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Permutable entire functions and Baker domains

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Abstract

Suppose that f and g are transcendental entire functions of finite order, without wandering domains, and suppose that $f \circ g = g \circ f$. Then the Julia sets of f and g coincide.

1. *Introduction*

If f is a transcendental entire function or a rational function of degree at least 2, the Fatou set N_f is the union of all open sets on which the iterates f^n, defined by $f^1 = f$ and $f^{n+1} = f \circ f^n$, form a normal family, the complement of N_f being the non-empty Julia set J_f. We refer the reader to [**3**, **9**] for comprehensive surveys of iteration theory.

It was proved by Julia [**7**, **9**] that if f and g are rational functions of degree at least two and f and g are permutable, that is, $f \circ g = g \circ f$, then $J_f = J_g$, and it is conjectured that this remains the case for transcendental entire functions f and g. The following result was proved by Baker [**2**].

Theorem A. *Suppose that f and g are permutable transcendental entire functions, and that U is a component of N_f. If it is not the case that f^n tends to infinity locally uniformly on U then $g(U) \subseteq N_f$.*

For if f^{n_k} tends to an analytic limit function ϕ on U then $f^{n_k} \circ g = g \circ f^{n_k}$ tends to $g \circ \phi$, so that $g(U)$ cannot meet J_f. Baker [**2**] went on to show that if there is no component of N_f on which f^n tends to infinity and the same holds with f replaced by g, then $J_f = J_g$. Further [**2**, lemma 4·5], if F and G are permutable transcendental entire functions with $F - G$ constant, then $J_F = J_G$.

In view of the classification of components of the Fatou set [**3**, theorem 6], there are only two ways in which f^n can tend to infinity locally uniformly on a component U_0 of N_f. The first is when U_0 is wandering, so that $f^n(U_0)$ is contained in a component U_n of N_f and $U_n \neq U_m$ for $m \neq n$. The second possibility is that for some $r \geqslant 0$ the image $f^r(U_0)$ is contained in a Baker domain V of f, for which $f^p(V) \subseteq V$ for some $p > 0$. In this second case U_0 must be unbounded and, by a result of Baker [**1**, **3**], simply connected.

The purpose of this note is to point out that, at least for entire functions of restricted growth, Baker domains do not present a problem as far as the conjecture on the Julia sets of permutable functions is concerned. We have the following result.

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THEOREM 1. *Let $\phi(r)$ be a continuous strictly increasing non-negative function on $[0, +\infty)$, such that*

$$\lim_{x\to+\infty} \left(\log \phi\left(D\phi^{-1}(x)\right) - x/2\right) = -\infty \tag{1}$$

for every constant $D > 1$. Let f and g be permutable transcendental entire functions such that $\log M(r, g) \leqslant \phi(r)$ for all large r. If V is a non-wandering component of N_f, then $g(V)$ is a subset of a non-wandering component of N_f, and $V \subseteq N_g$. If f does not have wandering domains, then $N_f \subseteq N_g$.

It seems likely that the theorem would remain true without the extra condition (1) on g, but it is easy to verify that (1) is satisfied by any function of form $\phi(r) = \exp_n(r)$, where $\exp_1 = \exp$ and $\exp_{n+1} = \exp \circ \exp_n$, so that the theorem applies even for functions of quite rapid growth. In particular, if f and g are permutable transcendental entire functions of finite order both without wandering domains then $J_f = J_g$. The proof is based on a combination of known estimates for iterates in Baker domains with an idea of Eremenko concerning points whose forward orbits tend to infinity.

2. *Estimates for functions with Baker domains*

We use estimates from lemma 7 of [**3**] but, since we require only a special case, we sketch a proof. Suppose that U is a Baker domain of f, necessarily unbounded and simply connected, such that f maps U into U and $f^n \to \infty$ on U. Let $w = h(u)$ map $\Delta = B(0, 1)$ analytically and univalently onto U. The hyperbolic metric [**8**] on Δ lifts to a metric $[w_1, w_2]$ on U which, with $w = h(u), u = h^{-1}(w)$, is the infimum over all curves C joining w_1 to w_2 in U of

$$\int_C \frac{|dw|}{|h'(u)|(1-|u|^2)} \geqslant \int_C \frac{|dw|}{c_0 \operatorname{dist}\{w, \partial U\}} \geqslant \int_C \frac{|dw|}{c_2(|w| + c_1)} \geqslant \int_{|w_1|}^{|w_2|} \frac{dt}{c_2(t + c_1)}$$

in which the positive constants c_j depend only on U and the first inequality follows from the Koebe one-quarter theorem. Let K be a compact subset of U. If z is in K then z and $f(z)$ both lie in the compact set $K_1 = K \cup f(K)$, and we have

$$c_4 \geqslant [z, f(z)] \geqslant [f^n(z), f^{n+1}(z)] \geqslant c_3 \log\left(\frac{|f^{n+1}(z)| + c_1}{|f^n(z)| + c_1}\right).$$

Since $f^n(z) \to \infty$ uniformly on K_1, we get

$$|f^{n+1}(z)| \leqslant A|f^n(z)|, \quad n \geqslant n_0, \quad z \in K, \tag{2}$$

in which n_0 and the constant A depend only on K.

Take z_0 in U and let σ be a path in U joining z_0 to $f(z_0)$. A path γ is formed as the union of the paths $f^n(\sigma), n = 0, 1, 2, \ldots$, and γ tends to infinity in U, with $f(\gamma) \subseteq \gamma$. Further, by (2) applied to σ,

$$|f(z)| \leqslant B|z| \quad \text{as} \quad z \to \infty \quad \text{on} \quad \gamma, \tag{3}$$

in which the constant B depends only on σ.

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LEMMA 1. *Let f be a transcendental entire function having at least one Baker domain. Then we have*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\frac{1}{2}}} > 0.$$

This follows at once from (3) and the classical cos $\pi\rho$ theorem [**6**, theorem 6·4].

3. *A method of Eremenko*

We require the following slight modification of a result of Eremenko [**4**] proving the existence of points for which the forward orbit tends to infinity.

LEMMA 2. *Let f be a transcendental entire function, having at least one Baker domain, and let $R > 0$. Then there exists w with $|w| > R$ such that*

$$f^n(w) \to \infty \quad \text{as} \quad n \to \infty, \quad \liminf_{n\to\infty} \frac{\log |f^{n+1}(w)|}{|f^n(w)|^{\frac{1}{2}}} > 0. \tag{4}$$

The proof of Lemma 2 proceeds as in Eremenko's method from [**4**], which is based on the Wiman–Valiron theory [**5**, **10**], but using Lemma 1. Let z_0 be large, with $|f(z_0)| = M(|z_0|, f)$, and for positive k define

$$D(z_0,k) = \{z\colon |\log|z/z_0|| < k/N(|z_0|), \quad |\arg(z/z_0)| < k/N(|z_0|)\},$$

with $N(r)$ the central index of f [**5**]. Provided $|z_0|$ lies outside a set E of finite logarithmic measure we have [**5**]

$$\log f(z) = \phi(z) + o(1), \quad \phi(z) = N(|z_0|)\log(z/z_0) + \log f(z_0), \quad z \in D(z_0, 32).$$

By Rouché's theorem, $\log f(z)$ is univalent on $D(z_0, 8)$ and the image of $D(z_0, 8)$ under $\log f(z)$ contains the region $\phi(D(z_0, 4))$. Thus $f(D(z_0, 8))$ covers the annulus $\{u\colon e^{-4}M(|z_0|, f) < |u| < e^4 M(|z_0|, f)\}$. Further, there are positive constants c_0, c_1, c_2 such that if $|z_0|$ is large enough we have, using Lemma 1,

$$\log|f(z)| \geqslant \log|f(z_0)| - c_0 \geqslant c_1|z_0|^{\frac{1}{2}} \geqslant c_2|z|^{\frac{1}{2}}, \quad z \in D(z_0, 8). \tag{5}$$

Choose z_1 with $|z_1| \notin E$ and with $|f(z_1)| = M(|z_1|, f)$ and $M(|z_0|, f)/2 \leqslant |z_1| \leqslant 2M(|z_0|, f)$. Repeating this process we obtain maximum modulus points z_j of f tending to infinity such that $D(z_{j+1}, 8) \subseteq f(D(z_j, 8))$. Let C_j be a subset of $D(z_j, 8)$ mapped univalently onto $D(z_{j+1}, 8)$ by f and define a branch of f^{-1} mapping $D(z_{j+1}, 8)$ onto C_j. Setting $B_j = f^{-j}(C_j)$ for $j = 1, 2, \ldots$, we have

$$B_{j+1} = f^{-j}f^{-1}(C_{j+1}) \subseteq f^{-j}(C_j^*) = B_j^*,$$

in which C_j^* and B_j^* are compact subsets of C_j and B_j respectively. Thus the closure of B_{j+1} is contained in B_j and we need only choose w lying in the intersection of the B_j. Since $f^n(w)$ lies in C_n the first assertion of (4) is obvious, while the second follows from (5).

4. *Proof of Theorem* 1

LEMMA 3. *Let f, g be permutable transcendental entire functions such that $\log M(r, g) \leqslant \phi(r)$ for all large r, where $\phi(r)$ satisfies* (1) *for every $D > 1$. Let U be a Baker domain of f such that $f^p(U) \subseteq U$. Then $g(U) \subseteq N_f$.*

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Proof. Since g and f^p are permutable we may assume that $p = 1$. Thus

$$f^n(g(U)) = g(f^n(U)) \subseteq g(U).$$

Suppose that the complement of $g(U)$ in the plane contains at most one point. Then we may choose z in U such that $w = g(z)$ satisfies (4), so that $x_n = \log |f^n g(z)|$ tends to infinity with

$$\log x_{n+1} \geqslant \tfrac{1}{2}x_n - O(1) \tag{6}$$

for all large n. However, since $f^n g(z) = g f^n(z)$ we have, by (2), the estimates $x_n \leqslant \phi(C^n)$ and, consequently, $\phi^{-1}(x_n) \leqslant C^n$ for some $C > 0$, independent of n, from the second of which it follows that we must have

$$\phi^{-1}(x_{n+1}) \leqslant 2C\phi^{-1}(x_n) \quad \text{and} \quad x_{n+1} \leqslant \phi(2C\phi^{-1}(x_n))$$

for arbitrarily large n. For these n we therefore have, using (6),

$$\tfrac{1}{2}x_n - O(1) \leqslant \log x_{n+1} \leqslant \log \phi(2C\phi^{-1}(x_n)),$$

which contradicts (1). Hence the complement of $g(U)$ in the plane contains at least two points and so (f^n) is normal on $g(U)$.

We may now complete the proof of the theorem. Let V be a non-wandering component of N_f, and assume that f^r maps V into a component U of N_f such that $f^p(U) \subseteq U$, where $r \geqslant 0$ and $p > 0$. Then $g(U)$ is contained in a component W of N_f, by Lemma 3 if U is a Baker domain and by Theorem A otherwise. Further,

$$f^{pm+r}(g(V)) = g(f^{pm+r}(V)) \subseteq g(U) \subseteq W \subseteq N_f,$$

which shows first that $g(V) \subseteq N_f$, by the complete invariance of N_f and, second, that $g(V)$ is contained in a non-wandering component of N_f. Repeating this argument we find that $g^m(V) \subseteq N_f$ and (g^m) is normal on V.

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