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Some formulae for generalized Rice polynomials

BY H. L. MANOCHA
Punjab Engineering College, Chandigarh, India

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Introduction. The aim of this note is to obtain some formulae for generalized Rice polynomials as defined by (2)

$$H_n^{(\alpha,\beta)}(\xi,p,x) = \frac{(1+\alpha)_n}{n!}\,{}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, \xi;\\ 1+\alpha, p;\end{matrix}\; x\right]. \tag{1}$$

For $\alpha = 0 = \beta$, (1) reduces to the original form (5)

$$H_n(\xi,p,x) = {}_3F_2\left[\begin{matrix}-n, n+1, \xi;\\ 1, p;\end{matrix}\; x\right]. \tag{2}$$

In the case $\xi = p$, (1) becomes a Jacobi polynomial defined by ((4), p. 254)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n;\\ 1+\alpha;\end{matrix}\; \frac{1-x}{2}\right] \tag{3}$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, -\beta-n;\\ -\alpha-\beta-2n;\end{matrix}\; \frac{2}{x+1}\right]. \tag{4}$$

The formulae that we prove now are in fact generating functions for various polynomials, or polynomial products.

2. The first formula to be proved is

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_m^{(\alpha,\beta)}(-n,p,x)\,t^n = (1-t)^{-\lambda} H_m^{(\alpha,\beta)}\left(\lambda, p, \frac{-xt}{1-t}\right). \tag{5}$$

Proof. By (1) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_m^{(\alpha,\beta)}(-n,p,x)\,t^n \\
&\quad= \frac{(1+\alpha)_m}{m!} \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(\lambda)_n (-m)_r (-n)_r (1+\alpha+\beta+m)_r}{n!\, r!\, (1+\alpha)_r (p)_r}\, t^n x^r \\
&\quad= \frac{(1+\alpha)_m}{m!} \sum_{r=0}^{m} \frac{(-m)_r (\lambda)_r (1+\alpha+\beta+m)_r}{r!\,(1+\alpha)_r (p)_r} (-xt)^r \sum_{n=0}^{\infty} \frac{(\lambda+r)_n}{n!}\, t^n \\
&\quad= \frac{(1+\alpha)_m}{m!} (1-t)^{-\lambda}\, {}_3F_2\left(-m, 1+\alpha+\beta+m, \lambda;\ 1+\alpha, p; \frac{-xt}{1-t}\right) \\
&\quad= (1-t)^{-\lambda} H_m^{(\alpha,\beta)}\left(\lambda, p, \frac{-xt}{1-t}\right).
\end{aligned}$$

For $\lambda = p$, (5) reduces to

$$\sum_{n=0}^{\infty} \frac{(p)_n}{n!} H_m^{(\alpha,\beta)}(-n,p,x)\,t^n = (1-t)^{-p} P_m^{(\alpha,\beta)}\left(1 + \frac{2xt}{1-t}\right). \tag{6}$$

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3. The second formula to be proved is

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} H_n^{(\alpha-n,\beta-n)}(\xi,p,x)\,t^n = F_2(\lambda;\ -\alpha,\xi;\ -\alpha-\beta,p;\ -t,xt). \tag{7}$$

Proof. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} H_n^{(\alpha-n,\beta-n)}(\xi,p,x)\,t^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(\lambda)_n (1+\alpha-n)_n}{n!\,(-\alpha-\beta)_n} {}_3F_2(-n, 1+\alpha+\beta-n, \xi;\ 1+\alpha-n, p;\ x)\,t^n \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k}(\lambda)_n(-\alpha)_{n-k}(\xi)_k}{(n-k)!\,k!\,(-\alpha-\beta)_{n-k}(p)_k}\,t^n x^k \\
&\quad = F_2(\lambda;\ -\alpha,\xi;\ -\alpha-\beta,p;\ -t,xt).
\end{aligned}$$

For $\xi = p$, it follows from (7) that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x)\,t^n = [1-\tfrac{1}{2}(1-x)\,t]^{-\lambda}\,{}_2F_1\left[\lambda, -\alpha;\ -\alpha-\beta;\ \frac{-t}{1-\frac{1}{2}(1-x)\,t}\right]. \tag{8}$$

4. The third formula to be proved is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(p)_k}{k!} H_m^{(\alpha,\beta)}(-k,p,x)\,H_n^{(\gamma,\delta)}(p+k,q,y)\,t^k &= \frac{(1+\alpha)_m(1+\gamma)_n}{m!\,n!}(1-t)^{-p} \\
&\times {}_3F_2(2)\left[\begin{matrix} p;\ -m, 1+\alpha+\beta+m;\ -n, 1+\gamma+\delta+n; \\ 1+\alpha, p;\ 1+\gamma, q; \end{matrix} \frac{-xt}{1-t}, \frac{y}{1-t}\right],
\end{aligned} \tag{9}$$

where the double hypergeometric function is in general defined by ((1), p. 270)

$$\begin{aligned}
{}_{p+1}F_p(2)\left[\begin{matrix} \alpha;\ \beta_1,\ldots,\beta_p;\ \beta_1',\ldots,\beta_p'; \\ \gamma_1,\ldots,\gamma_p;\ \gamma_1'\ldots,\gamma_p'; \end{matrix} x,y\right] \\
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m\cdots(\beta_p)_m(\beta_1')_n\cdots(\beta_p')_n}{m!\,n!\,(\gamma_1)_m\cdots(\gamma_p)_m(\gamma_1')_n\cdots(\gamma_p')_n}\,x^m y^n.
\end{aligned}$$

Proof. We start with the L.H.S.

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(p)_k}{k!} H_m^{(\alpha,\beta)}(-k,p,x)\,H_n^{(\gamma,\delta)}(p+k,q,y)\,t^k \\
&\quad = \frac{(1+\gamma)_n}{n!} \sum_{k=0}^{\infty}\sum_{r=0}^{n} \frac{(p)_{k+r}(-n)_r(1+\gamma+\delta+n)_r}{k!\,r!\,(1+\gamma)_r(q)_r} H_m^{(\alpha,\beta)}(-k,p,x)\,t^k y^r \\
&\quad = \frac{(1+\gamma)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r(p)_r(1+\gamma+\delta+n)_r}{r!\,(1+\gamma)_r(q)_r} y^r \sum_{k=0}^{\infty} \frac{(p+r)_k}{k!} H_m^{(\alpha,\beta)}(-k,p,x)\,t^k \\
&\quad = \frac{(1+\alpha)_m(1+\gamma)_n}{m!\,n!}(1-t)^{-p} \sum_{r=0}^{n} \frac{(-n)_r(p)_r(1+\gamma+\delta+n)_r}{r!\,(1+\gamma)_r(q)_r} \left(\frac{y}{1-t}\right)^r \\
&\qquad \times {}_3F_2\left(-m, p+r, 1+\alpha+\beta+m;\ 1+\alpha, p;\ \frac{-xt}{1-t}\right) \quad \text{by} \quad (5) \\
&\quad = \frac{(1+\alpha)_m(1+\gamma)_n}{m!\,n!}(1-t)^{-p} \\
&\qquad \times {}_3F_2(2)\left[\begin{matrix} p;\ -m, 1+\alpha+\beta+m;\ -n, 1+\gamma+\delta+n; \\ 1+\alpha, p;\ 1+\gamma, q; \end{matrix} \frac{-xt}{1-t}, \frac{y}{1-t}\right],
\end{aligned}$$

which completes the proof of (9).

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In case $\alpha = \beta = 0 = \gamma = \delta$, (9) yields

$$\sum_{k=0}^{\infty} \frac{(p)_k}{k!} H_m(-k, p, x)\, H_n(p+k, q, y)\, t^k = (1-t)^{-p}\, {}_3F_2(2)\left[\begin{matrix} p;\ -m\ 1+m;\ -n, 1+n; \\ 1, p;\ 1, q; \end{matrix} \frac{-xt}{1-t}, \frac{y}{1-t}\right]. \quad (10)$$

5. The fourth formula to be proved is

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(-\alpha-\beta)_n} H_n^{(\alpha-n, \beta-n)}(\xi, p, x)\, P_{m+n}^{(\gamma-n, \delta-n)}(y)\, t^n = (1+\gamma+\delta+m)_m [\tfrac{1}{2}(y-1)]^m \left[\frac{y-1}{y+1}\right]^{\delta} \times F_A\left[-\gamma-\delta-m;\ -\alpha, \xi, -\delta-m;\ -\alpha-\beta, p, -\gamma-\delta-2m; \times \tfrac{1}{2}(1+y)t, -\tfrac{1}{2}x(1+y)t, \frac{2}{1-y}\right], \quad (11)$$

where

$$F_A(\alpha;\ \beta, \beta', \beta'';\ \gamma, \gamma', \gamma'';\ x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta)_m (\beta')_n (\beta'')_p}{m!\, n!\, p!\, (\gamma)_m (\gamma')_n (\gamma'')_p} x^m y^n z^p.$$

Proof. Starting with the L.H.S. we have

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(-\alpha-\beta)_n} H_n^{(\alpha-n, \beta-n)}(\xi, p, x)\, P_{m+n}^{(\gamma-n, \delta-n)}(y)\, t^n = (1+\gamma+\delta+m)_m [\tfrac{1}{2}(y+1)]^m \sum_{n=0}^{\infty} \frac{(-\gamma-\delta-m)_n}{(-\alpha-\beta)_n} H_n^{(\alpha-n, \beta-n)}(\xi, p, x) \times {}_2F_1\left(-m-n, -\delta-m;\ -\gamma-\delta-2m;\ \frac{2}{y+1}\right) [-\tfrac{1}{2}(1+y)t]^n.$$

Using Euler's transformation ((3), p. 60)

$${}_2F_1(\alpha, \beta;\ \gamma;\ x) = (1-x)^{-\alpha}\, {}_2F_1\left(\alpha, \gamma-\beta;\ \gamma;\ -\frac{x}{1-x}\right),$$

it becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(m+n)!}{(-\alpha-\beta)_n} H_n^{(\alpha-n, \beta-n)}(\zeta, p, x)\, P_{m+n}^{(\gamma-n, \delta-n)}(y)\, t^n \\
&= (1+\gamma+\delta+m)_m [\tfrac{1}{2}(y-1)]^m \left(\frac{y-1}{y+1}\right)^{\delta} \sum_{k=0}^{\infty} \frac{(-\gamma-\delta-m)_k (-\delta-m)_k}{k!\, (-\gamma-\delta-2m)_k} \left(\frac{2}{1-y}\right)^k \\
&\quad \times \sum_{n=0}^{\infty} \frac{(-\gamma-\delta-m+k)_n}{(-\alpha-\beta)_n} H_n^{(\alpha-n, \beta-n)}(\xi, p, x) [-\tfrac{1}{2}(1+y)t]^n \\
&= (1+\gamma+\delta+m)_m [\tfrac{1}{2}(y-1)]^m \left(\frac{y-1}{y+1}\right)^{\delta} \sum_{k=0}^{\infty} \frac{(-\gamma-\delta-m)_k (-\delta-m)_k}{k!\, (-\gamma-\delta-2m)_k} \left(\frac{2}{1-y}\right)^k \\
&\quad \times F_2[-\gamma-\delta-m+k;\ -\alpha, \xi;\ -\alpha-\beta, p;\ \tfrac{1}{2}(1+y)t, -\tfrac{1}{2}x(1+y)t] \quad \text{by} \quad (7) \\
&= (1+\gamma+\delta+m)_m [\tfrac{1}{2}(y-1)]^m \left(\frac{y-1}{y+1}\right)^{\delta} \\
&\quad \times F_A\left[-\gamma-\delta-m;\ -\alpha, \zeta, -\delta-m;\ -\alpha-\beta, p, -\gamma-\delta-2m; \times \tfrac{1}{2}(1+y)t, -\tfrac{1}{2}x(1+y)t, \frac{2}{1-y}\right].
\end{aligned}$$

This completes the proof of (11).

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Now putting $p = \xi$ and replacing x by $\frac{1}{2}(1-x)$, it follows that

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x)\, P_{m+n}^{(\gamma-n,\delta-n)}(y)\, t^n$$

$$= (1+\gamma+\delta+m)_m [\tfrac{1}{2}(y-1)]^m \left(\frac{y-1}{y+1}\right)^{\delta} [1+\tfrac{1}{4}(1-x)(1+y)t]^{\gamma+\delta+m}$$

$$\times F_2\Bigg[-\gamma-\delta-m;\ -\alpha,\ -\delta-m;\ -\alpha-\beta,\ -\gamma-\delta-2m;$$

$$\times \frac{2(1+y)t}{4+(1-x)(1+y)t},\ \frac{8}{(1-y)\{4+(1-x)(1+y)t\}}\Bigg]. \quad (12)$$

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