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Nonlinear Oscillations Analysis of the Elevator Cable in a Drum Drive Elevator System

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Abstract. This paper aims to investigate nonlinear oscillations of an elevator cable in a drum drive. The governing equation of motion of the objective system is developed by virtue of Lagrangian's method. A complicated term is broached in the governing equation of the motion of the system owing to existence of multiplication of a quadratic function of velocity with a sinusoidal function of displacement in the kinetic energy of the system. The obtained equation is an example of a well-known category of nonlinear oscillators, namely, non-natural systems. Due to the complex terms in the governing equation, perturbation methods cannot directly extract any closed form expressions for the natural frequency. Unavoidably, different non-perturbative approaches are employed to solve the problem and to elicit a closed-form expression for the natural frequency. Energy balance method, modified energy balance method and variational approach are utilized for frequency analyzing of the system. Frequency-amplitude relationships are analytically obtained for nonlinear vibration of the elevator's drum. In order to examine accuracy of the obtained results, exact solutions are numerically obtained and then compared with those obtained from approximate closed-form solutions for several cases. In a parametric study for different nonlinear parameters, variation of the natural frequencies against the initial amplitude is investigated. Accuracy of the three different approaches is then discussed for both small and large amplitudes of the oscillations.

AMS subject classifications: 70K30, 70E55

Key words: Nonlinear oscillation, perturbation methods, non-perturbative approach, frequency-amplitude relationship.

1 Introduction

The nonlinearity and complexity of the real world phenomena enforce leading edge scientists to develop innovative ways for understanding of the enigmatic behavior of the

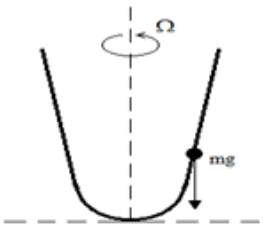
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nature. There is a broad compendium of detailed information and knowledge regarding nonlinear systems in different outstanding books. Inevitably, a number of researchers have devoted their time and effort to find potent approaches for investigating of the nonlinear phenomena. Owing to the difficulty of the corresponding governing equations, a plenty of approaches have been developed so far to solve such sophisticated problems. As the earliest effort in the mysterious and wondrous road of the investigation of the nonlinear phenomena, perturbation-based methods can be referred [1,2]. The abovementioned types of straightforward methods have been widely used and modified mainly by Nayfeh et al. [1–3] and recently by other researchers [4–12]. They have been also employed and ameliorated for analyzing of diverse types of nonlinear structural systems. Adomian Decomposition method [16], Variational iteration method [17] and Homotopy analysis method [18–20] have been recently broached by researchers to solve nonlinear systems. In comparison with perturbation methods, more profound knowledge and perception can be provided using the aforesaid new analytical methods [21–26]. Hamiltonian approach [27], Energy Balance Method [28], Modified Energy Balance Method [29], Max-Min approach [30], Variational approach [31] and frequency amplitude formulation [32] can be cited as newly developed non-perturbative approaches. Efficiency and potency of these analytical methods have been proved in variety of complicated nonlinear cases, namely, non-natural systems, non-Hamiltonian systems, fractional order systems and generalized Duffing systems [33–42]. Many different kinds of non-natural systems have been analyzed so far by several researchers [8, 43–50]. The following Examples 1.1-1.6 (with models and governing equations) of non-natural systems succinctly delineates a plenty of well-known non-natural systems which have been analytically investigated so far. In the present study, three different non-perturabtive approaches are employed to extract frequency-amplitude relationship for nonlinear oscillations of an elevator cable in a drum drive elevator system. The governing equation of the nonlinear system is developed using Lagrangian's method. In accordance with the obtained ODE, it can be mentioned that the considered system is a non-natural structure.

Example 1.1. Motion of a particle on rotating parabola [8, 43, 44].



The governing equation is

$$(1+4p^2x^2)\ddot{x}+\Lambda x+4p^2\dot{x}^2x=0.$$

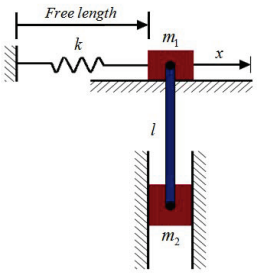
The motion of the system described by the following parabola in which p is a positive

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constant

$$z = px^2.$$

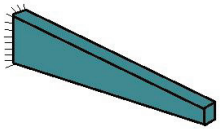
Example 1.2. Vibration of two-mass spring system [8].



The governing equation is

$$(m_1 + \frac{m_2 x^2}{l^2 - x^2})\ddot{x} + \frac{m_2 l^2 x \dot{x}^2}{(l^2 - x^2)^2} + kx + m_2 g \frac{x}{(l^2 - x^2)^{1/2}} = 0.$$

Example 1.3. Vibration of a tapered beam [45].

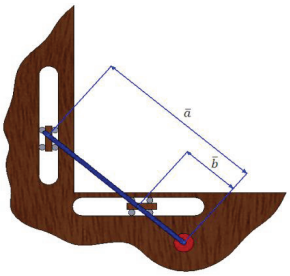


The governing equation is

$$\ddot{u} + \alpha_0 u + \alpha_1 u^3 + \beta_1 [u^2 \ddot{u} + u \dot{u}^2] = 0.$$

α_0, α_1, β_1 are constant parameters which can be obtained using the Galerkin procedure from the governing equation of the system.

Example 1.4. Vibration of the trammel pendulum [46].



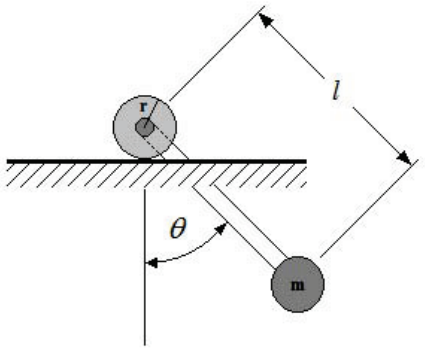
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The governing equation is

$$(\bar{a}\cos^2\theta+\bar{b}\sin^2\theta)\ddot{\theta}+0.5(\bar{b}^2-\bar{a}^2)\dot{\theta}^2+g\bar{b}\sin\theta=0.$$

Define g, θ, g is gravitational acceleration, θ represents rotational oscillations of the trammel pendulum.

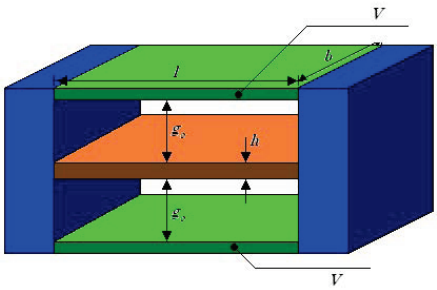
Example 1.5. Vibration of a pendulum attached to a rolling wheel [8].



The governing equation is

$$(l^2+r^2-2rl\cos\theta)\ddot{\theta}+rl\sin\theta\dot{\theta}^2+gl\sin\theta=0.$$

Example 1.6. Vibration of a micro-electromechanical system [47].



The governing equation is

$$\ddot{x}(a_1x^4+a_2x^2+a_3)+a_4x+a_5x^3+a_6x^5+a_7x^7=0.$$

α_0, α_1, α_2, α_3, α_4, α_5, α_6, α_7 are constant parameters which can be obtained using the Galerkin procedure from the governing equation of the system.

The energy balance method, modified energy balance method and variational approach are applied to delineate the frequency-amplitude relationship of the objective

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problem. Three different and precise frequency-amplitude relationships are analytically achieved for large oscillations of the elevator's drum. Parameter expansion method which was already employed in an earlier paper [50] is actually a specific type of the perturbation approach [51]. All of the employed approaches are very straightforward in comparison with parameter expansion method, and they are very efficient for solving the governing equations without any demand to Taylor series expansion. Owing to the above-mentioned capability of the employed methods, the obtained solutions herein are more accurate than those obtained from parameter expansion method. The exact solutions (in addition to numerical solutions [50]) of the target system have been developed in this study in order to verify the obtained analytical solutions. In order to avere the axiomatic amelioration of the obtained analytical solutions, they are tabulated to be compared with those found from the exact numerical solutions. It is corroborated the obtained analytical solutions are palpable improvement and very contiguous to the exact response of the system.

It is demonstrated that the utilized methods are rather reliable and efficient for elicitation of the periodic solutions of the corresponding non-natural systems. A parametric study is then carried out and natural-frequencies are plotted against nonlinear parameters. In addition, the frequency ratio (defined as the ratio of the nonlinear natural frequency to the linear one) is obtained and discussed for different initial amplitudes. The most optimal technique for both small and large amplitudes of oscillations is eventually recognized.

2 Mathematical modeling

A physical model of an elevator cable in a drum drive system is schematically shown in Fig. 1.

To derive the governing equation of motion of the system, the Lagrangian method is

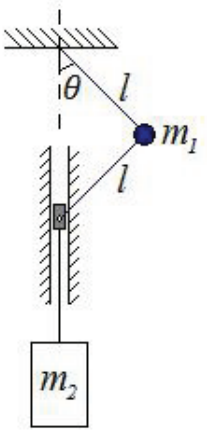


Figure 1: Elevator cable in a drum drive elevator system [50].

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applied, i.e.,

$$T=l^2\dot{\theta}^2\Big(\frac{1}{2}m_1+2m_2\sin^2\theta\Big), \tag{2.1a}$$

$$U=gl(1-\cos\theta)(m_1+2m_2), \tag{2.1b}$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\theta}}\Big)-\frac{\partial L}{\partial\theta}=0,\quad L=T-U. \tag{2.1c}$$

Where T and U stand for the kinetic and potential energy of the system, respectively. Thus, the governing equation is obtained as

$$l\ddot{\theta}(m_1+4m_2\sin^2\theta)+2l\dot{\theta}^2\sin2\theta+g(m_1+2m_2)\sin\theta=0 \tag{2.2}$$

or

$$\ddot{\theta}(1+2\alpha\sin^2\theta)+\alpha\dot{\theta}^2\sin2\theta+\beta\sin\theta=0,\quad \text{where } \alpha=\frac{2m_2}{m_1},\quad \beta=\frac{g}{l}(1+\alpha). \tag{2.3}$$

It is important to note that the nonlinear coefficient that appears in the inertial part of the above equation cannot be ignored. The magnitude of α is reasonably large in real applications and hence one must avoid such a simplification.

3 Solution procedure

3.1 Energy balance method

Eq. (2.3) is analytically solved using the EBM in this section. The corresponding variational principle can be obtained as

$$J=\int_0^t\Big[-\frac{1}{2}\dot{\theta}^2(1+2\alpha\sin^2\theta)-\beta\cos\theta\Big]dt. \tag{3.1}$$

Supporting Hamiltonian is obtained to be

$$H=\frac{1}{2}\dot{\theta}^2(1+2\alpha\sin^2\theta)-\beta\cos\theta=-\beta\cos A. \tag{3.2}$$

The initial conditions are assumed to be

$$\theta(0)=A,\quad \dot{\theta}(0)=0. \tag{3.3}$$

Satisfying Eq. (3.3), the function $\theta=A\cos\omega t$ is taken into account to be the trail function. The following residual function is then obtained

$$R(t)=\frac{1}{2}A^2\omega^2\sin^2\omega t(1+2\alpha\sin^2(A\cos\omega t))-\beta\cos(A\cos\omega t)+\beta\cos(A)=0. \tag{3.4}$$

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According to the standard EBM procedure and employing the collocation method at $\omega t \to \pi/4$, the frequency-amplitude relationship is analytically obtained as

$$\omega_{EBM} = \frac{2}{A}\sqrt{\frac{\beta\left(\cos\left(\frac{\sqrt{2}}{2}A\right)-\cos(A)\right)}{1+2\alpha\sin^2\left(\frac{\sqrt{2}}{2}A\right)}}. \tag{3.5}$$

Here the Galerkin-Petrov technique is combined with the standard EBM to find more accurate solution. Accordingly, integral of the residue vanishes with the kernel of the trail function so that

$$\tilde{R} = \int_0^{T/4} R(t)\cos\omega t dt = 0. \tag{3.6}$$

Substituting Eq. (3.4) into Eq. (3.6), gives

$$\tilde{R} = \int_0^{\pi/2}\left[\frac{1}{2}A^2\omega^2\sin^2 t\left(1+2\alpha\sin^2(A\cos t)\right)-\beta\cos(A\cos(t))+\beta\cos(A)\right]\cos t dt = 0. \tag{3.7}$$

Consequently, the nonlinear frequency of the system obtains as

$$\omega_{MEBM} = \frac{1}{A}\sqrt{\frac{2\beta\int_0^{\pi/2}[\cos(A\cos t)-\cos A]\cos t dt}{\int_0^{\pi/2}[\sin^2 t(1+2\alpha\sin^2(A\cos t))]\cos t dt}}. \tag{3.8}$$

3.2 Variational approach

Variational approach is applied to solve Eq. (2.3) in this section. According to the principles of the method [31] and using Eq. (3.1), the variational formulation for the system is obtained as

$$J(A) = \int_0^{T/4}\left[-\frac{1}{2}A^2\omega^2\sin^2\omega t(1+2\alpha\sin^2(A\cos\omega t))-\beta\cos(A\cos\omega t)\right]dt. \tag{3.9}$$

The stationary condition with respect to A reads

$$\begin{aligned}\frac{\partial J}{\partial A} = \int_0^{T/4}\big[&-A\omega^2\sin^2\omega t(1+2\alpha\sin^2(A\cos\omega t)+\alpha A\sin(2A\cos\omega t)\cos\omega t)\\ &+\beta\sin(A\cos\omega t)\cos\omega t\big] = 0.\end{aligned} \tag{3.10}$$

From the above equation, the closed form frequency-amplitude is given by

$$\omega_{VA} = \sqrt{\frac{\beta\int_0^{\pi/2}\sin(A\cos t)\cos t dt}{\int_0^{\pi/2}A\sin^2 t(1+2\alpha\sin^2(A\cos t)+\alpha A\sin(2A\cos t)\cos t)dt}}. \tag{3.11}$$

Besides, by using the elliptic integration technique the exact numerical value of natural frequency is obtained from Eq. (2.3) to be

$$\omega_E = \frac{\pi}{2A}\left[\int_0^{\pi/2}\left(\frac{(1+2\alpha\sin^2(A\sin\varphi))\cos\varphi}{2\beta(\cos(A\sin\varphi)-\cos A)}\right)^{1/2}d\varphi\right]^{-1}. \tag{3.12}$$

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4 Discussion and numerical results

In this part, a numerical analysis is utilized to examine validity of the obtained results. Influences of varying amplitude are evaluated on the natural frequencies of the system. Furthermore, effects of the nonlinear parameters on the frequency responses of the elevator's drum are studied. Figs. 2 to 4 demonstrate time responses of the system for different initial amplitudes and nonlinear parameters in a full period of oscillation. It is seen that, variational approach is the most accurate technique for analyzing of such a system.

The modified energy balance method furnishes more accurate solution than the classical EBM in case of the large amplitude of the oscillations.

In contrast, classical EBM predicts more precise results in small amplitude oscillations and in weakly nonlinear systems. To give a clearer picture of this accuracy analysis, the achieved natural frequencies are listed in Table 1.

As evidenced by results of Table 1, the variational approach yields the most optimal solution method for obtaining the natural frequencies of the elevator's drum. Besides, Figs. 5 and 6 represent phase-plane trajectories of the target system with respect to differ-

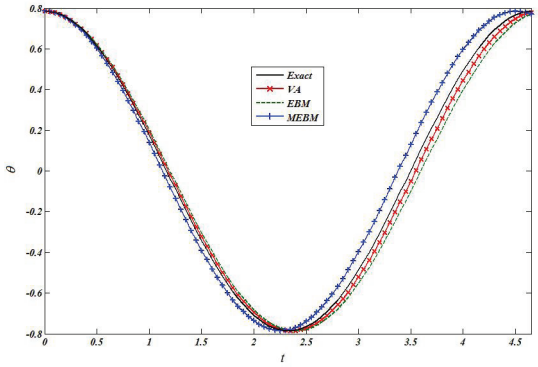


Figure 2: Comparison of the system response obtained by different methods for $A=\pi/4$, $\alpha=2$, $\beta=4$.

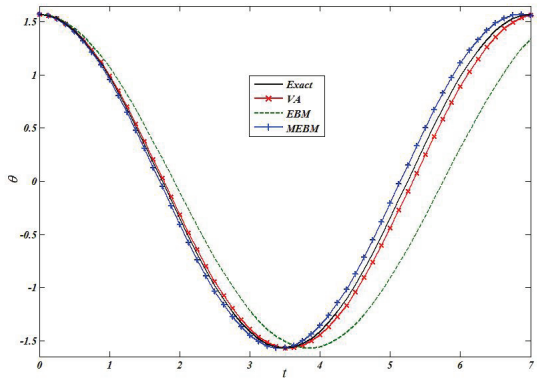


Figure 3: Comparison of the system response obtained by different methods for $A=\pi/2$, $\alpha=2$, $\beta=4$.

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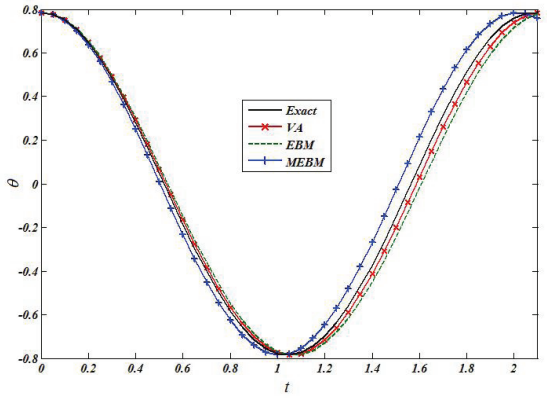


Figure 4: Comparison of the system response obtained by different methods for $A=\pi/4$, $\alpha=2$, $\beta=20$.

ent parameters. It is found that nonlinear equation of the objective system has a conservative type trajectory. Moreover, *C* it is revealed that the elevator's drum is a Hamiltonian system. The obtained solutions are also verified in the phase-plane domain. Nonlinear natural frequencies are plotted with respect to a set of varying nonlinear parameters in Figs. 7 and 8. These figures show that (a) by an increase in the nonlinear parameter α (mass ratio), value of the nonlinear natural frequency increases and (b) the accuracy of the modified energy balance method is enhanced when the nonlinearity index increases. The variational approach is found to be the most accurate method for solving the objective non-natural system problem.

Non-dimensional frequency ratios are plotted against the variations of the initial conditions in Figs. 9 and 10. As evidenced by those figures, value of the frequency ratios decreases by an increase in the initial amplitude of the oscillation. The abovementioned result plays a very crucial role in the dynamicl behavior of the objective system as well as

Table 1: Natural frequencies (rad/s) and relative errors for various lengths of connecting link $(A=\pi/4, \alpha=2)$.

$l(m)$	ω_{VA} (Relative Errors %)	ω_{EBM} (Relative Errors %)	ω_{EMEBM} (Relative Errors %)	ω_E
0.1	9.394505 (1.427159)	9.268540 (2.748862)	9.857774 (3.433735)	9.530521
0.5	4.201350 (1.428298)	4.145017 (2.749983)	4.408530 (3.432541)	4.262228
1	2.9708033 (1.428297)	2.930969 (2.749985)	3.117302 (3.432543)	3.013850
5	1.328583 (1.428701)	1.310769 (2.750383)	1.394099 (3.432119)	1.3478402
10	0.939450 (1.428703)	0.926854 (2.750383)	0.985777 (3.432119)	0.953067
15	0.767058 (1.428804)	0.756773 (2.750482)	0.804884 (3.432013)	0.778177

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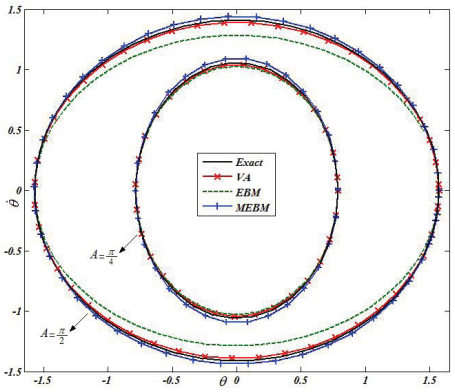


Figure 5: Phase-plane trajectories of the system $A=\pi/4,\ \pi/2,\ \alpha=2,\ \beta=4$.

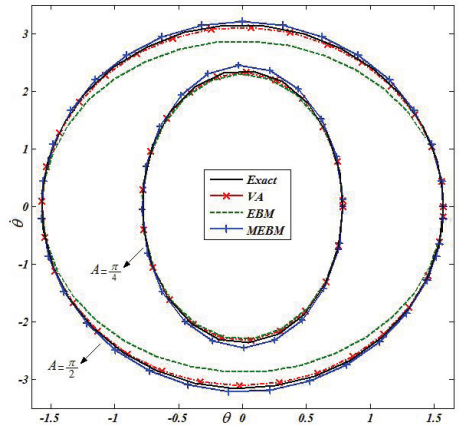


Figure 6: Phase-plane trajectories of the system $A=\pi/4,\ \pi/2,\ \alpha=2,\ \beta=20$.

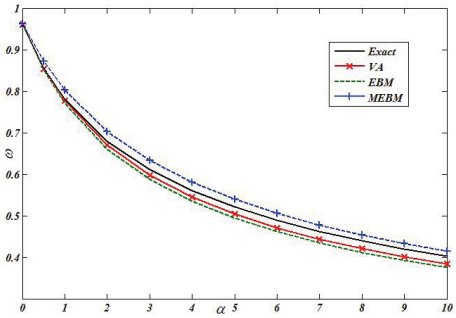


Figure 7: Variation of natural frequency with respect to α for $A=\pi/4,\ \beta=1$.

real applications. Indeed, it should be highlighted that the real natural frequency is less than those found from linearization theories.

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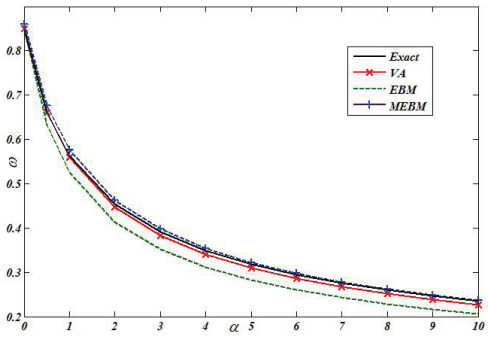


Figure 8: Variation of natural frequency with respect to α for $A=\pi/2$, $\beta=1$.

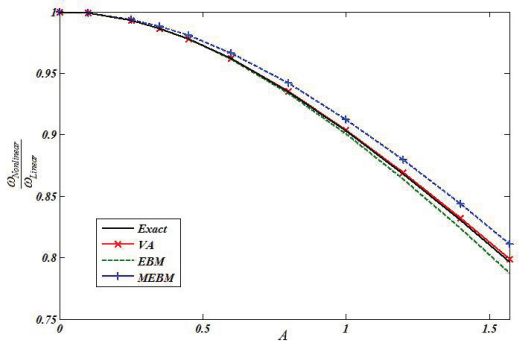


Figure 9: Variation of frequency ratio with respect to initial amplitude $(\alpha=0.1,\ \beta=2)$.

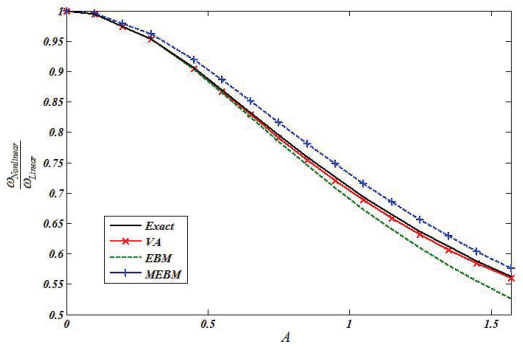


Figure 10: Variation of frequency ratio with respect to initial amplitude $(\alpha=1,\ \beta=10)$.

5 Conclusions

In this paper, nonlinear oscillations of an elevator cable in a drum drive elevator system was fully scrutinized. In order to obtain the governing equation of motion of the objective system, Lagrangian's method was exerted. Three non-perturbative approaches, namely, variational approach, classical energy balance method and modified energy

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balance method were performed to obtain frequencyCamplitude relationship of the target system. The extracted closed form solutions were implemented as a potent tool to meticulously investigate dynamical behavior of the considered system with respect to the initial amplitudes as well as nonlinear parameters. Exact numerical solutions were obtained to analyze potency and validity of the employed approaches. The following points can be highlighted as chief conclusions of the present study:

1. Variational approach is the most optimal and efficient approach for the elicitation of the frequency-amplitude relationship of the oscillations of the elevator's drum among utilized methods.
2. The effectiveness and potency of the modified energy balance method were proved in comparison with the classical energy balance method when the target system oscillates with both large amplitude and strong nonlinearity.
3. It has been represented that an increment in the nonlinear factor α (mass ratio), results in decrement of the value of the natural frequency of the objective system.
4. The value of the natural frequency decreases by an enhancement of the initial amplitude.
5. The conservative behavior of the system was C delineated using phase plane trajectories.
6. In accordance with the exact numerical solutions, it was depicted that the exerted methods all have excellent accuracy and reliability.

References

[1] A. H. NAYFEH, Problems in Perturbation, Wiley, New York, 1985.
[2] A. H. NAYFEH, Perturbation Methods, Wiley, New York, 1973.
[3] A. H. NAYFEH AND D. T. MOOK, Nonlinear Oscillations, Wiley, New York, 1979.
[4] E. J. HINCH, Perturbation Methods, Cambridge University Press, London, 1991.
[5] D. YOUNESIAN, E. ESMAILZADEH AND R. SEDAGHATI, *Existence of periodic solutions for generalized form of Mathieu equation,* Nonlinear Dyn., 39 (2005), pp. 335–348.
[6] I. KOVACIC AND M. J. BRENNAN, The Duffing Equation Nonlinear Oscillators and Their Behavior, Wiley, New York, 2011.
[7] D. YOUNESIAN, E. ESMAILZADEH AND R. SEDAGHATI, *Asymptotic solutions and stability analysis for generalized non-homogeneous Mathieu equation,* Commun. Nonlinear Sci. Numer. Sim., 12 (2007), pp. 58–71.
[8] M. BELHAQ AND F. LAKRAD, *The elliptic multiple scales method for a class of autonomous strongly non-linear oscillators,* J. Sound Vibration, 234 (2000), pp. 547–553.
[9] H. N. ARAFAT AND A. H. NAYFEH, *Non-linear responses of suspended cables to primary resonance excitations,* J. Sound Vibration, 266 (2003), pp. 325–354.
[10] A. H. NAYFEH, *The response of multidegree-of-freedom systems with quadratic non-linearities to a harmonic parametric resonance,* J. Sound Vibration, 90 (1983), pp. 237–244.
[11] H. N. ARAFAT AND A. H. NAYFEH, *Non-linear responses of suspended cables to primary resonance excitations,* J. Sound Vibration, 266 (2003), pp. 325–354.

https://doi.org/10.4208/aamm.2013.m225 Published online by Cambridge University Press

[12] A. H. NAYFEH, *The response of non-linear single-degree-of-freedom systems to multifrequency excitations,* J. Sound Vibration, 102 (1985), pp. 403–414.
[13] R. H. RAND, S. M. SAH AND M. K. SUCHORSKY, *Fractional Mathieu equation,* Commun. Nonlinear Sci. Numer. Sim., 15 (2010), pp. 3254–3262.
[14] D. W. STORTI AND R. H. RAND, *Dynamics of two strongly coupled van der pol oscillators,* Int. J. Non-Linear Mech., 17 (1982), pp. 143–152.
[15] R. RAND, K. GUENNOUN AND M. BELHAQ, *Resonance in the quasi-periodic Mathieu equation,* Nonlinear Dyn., 31 (2003), pp. 367–374.
[16] G. ADOMIAN, *A review of the decomposition method in applied mathematics,* J. Math. Anal. Appl., 135 (1988), pp. 501–544.
[17] J. H. HE, G. CH. WU AND F. AUSTIN, *The variational iteration method which should be followed,* Nonlinear Sci. Lett. A, 1 (2010), pp. 1–30.
[18] S. J. LIAO, Beyond Perturbation-Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC, Boca Raton, 2003.
[19] S. J. LIAO, *On the homotopy analysis method for nonlinear problems,* Appl. Math. Comput., 147 (2004), pp. 499–513.
[20] S. J. LIAO, *A kind of approximate solution technique which does not depend upon small parameters: a special example,* Int. J. Non-Linear Mech., 30 (1995), pp. 371–380.
[21] D. YOUNESIAN, H. ASKARI, Z. SAADATNIA AND A. YILDIRIM, *Analytical solution for nonlinear wave propagation in shallow media using the variational iteration method,* Waves Random Complex Media, (2012), doi:10.1080/17455030.2011.633578.
[22] D. YOUNESIAN, Z. SAADATNIA AND H. ASKARI, *Analytical solutions for free oscillations of beams on nonlinear elastic foundation using the variational iteration method,* J. Theor. Appl. Mech., 50 (2012), pp. 639–652.
[23] A. M. WAZWAZ, *The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations,* Int. J. Comput. Math., 87 (2010), pp. 1131–1141.
[24] H. CHEN, J. SHEN AND Z. ZHOU, *On the application of the homotopy analysis method to limit cycles approximation in planar self-excited systems,* Commun. Nonlinear Sci. Numer. Sim., 17 (2012), pp. 4494–4507.
[25] J. BIAZAR, M. GHOLAMI PORSHOKUHI AND B. GHANBARI, *Extracting a general iterative method from an Adomian decomposition method and comparing it to the variational iteration method,* Comput. Math. Appl., 59 (2010), pp. 622–628.
[26] L. XU, *Variational iteration method for solving integral equations,* Comput. Math. Appl., 54 (2007), pp. 1071–1078.
[27] J. H. HE, *Preliminary report on the energy balance for nonlinear oscillations,* Mech. Res. Commun., 29 (2002), pp. 107–111.
[28] J. H. HE, *Hamiltonian approach to nonlinear oscillators,* Phys. Lett. A, 374 (2010), pp. 2312–2314.
[29] M. KALAMIYAZDI, Y. KHAN, M. MADANI, H. ASKARI, Z. SAADATNIA AND A. YILDIRIM, *Analytical solutions for autonomous conservative nonlinear oscillator,* Int. J. Nonlinear Sci. Numer. Sim., 11 (2010), pp. 979–984.
[30] J. H. HE, *Max-min approach to nonlinear oscillators,* Int. J. Nonlinear Sci. Numer. Sim., 9 (2008), pp. 207–210.
[31] J. H. HE, *Variational approach for nonlinear oscillators,* Chaos Solitons Fractals, 34 (2007), pp. 1430–1439.
[32] J. H. HE, *An Improved amplitude-frequency formulation for nonlinear oscillators,* Int. J. Nonlinear Sci. Numer. Sim., 9 (2008), pp. 211–212.
[33] H. ASKARI, M. K. YAZDI AND Z. SAADATNIA, *Frequency analysis of nonlinear oscillators with*

https://doi.org/10.4208/aamm.2013.m225 Published online by Cambridge University Press

rational restoring force via He's energy balance method and He's variational approach, Nonlinear Sci. Lett. A, 1 (2010), pp. 425–430.

[34] H. Askari, Z. Saadatnia, D. Younesian, A. Yildirim and M. KalamiYazdi, *Approximate periodic solutions for the Helmholtz-Duffing equation,* Comput. Math. Appl., 62 (2011), pp. 3894–3901.

[35] L. Cveticanin, M. KalamiYazdi and H. Askari, *Analytical approximations to the solutions for a generalized oscillator with strong nonlinear terms,* J. Eng. Math., DOI 10.1007/s10665-012-9542-4.

[36] A. Yildirim, H. Askari, Z. Saadatnia, M. KalamiYazdi and Y. Khan, *Analysis of nonlinear oscillations of a punctual charge in the electric field of a charged ring via a Hamiltonian approach and the energy balance method,* Comput. Math. Appl., 62 (2011), pp. 486–490.

[37] L. Cveticanin, M. KalamiYazdi, Z. Saadatnia and H. Askari, *Application of Hamiltonian approach to the generalized nonlinear oscillator with fractional power,* Int. J. Nonlinear Sci. Numer. Sim., 11 (2010), pp. 997–1002.

[38] A. Yildirim, H. Askari, M. Kalami Yazdi and Y. Khan, *A relationship between three analytical approaches to nonlinear problems,* Appl. Math. Lett., doi:10.1016/j.aml.2012.02.001.

[39] A. Yildirim, Z. Saadatnia and H. Askari, *Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms,* Math. Comput. Model., 54 (2011), pp. 697–703.

[40] H. Askari, Z. SaadatNia and A. Yildrim, *Application of higher order Hamiltonian approach to nonlinear vibrating systems,* J. Theor. Appl. Mech., (2012), in Press.

[41] D. Younesian, Z. Saadatnia and H. Askari, *Analytical solutions for free oscillations of beams on nonlinear elastic foundations using the variational iteration method,* J. Theor. Appl. Mech., 50 (2012), pp. 639–652.

[42] A. Mohammadpour, E. Rokni, M. Fooladi and A. Kimiaeifar, *Approximate analytical solution for Bernoulli-Euler beams under different boundary conditions with non-linear Winkler type foundation*, J. Theor. Appl. Mech., 50 (2012), pp. 339–355.

[43] V. Marincaa and N. Herişanua, *Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method,* J. Sound Vibration, 329 (2010), pp. 1450–1459.

[44] B. S. Wu, C. W. Lim and L. H. He, *A new method for approximate analytical solutions to nonlinear oscillations of nonnatural systems,* Nonlinear Dyn., 32 (2003), pp. 1–13.

[45] M. Bayat, I. Pakar and M. Bayat, *Analytical study on the vibration frequencies of tapered beams,* Latin American J. Solids Structures, 8 (2011), pp. 149–162.

[46] M. I. Salem, V. H. Mucino, E. Saunders and M. Gautam, *Lateral sloshing in partially filled elliptical tanker trucks using a trammel pendulum,* Int. J. Heavy Vehicle Syst., 16 (2009), pp. 207–224.

[47] Y. Fu, J. Zhang and L. Wan, *Application of the energy balance method to a nonlinear oscillator arising in the microelectromechanical system (MEMS),* Current Appl. Phys., 11 (2011), pp. 482–485.

[48] N. Herisanu and V. Marinca, *Explicit analytical approximation to large-amplitude non-linear oscillations of a uniform cantilever beam carrying an intermediate lumped mass and rotary inertia,* Meccanica, 45 (2010), pp. 847–855.

[49] M. N. Hamdan and M. H. F. Dado, *Large amplitude free vibrations of a uniform cantilever beam carrying an intermediate lumped mass and rotary inertia,* J. Sound Vibration, 206 (1997), pp. 151–168.

[50] R. Mirabdollah Yani, R. Ghodsi and E. Darabi, *A closed-form solution for nonlinear*

https://doi.org/10.4208/aamm.2013.m225 Published online by Cambridge University Press

oscillation and stability analyses of the elevator cable in a drum drive elevator system experiencing free vibration, Commun. Nonlinear Sci. Numer. Sim., 17 (2012), pp. 4467–4484.

[51] D. H. SHOU AND J. H. HE, *Application of parameter-expanding method to strongly nonlinear oscillators*, Int. J. Nonlinear Sci. Numer. Sim., 8 (2007), pp. 121–124.

https://doi.org/10.4208/aamm.2013.m225 Published online by Cambridge University Press