l)}$$

$$= \frac{(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j}{\omega_l}\left(\frac{1 + \sum_{j=k}^{T-1} w_{j-k}(\hat{\gamma}_j - \gamma_j)\Big/\sum_{j=k}^{T-1} w_{j-k}\gamma_j}{1 + (\hat{\omega}_l - \omega_l)/\omega_l}\right). \tag{A.16}$$

It follows from Lemma 7 that there exists some J such that $\gamma_j > 0$ for all $j \geq J$, so that $(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j$ is positive for large enough k and has order

$$(T-k)^{1/2}\sum_{j=k}^{T-1} \frac{j^{2d}}{j(j-k+1)} \geq (T-k)^{1/2}k^{2d}\sum_{j=k}^{T-1} \frac{1}{j(j-k+1)}. \tag{A.17}$$

Now

$$\sum_{j=k}^{T-1} \frac{1}{j(j-k+1)} = \sum_{j=1}^{\infty} \frac{1}{j(j+k-1)} - \sum_{j=T-k+1}^{\infty} \frac{1}{j(j+k-1)},$$

where

$$\sum_{j=1}^{\infty} \frac{1}{j(j+k-1)} \sim \frac{1}{k-1}\sum_{j=1}^{k-1} \frac{1}{j} \sim \frac{1}{k}\ln k,$$

and by CS inequality

$$\left|\sum_{j=T-k+1}^{\infty} \frac{1}{j(j+k-1)}\right| \leq \sum_{j=T-k+1}^{\infty} \frac{1}{j^2} \sim \frac{1}{T},$$

so that

$$(T-k)^{1/2}k^{2d}\sum_{j=k}^{T-1} \frac{1}{j(j-k+1)} \sim T^{1/2+d}\frac{1}{k}\ln k \sim T^d \ln T.$$

Also

$$\omega_l^2 = \sum_{i=-l}^{l}\left(\sum_{h=0}^{l} w_h\gamma_{h+i}\right)^2 = \sum_{h=0}^{l}\sum_{j=0}^{l} w_h w_j \sum_{i=-l}^{l} \gamma_{h+i}\gamma_{j+i} \leq \left(\sum_{h=0}^{l} w_h\right)^2\left(\sum_{i=-2l}^{2l} \gamma_i^2\right),$$

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which has order

$$(\ln l)^2, \qquad \text{for } 0 < d < \tfrac{1}{4},$$
$$(\ln l)^3, \qquad \text{for } d = \tfrac{1}{4},$$
$$(\ln l)^2 l^{4d-1}, \qquad \text{for } \tfrac{1}{4} < d < \tfrac{1}{2}.$$

The ratio $\omega_l^{-1}(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j$, the leading term in (A.16), therefore has order at least

$$\frac{T^d \ln T}{\ln l}, \frac{T^d \ln T}{(\ln l)^{3/2}}, \frac{T^d \ln T}{l^{2d-1/2}(\ln l)},$$

and, because $l/T^{1/2} \to 0$, the ratio diverges to $+\infty$. To conclude from this that $\tilde{S}_k^*$ diverges to $+\infty$ we now argue that the second term in (A.16) converges in probability to $+1$.

In the numerator we have the term

$$\frac{\sum_{j=k}^{T-1} w_{j-k}(\hat{\gamma}_j - \gamma_j)}{\sum_{j=k}^{T-1} w_{j-k}\gamma_j} = \frac{(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}(\hat{\gamma}_j - \tilde{\gamma}_j)}{(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j} + \frac{(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}(\tilde{\gamma}_j - \gamma_j)}{(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j}.$$

To show that the first term disappears, we have by (A.15) that

$$E\left|(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}(\hat{\gamma}_j - \tilde{\gamma}_j)\right| \le (T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\cdot 3pT^{-1}\sum_{t=1-T}^{T-1}|\gamma_t|$$
$$= O(T^{2d-1/2}\ln T),$$

whereas the denominator $(T-k)^{1/2}\sum_{j=k}^{T-1} w_{j-k}\gamma_j$ has been shown to have a rate of divergence of at least $k^{2d}\ln k$, which is rate $T^d \ln T$ with $k = O(T^{1/2})$, and $(T^{2d-1/2}\ln T)/(T^d \ln T) \to 0$ for all $d < \frac{1}{2}$. The second term disappears similarly because $E|\tilde{\gamma}_j - \gamma_j| = O(T^{-1/2\vee(2d-1)})$. The disappearance of $(\hat{\omega}_l - \omega_l)/\omega_l$ in the denominator in (A.16) also follows by similar reasoning.

For $d > 0.5$ we first note that

$$\sum_{j=-l}^{l} h_j \le \frac{\pi^2}{6} + 2\left(\sum_{j=1}^{l}\frac{1}{j}\right)^2 \le \frac{\pi^2}{6} + 2(\ln l)^2,$$

so that for some $c > 0$

$$\hat{\omega}_l^2 \le \hat{\gamma}_0^2(2l+1)\sum_{j=-l}^{l} h_j \le c^{-1/2} l(\ln l)^2 \hat{\gamma}_0^2. \qquad \textbf{(A.18)}$$

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Next consider

$$\sum_{j=k}^{T-1} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0} = \sum_{j=k}^{\lfloor T^{1-\varepsilon}\rfloor} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0} + \sum_{j=\lfloor T^{1-\varepsilon}\rfloor+1}^{T-1} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0}$$

$$\geq \sum_{j=k}^{\lfloor T^{1-\varepsilon}\rfloor} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0} - \left|\sum_{j=\lfloor T^{1-\varepsilon}\rfloor+1}^{T-1} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0}\right| \tag{A.19}$$

for some $0 < \varepsilon < \frac{1}{2}$ (the inequality uses $a + b \geq a - |b|$ because $b + |b| \geq 0$). Now

$$\frac{\hat{\gamma}_j}{\hat{\gamma}_0} - 1 = \frac{\sum_{t=j+1}^{T}(\hat{z}_t - \hat{z}_{t-j})\hat{z}_{t-j} - \sum_{t=T-j+1}^{T}\hat{z}_t^2}{\sum_{t=1}^{T}\hat{z}_t^2},$$

and clearly

$$\frac{\sum_{t=T-j+1}^{T}\hat{z}_t^2}{\sum_{t=1}^{T}\hat{z}_t^2} \xrightarrow{p} 0$$

uniformly for all $j \leq T^{1-\varepsilon}$. Using equation (22) of Tanaka (1999) and the CMT we obtain

$$T^{1/2-d}\hat{z}_{[Ts]} \xrightarrow{d} \phi\left(F_{d-1}(s) - X(s)\left(\int_0^1 X(r)X(r)'\,dr\right)^{-1}\int_0^1 X(r)F_{d-1}(r)\,dr\right)$$

$$:= \phi\hat{F}_{d-1}(s),$$

and hence

$$T^{-2d}\sum_{t=1}^{T}\hat{z}_t^2 \xrightarrow{d} \phi^2\int_0^1 \hat{F}_{d-1}(s)^2\,ds,$$

where F_{d-1} is given by Tanaka's equation (24) and ϕ by his equation (13). Also

$$T^{-2d}\sum_{t=j+1}^{T}\left(\sum_{i=0}^{j-1}\Delta\hat{z}_{t-i}\right)\hat{z}_{t-j} = T^{-1}\sum_{t=j+1}^{T}\left(T^{1/2-d}\sum_{i=0}^{j-1}\Delta\hat{z}_{t-i}\right)T^{1/2-d}\hat{z}_{t-j} \xrightarrow{p} 0$$

uniformly for all $j \leq T^{1-\varepsilon}$. Thus,

$$\frac{T^{-2d}\sum_{t=j+1}^{T}(\hat{z}_t - \hat{z}_{t-j})\hat{z}_{t-j}}{T^{-2d}\sum_{t=1}^{T}\hat{z}_t^2} = \frac{T^{-2d}\sum_{t=j+1}^{T}\left(\sum_{i=0}^{j-1}\Delta\hat{z}_{t-i}\right)\hat{z}_{t-j}}{T^{-2d}\sum_{t=1}^{T}\hat{z}_t^2} \xrightarrow{p} 0,$$

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and hence

$$\frac{\hat{\gamma}_j}{\hat{\gamma}_0} \xrightarrow{p} 1$$

uniformly for all $j \leq T^{1-\varepsilon}$. Therefore we write the first term in (A.19) as

$$\begin{aligned}\sum_{j=k}^{\lfloor T^{1-\varepsilon}\rfloor} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0} &= \sum_{j=1}^{\lfloor T^{1-\varepsilon}\rfloor-k+1}\frac{1}{j} + \sum_{j=1}^{\lfloor T^{1-\varepsilon}\rfloor-k+1}\frac{1}{j}\left(\frac{\hat{\gamma}_{j+k-1}}{\hat{\gamma}_0}-1\right)\\ &= \ln(T^{1-\varepsilon}) + o_p(\ln(T^{1-\varepsilon}))\\ &= (1-\varepsilon)\ln T + o_p(\ln T). \end{aligned} \tag{A.20}$$

Next we consider the second term in (A.19), which, using $|\hat{\gamma}_j/\hat{\gamma}_0| \leq 1$ for all j, can be written

$$\begin{aligned}\left|\sum_{j=\lfloor T^{1-\varepsilon}\rfloor-k+1}^{T-k-1} w_j\frac{\hat{\gamma}_{j+k}}{\hat{\gamma}_0}\right| &\leq \sum_{j=\lfloor T^{1-\varepsilon}\rfloor-k+2}^{T-k}\frac{1}{j} = \ln T - \ln(T^{1-\varepsilon}) + o(\ln(T^{1-\varepsilon}))\\ &= \varepsilon\ln T + o(\ln T). \end{aligned} \tag{A.21}$$

Combining (A.20) and (A.21) in (A.19) gives

$$\sum_{j=k}^{T-1} w_{j-k}\frac{\hat{\gamma}_j}{\hat{\gamma}_0} \geq (1-2\varepsilon)\ln T + o_p(\ln T), \tag{A.22}$$

and then combining (A.22) and (A.18) gives

$$\hat{S}_k^* \geq cl^{-1/2}(\ln l)(T-k)^{1/2}((1-2\varepsilon)\ln T + o_p(\ln T)),$$

so that $\hat{S}_k^*$ diverges at least as fast as $(T^{1/2}\ln T)/(l^{1/2}\ln l)$ as claimed. ■

A.3. Technical Lemmas.

LEMMA 5. *If* $\sum_{j=0}^{\infty} w_j^2 < \infty$ *then* $\sum_{i=-\infty}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h+i})^2 < \infty$. *Note that this also immediately implies* $\sum_{i=-\infty}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h-i})^2 < \infty$.

Proof. For any $n > 0$

$$\begin{aligned}\sum_{i=-n}^{n}\left(\sum_{h=0}^{n}|w_h|\bar{\gamma}_{h+i}\right)^2 &= \sum_{i=-n}^{n}\sum_{j=0}^{n}\sum_{h=0}^{n}|w_j||w_h|\bar{\gamma}_{j+i}\bar{\gamma}_{h+i}\\ &\leq 2\sum_{i=-n}^{n}\sum_{j=0}^{n}\sum_{h=0}^{n-j}|w_j||w_{j+h}|\bar{\gamma}_{j+i}\bar{\gamma}_{h+j+i}\\ &= 2\sum_{h=0}^{n}\sum_{j=0}^{n-h}|w_j||w_{j+h}|\sum_{i=-n+j}^{n+j}\bar{\gamma}_i\bar{\gamma}_{h+i}\\ &\leq 2\left(\sum_{j=0}^{n}w_j^2\right)\sum_{h=0}^{n}\sum_{i=-n}^{2n}\bar{\gamma}_i\bar{\gamma}_{h+i}\\ &\leq 2\left(\sum_{j=0}^{n}w_j^2\right)\left(\sum_{i=-n}^{3n}\bar{\gamma}_i\right)^2\\ &\leq 2\left(\sum_{j=0}^{\infty}w_j^2\right)\left(\sum_{i=-\infty}^{\infty}\bar{\gamma}_i\right)^2. \end{aligned}$$

■

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LEMMA 6. *For $g_{k,t,k-i}$ defined following (A.1),*

(i) if $0 \le s \le k-1$, $\sum_{i=s+1}^{\infty} g_{k,t,k-i}^2 \le \bar{\omega}^2 = \sum_{i=-\infty}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h+i})^2$,
(ii) if $s \ge k$, $\sum_{i=s+1}^{\infty} g_{k,t,k-i}^2 \le \sum_{i=s-k+1}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h-i})^2$ uniformly in t, and if $k \to \infty$ as $T \to \infty$ such that $k/T \to 0$ then
(iii) $(T-k)^{-1}\sum_{t=k+1}^{T}(\sum_{i=1}^{\infty} g_{k,t,k-i}^2) \to \omega^2$ as $T \to \infty$.

Proof. (i) For $0 \le s \le k-1$

$$\sum_{i=s+1}^{\infty} g_{k,t,k-i}^2 = \sum_{i=s+1}^{k}\left(\sum_{j=k-i}^{k-1} c_j f_{t-k,i-k+j}\right)^2 + \sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{k-1} c_j f_{t-k,i-k+j}\right)^2,$$

where

$$\begin{aligned}\sum_{i=s+1}^{k}\left(\sum_{j=k-i}^{k-1} c_j f_{t-k,i-k+j}\right)^2 &\le \sum_{i=s+1}^{k}\left(\sum_{j=k-i}^{k-1}\sum_{h=0}^{i-k+j}|w_h||c_j||c_{i-k+j-h}|\right)^2 \\ &= \sum_{i=s+1}^{k}\left(\sum_{h=0}^{i-1}|w_h|\sum_{j=h}^{i-1}|c_{j+k-i}||c_{j-h}|\right)^2 \\ &\le \sum_{i=s+1}^{k}\left(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{k-i+h}\right)^2 \\ &\le \sum_{i=0}^{\infty}\left(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{i+h}\right)^2, \qquad \textbf{(A.23)}\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{k-1} c_j f_{t-k,i-k+j}\right)^2 &\le \sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{k-1}\sum_{h=0}^{i-k+j}|w_h||c_j||c_{i-k+j-h}|\right)^2 \\ &\le \sum_{i=1}^{\infty}\left(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h-i}\right)^2, \qquad \textbf{(A.24)}\end{aligned}$$

and so adding (A.23) and (A.24) gives the required bound.

(ii) For $s \ge k$

$$\sum_{i=s+1}^{\infty} g_{k,t,k-i}^2 = \sum_{i=s+1}^{\infty}\left(\sum_{j=0}^{k-1} c_j f_{t-k,i-k+j}\right)^2,$$

and the bound follows similarly to (A.24).

(iii) The sum can be written

$$(T-k)^{-1}\sum_{t=1}^{T-k} x_{T,t}, \qquad \textbf{(A.25)}$$

where $x_{T,t} = \sum_{i=1}^{\infty} g_{k,t+k,k-i}^2$. We can validly define $x_{T,\infty} = \sum_{i=1}^{\infty} g_{k,\infty,k-i}^2$ where $g_{k,\infty,k-i} = \sum_{j=(k-i)\vee 0}^{k-1} c_j f_{\infty,i-k+j}$, $f_{\infty,i} = \sum_{h=0}^{i} w_h c_{i-h}$ and $x_{T,\infty}$ exists by part (i). We now show that

$$|x_{T,t} - x_{T,\infty}| \le c_t, \qquad \textbf{(A.26)}$$

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where $c_t = \sum_{i=-\infty}^{\infty}(\sum_{h=t}^{\infty}|w_h|\bar{\gamma}_{h-i})^2$. To do this, first write

$$|x_{T,t} - x_{T,\infty}| = \left|\sum_{i=1}^{\infty}(g^2_{k,t+k,k-i} - g^2_{k,\infty,k-i})\right|$$

$$\leq 2x^{1/2}_{T,\infty}\left(\sum_{i=1}^{\infty}(g_{k,t+k,k-i} - g_{k,\infty,k-i})^2\right)^{1/2} + \sum_{i=1}^{\infty}(g_{k,t+k,k-i} - g_{k,\infty,k-i})^2.$$

Now

$$\sum_{i=1}^{\infty}(g_{k,t+k,k-i} - g_{k,\infty,k-i})^2 = \sum_{i=1}^{\infty}\left(\sum_{j=(k-i)\vee 0}^{k-1} c_j(f_{t,i-k+j} - f_{\infty,i-k+j})\right)^2$$

$$= \sum_{i=1}^{\infty}\left(\sum_{j=0\vee(i-k)}^{i-1} c_{j+k-i}(f_{t,j} - f_{\infty,j})\right)^2, \tag{A.27}$$

where

$$f_{t,j} - f_{\infty,j} = \sum_{h=0}^{j\wedge(t-1)} w_h c_{j-h} - \sum_{h=0}^{j} w_h c_{j-h} = \begin{cases} 0, & j \leq t-1, \\ -\sum_{h=t}^{j} w_h c_{j-h}, & j > t-1. \end{cases} \tag{A.28}$$

Substituting (A.28) into (A.27) gives

$$\sum_{i=1}^{\infty}(g_{k,t+k,k-i} - g_{k,\infty,k-i})^2 = \sum_{i=t+1}^{\infty}\left(\sum_{j=t}^{i-1} c_{j+k-i}\sum_{h=t}^{j} w_h c_{j-h}\right)^2$$

$$= \sum_{i=t+1}^{\infty}\left(\sum_{h=t}^{i-1} w_h \sum_{j=h}^{i-1} c_{j+k-i}c_{j-h}\right)^2$$

$$\leq \sum_{i=-\infty}^{\infty}\left(\sum_{h=t}^{\infty}|w_h|\bar{\gamma}_{h-i}\right)^2 = c_t,$$

as required for (A.26). Combining (A.25) and (A.26) shows

$$\left|(T-k)^{-1}\sum_{t=1}^{T-k} x_{T,t} - x_{T,\infty}\right| \leq (T-k)^{-1}\sum_{t=1}^{T-k} c_t \to 0$$

as $T \to \infty$ by Cesaro summation, because Lemma 5 implies that $c_t \to 0$ as $t \to \infty$.

It is therefore left to show that $x_{T,\infty} = \sum_{i=1}^{\infty} g^2_{k,\infty,k-i} \to \omega^2$ as $T \to \infty$. Now substituting for $g^2_{k,\infty,k-i}$ and $f_{\infty,j-i}$ with some rearrangement gives

$$\sum_{i=1}^{\infty} g^2_{k,\infty,k-i} = \sum_{i=-\infty}^{k-1}\left(\sum_{j=i\vee 0}^{k-1} c_j f_{\infty,j-i}\right)^2 = \sum_{i=-\infty}^{k-1}\left(\sum_{h=0}^{k-i-1} w_h \gamma_{k,i+h}\right)^2,$$

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where $\gamma_{k,i+h} = \sum_{j=0\vee(i+h)}^{k-1} c_j c_{j-i-h}$. Thus

$$\left|\sum_{i=1}^{\infty} g_{k,\infty,k-i}^2 - \omega^2\right| \le \left|\sum_{i=-\infty}^{k-1}\left[\left(\sum_{h=0}^{k-i-1} w_h\gamma_{k,h+i}\right)^2 - \left(\sum_{h=0}^{\infty} w_h\gamma_{h+i}\right)^2\right]\right| + \left|\sum_{i=k}^{\infty}\left(\sum_{h=0}^{\infty} w_h\gamma_{h+i}\right)^2\right|.$$

The second term on the right-hand side is bounded by $\sum_{i=k}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h+i})^2$, which, in view of Lemma 5, clearly disappears as $k \to \infty$. So it remains to consider the first term, which can be seen to be bounded by $2\omega r_k^{1/2} + r_k$ where

$$\begin{aligned} r_k &= \sum_{i=-\infty}^{k-1}\left(\sum_{h=0}^{k-i-1} w_h\gamma_{k,h+i} - \sum_{h=0}^{\infty} w_h\gamma_{h+i}\right)^2 \\ &= \sum_{i=-\infty}^{k-1}\left(\sum_{h=0}^{k-i-1} w_h(\gamma_{k,h+i} - \gamma_{h+i}) - \sum_{h=k-i}^{\infty} w_h\gamma_{h+i}\right)^2 \\ &\le \sum_{i=-\infty}^{k-1}\left(\sum_{h=0}^{k-i-1} w_h(\gamma_{k,h+i} - \gamma_{h+i})\right)^2 + \sum_{i=-\infty}^{k-1}\left(\sum_{h=k-i}^{\infty} w_h\gamma_{h+i}\right)^2 \end{aligned}$$

and $\gamma_{k,h+i} - \gamma_{h+i} = -\sum_{j=k}^{\infty} c_j c_{j-i-h}$. Rearrangement of the second term shows that it is bounded by $\sum_{i=k}^{\infty}(\sum_{h=0}^{\infty}|w_h|\bar{\gamma}_{h-i})^2$, which disappears as $k \to \infty$ by Lemma 5. The first term is bounded by

$$\begin{aligned} &\sum_{i=-\infty}^{k-1}\left(\sum_{h=0}^{k-i-1}|w_h|\sum_{j=k}^{\infty}|c_j||c_{j-i-h}|\right)^2 \\ &\quad= \sum_{i=1}^{\infty}\left(\sum_{h=0}^{i-1}|w_h|\sum_{j=k}^{\infty}|c_j||c_{j-k+i-h}|\right)^2 \\ &\quad\le 2\sum_{i=1}^{\infty}\sum_{h=0}^{i-1}\sum_{p=0}^{i-h-1}\sum_{j=k}^{\infty}\sum_{q=j}^{\infty}|w_h||w_{h+p}||c_j||c_{j-k+i-h}||c_q||c_{q-k+i-p-h}| \\ &\quad= 2\sum_{h=1}^{\infty}\sum_{p=1}^{h}\sum_{j=k}^{\infty}\sum_{q=j}^{\infty}\sum_{i=0}^{\infty}|w_i||w_{i+p}||c_j||c_{j-k+h}||c_q||c_{q-k+p}| \\ &\quad\le 2\left(\sum_{i=0}^{\infty} w_i^2\right)\left(\sum_{h=1}^{\infty}\sum_{j=k}^{\infty}|c_j||c_{j-k+h}|\right)^2 \\ &\quad\le 2\left(\sum_{i=0}^{\infty} w_i^2\right)\left(\sum_{j=k}^{\infty}|c_j|\right)^2\left(\sum_{h=1}^{\infty}|c_j|\right)^2, \end{aligned}$$

which disappears for $k \to \infty$ as c_j is absolutely summable. Thus $r_k \to 0$ and $\sum_{i=1}^{\infty} g_{k,\infty,k-i}^2 \to \omega^2$. ■

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LEMMA 7. *Under the conditions of Theorem 2 with* $0 < d < \frac{1}{2}$ *the autocovariance function of* z_t*, denoted* γ_h*, satisfies*

$$h^{1-2d}\gamma_h \to \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \sum_{j=-\infty}^{\infty} \gamma_{u,j} > 0,$$

where $\gamma_{u,j}$ *denotes the autocovariance function of* u_t*.*

Proof. The proof follows that of Theorem 13.2.2 of Brockwell and Davis (1991). We have

$$\gamma_h = \sum_{j=-\infty}^{\infty} \gamma_{u,j}\gamma_{x,h-j},$$

where $\gamma_{x,j}$ is the autocovariance function of the fractional noise process $(1-L)^d x_t = e_t$ where e_t is unit variance white noise. We can write

$$h^{1-2d}\gamma_h = h^{1-2d} \sum_{|j|\le h^{1-2d}} \gamma_{u,j}\gamma_{x,h-j} + h^{1-2d} \sum_{|j|>h^{1-2d}} \gamma_{u,j}\gamma_{x,h-j}. \tag{A.29}$$

The second of these terms satisfies

$$\left| h^{1-2d} \sum_{|j|>h^{1-2d}} \gamma_{u,j}\gamma_{x,h-j} \right| \le |\gamma_{x,0}| \cdot h^{1-2d} \sum_{|j|>h^{1-2d}} |\gamma_{u,j}|$$

$$\le |\gamma_{x,0}| \cdot \sum_{|j|>h^{1-2d}} |j||\gamma_{u,j}|,$$

which disappears as $h \to \infty$ because $\sum_{j=1}^{\infty} j|c_j| < \infty$ implies that $\sum_{j=1}^{\infty} j|\gamma_j| < \infty$. Next,

$$h^{1-2d}\gamma_{x,h-j} \to \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}$$

uniformly in j for $|j| \le h^{1-2d}$ because $1-2d < 1$, and so the first term in (A.29) satisfies

$$h^{1-2d} \sum_{|j|\le h^{(1-2d)}} \gamma_{u,j}\gamma_{x,h-j} \to \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \sum_{j=-\infty}^{\infty} \gamma_{u,j}$$

as required. ■

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