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Abstract orthogonality and orthocomplementation

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Abstract. The notion of orthogonality is axiomatically defined on a poset. Various notions of orthocomplementation are distinguished and conditions are given in order to induce an orthocomplementation from an orthogonality and vice versa. Subsequently $\perp$-modular $\perp$-poset are defined and the set of morphisms between two posets with orthogonality is briefly discussed. Given the notions of additive monoid and of positive semi-ring, an orthogonality relation is introduced on the set of idempotent elements of a positive semi-ring. Finally, the obtained results are applied to the set of idempotent and absorbent endomorphisms of an additive monoid.

Introduction. Orthomodular lattices are assuming a bigger and bigger importance both in pure mathematics, because of the richness of such algebraic structure, and in other disciplines such as logic, probability calculus and theoretical physics. According to Lorenzen (1) the orthomodular lattices may be considered operative logic and from many points of view they appear as a possible syntatic structure which generalizes the Boolean structure of propositional calculus, more connected to a strictly traditional logic. In a similar direction Varadarajan (2) has defined the generalized notion of probability in which the probability function is defined, not longer on a Boolean lattice, but on an orthomodular lattice. According to Birkhoff–Von Neumann (3) the difference of behaviour between classical and quantum mechanics is essentially contained in the different algebraic structure of the lattices associated to yes–no experiments, which is Boolean in the former case and orthomodular in the latter. In this context the requirement of orthomodularity is strictly bound to the fact that this property is verified in the lattice of subspaces of a Hilbert space. In all these cases the structure of an orthomodular lattice has proved to be a considerable generalization of the Boolean lattice. In orthomodular lattices the notion of orthogonality appears as a notion intrinsically contained in the existence of the orthocomplementation. Recent works, (4) and (5), point out the possibility of a further weakening of the structure, because (at least in the case of quantum mechanics) the existence of complement and of 'meet' and 'join' appear doubtful in the set of yes–no experiments.

We introduce the notion of orthogonality in this work in such a way that its result is more primitive, in the sense that it can be defined on a poset which is not necessarily orthocomplemented. We also prove that the existence of orthocomplementation in a poset with orthogonality is essentially a requirement of completeness.

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In the last part of this work a remarkable example of the application of the orthogonality relation is given in view of a utilization for an axiomatic formulation of quantum mechanics.

I. *Orthocomplementation.* DEFINITION 1·1. *Weak degenerate orthocomplementation (wdo). Let* $(\mathscr{P}, \leqslant)$ *be a poset. We shall say that a mapping* $': \mathscr{P} \to \mathscr{P}$ *is a weak degenerate orthocomplementation (or wdo) iff it satisfies the conditions*:

(1) $\qquad x \leqslant x'', \quad \forall x \in \mathscr{P},$

(2) $\qquad x \leqslant y \quad$ *implies* $\quad y' \leqslant x'.$

A weak degenerate orthocomplemented poset (or wdop) is a poset with a wdo. The set $\mathscr{N}' = \{x \in \mathscr{P} : x \leqslant x'\}$ *is said to be the kernel of the orthocomplementation. Obviously if* $\mathscr{P}$ *has the least element* 0 *then* $0 \in \mathscr{N}'$.

PROPOSITION 1·1. *Let* $(\mathscr{P}, \leqslant, ')$ *be a wdop. Then the following propositions are verified*:

(i) $x' = x'''$, $\forall x \in \mathscr{P}$,

(ii) $x \in \mathscr{N}'$ *and* $x_0 \leqslant x$ *imply* $x_0 \in \mathscr{N}'$,

(iii) *If* $x \vee y$ *exists then* $x' \wedge y'$ *exists and* $x' \wedge y' = (x \vee y)'$.

Proof. (i) $x \leqslant x''$, $\forall x \in \mathscr{P}$ implies $(x') \leqslant (x')''$ and then $x' \leqslant x'''$, $\forall x \in \mathscr{P}$; but from $x \leqslant x''$, $\forall x \in \mathscr{P}$ we have $(x'')' \leqslant x'$ and then $x''' \leqslant x'$, $\forall x \in \mathscr{P}$. From these two results $x' = x'''$, $\forall x \in \mathscr{P}$ follows.

(ii) From the hypothesis we have $x_0 \leqslant x'$ and from this $x_0 \leqslant x \leqslant x'' \leqslant x_0'$, that is $x_0 \in \mathscr{N}'$.

(iii) From $x \leqslant x \vee y$ and $y \leqslant x \vee y$ we have $(x \vee y)' \leqslant x'$ and $(x \vee y)' \leqslant y'$; thus $(x \vee y)'$ is a lower bound of $\{x', y'\}$. Let z be such that $z \leqslant x'$ and $z \leqslant y'$, then we have $x \leqslant x'' \leqslant z'$ and $y \leqslant y'' \leqslant z'$ and so $x \vee y \leqslant z'$. From this relation $z \leqslant z'' \leqslant (x \vee y)'$ follows. Therefore we conclude that $(x \vee y)'$ is the g.l.b. of $\{x', y'\}$, that is $x' \wedge y' = (x \vee y)'$.

PROPOSITION 1·2. *In a wdop the following propositions*

(i) $z \leqslant x$ *and* $z \leqslant x'$ *imply* $z \in \mathscr{N}'$,

(ii) $x \leqslant z$ *and* $x' \leqslant z$ *imply* $z' \in \mathscr{N}'$

are equivalent and verified.

Proof. (i) $\Rightarrow$ (ii). $x \leqslant z$ and $x' \leqslant z$ imply $z' \leqslant x'$ and $z' \leqslant (x')'$. Thus from (i) $z' \in \mathscr{N}'$ follows.

(ii) $\Rightarrow$ (i). $z \leqslant x$ and $z \leqslant x'$ imply $x' \leqslant z'$ and $(x')' \leqslant z'$. Thus from (ii) $z'' \in \mathscr{N}'$ follows, but since $z \leqslant z''$ (ii) Proposition 1·1 implies $z \in \mathscr{N}'$.

Finally, let $z \leqslant x$ and $z \leqslant x'$, then $x' \leqslant z'$ and $z \leqslant x'$ imply $z \in \mathscr{N}'$.

PROPOSITION 1·3. *Let* $(\mathscr{P}, \leqslant, 0, ')$ *be a wdop with the least element* 0. *Then the greatest element* **1** *of* $\mathscr{P}$ *exists and* $\mathbf{1} = 0'$.

Proof. From $0 \leqslant x$, $\forall x \in \mathscr{P}$ we have $x' \leqslant 0'$, $\forall x \in \mathscr{P}$. Taking in particular $x = \bar{x}'$, $\forall \bar{x} \in \mathscr{P}$ is $\bar{x} \leqslant \bar{x}'' \leqslant 0'$, $\forall \bar{x} \in \mathscr{P}$ and so the element $0' = \mathbf{1}$ is such that $x \leqslant \mathbf{1}$, $\forall x \in \mathscr{P}$.

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PROPOSITION 1·4. *In a wdop $\mathscr{P}$ the following propositions are equivalent*:
(i) $': \mathscr{P} \to \mathscr{P}$ *is surjective*,
(ii) $': \mathscr{P} \to \mathscr{P}$ *is bijective*,
(iii) $x = x''$, $\forall x \in \mathscr{P}$.

Proof (i) $\Rightarrow$ (iii). If $'$ is a surjective mapping then for $\forall x \in \mathscr{P}$, $\bar{x} \in \mathscr{P}$ exists such that $x = \bar{x}'$. From this property $x'' = \bar{x}'''$ follows. But (ii) Proposition 1·1 implies $\bar{x}''' = \bar{x}'$ and thus $x = x''$.
(iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are easy to prove.

DEFINITION 1·2. *Strong degenerate orthocomplementation* (*sdo*). *A wdo defined on a poset* $(\mathscr{P}, \leqslant)$ *is called strong degenerate orthocomplementation* (*or sdo*) *iff one of the three equivalent propositions of Proposition* 1·4 *is satisfied. In this case* $(\mathscr{P}, \leqslant, ')$ *is called a strong degenerate orthocomplemented poset* (*or sdop*).

PROPOSITION 1·5. *In a sdop if* $x \wedge y$ *exists then* $x' \vee y'$ *exists and* $x' \vee y' = (x \wedge y)'$.

Proof. This property can be proved with a proceeding like that of (iii) Proposition 1·1.

DEFINITION 1·3. *Weak* (*or strong*) *orthocomplementation* (*w*(*s*)*o*). *A weak* (*or strong*) *degenerate orthocomplementation defined on a poset* $(\mathscr{P}, \leqslant, 0)$ *with the least element* 0 *is called a weak* (*or strong*) *orthocomplementation* (*or w*(*s*)*o*) *iff* $\mathscr{N}' = \{0\}$. *In this case* $(\mathscr{P}, \leqslant, 0, ')$ *is said to be a weak* (*or strong*) *orthocomplemented poset* (*or w*(*s*)*op*).
In the following a poset $(\mathscr{P}, \leqslant)$ *with the least element* 0 *will be denoted with* $(\mathscr{P}, \leqslant, 0)$.
Sometimes a weak (*or strong*) *orthocomplementation is also called weak* (*or strong*) *non-degenerate orthocomplementation.*

PROPOSITION 1·6. *A w*(*s*)*dop* $(\mathscr{P}, \leqslant, 0, ')$ *is a w*(*s*)*op if and only if the following property is satisfied*:

(3) $\quad x \wedge x'$ *exists in* $\mathscr{P}$ *and* $x \wedge x' = 0$, $\forall x \in \mathscr{P}$.

Proof. If $\mathscr{P}$ is characterized by the property $\mathscr{N}' = \{0\}$ then (i) Proposition 1·2 immediately implies (3). On the contrary let (3) be satisfied. If $x_0 \in \mathscr{P}$ is such that $x_0 \in \mathscr{N}'$, i.e. $x_0 \leqslant x_0'$, then $x_0 \wedge x_0' = x_0$ and thus $x_0 = 0$.

PROPOSITION 1·7. *Let* $(\mathscr{P}, \leqslant, 0, ')$ *be a sop. Then it follows that*:

$$x \vee x' \quad \textit{exists and} \quad x \vee x' = \mathbf{1},\ \forall x \in \mathscr{P}.$$

Proof. If $x \leqslant z$ and $x' \leqslant z$, then $z' \in \mathscr{N}' = \{0\}$ implies $z'' = 0' = \mathbf{1}$ by Proposition 1·4. Thus from $z'' = z$ we have that $x \vee x'$ exists and $x \vee x' = \mathbf{1}$, $\forall x \in \mathscr{P}$.

Remark. It follows from Proposition 1·4, Proposition 1·5 and Proposition 1·6 that we can define strong orthocomplementation, or orthocomplementation for short, on a poset $(\mathscr{P}, \leqslant, 0)$ as a mapping $': \mathscr{P} \to \mathscr{P}$ such that:
(1) $x = x''$, $\forall x \in \mathscr{P}$,
(2) $x \leqslant y$ implies $y' \leqslant x'$,
(3) $x \wedge x'$ exists and $x \wedge x' = 0$, $\forall x \in \mathscr{P}$.

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Thus orthocomplementation has the properties:

(i) $'$ is a bijection from $\mathscr{P}$ onto $\mathscr{P}$,

(ii) $\mathbf{1}$ exists in $\mathscr{P}$ and $\mathbf{1} = 0'$,

(iii) $x \wedge x'$ exists and $x \vee x' = \mathbf{1}$, $\forall x \in \mathscr{P}$.

DEFINITION 1·4. *Orthoposet. A weak degenerate orthoposet is a triple* $(\mathscr{P}, \leqslant, ')$ *which satisfies the requirements*:

(1) $'$ *is a weak degenerate orthocomplementation,*

(2) $x \leqslant y'$ *implies that* $x \wedge y$ *and* $x \vee y$ *exist.*

Other definitions relative to strong or non-degenerate orthocomplementations are consequently given.

As an example of wdop we can consider a Hilbert space H and denote by $(\mathscr{P}(H), \subseteq)$ the power set of H ordered by the set inclusion. The mapping from $\mathscr{P}(H)$ to $\mathscr{P}(H)$ which associates to each non-empty set $S \in \mathscr{P}(H)$ the element

$$S' = \{x \in H\colon (x|s) = 0,\ \forall s \in S\}$$

of $\mathscr{P}(H)$ and to the empty set $\varnothing \in \mathscr{P}(H)$ the element $\varnothing' = H$ of $\mathscr{P}(H)$ is a wdo, because it is easy to prove that (1) and (2) in Definition 1·1 are satisfied and that $\mathscr{N}' = \{\varnothing, \{0\}\}$. Obviously $(\mathscr{P}(H), \varnothing, \subseteq)$ is a weak degenerate orthoposet. Let $(\mathscr{C}(H), \subseteq)$ be the set of all closed (non-empty) linear subspaces of H ordered by the set inclusion, then the restriction to $\mathscr{C}(H)$ of the above mapping is an orthocomplementation. If we consider the set $\pi(H)$ of all projections on H, i.e. of all idempotent ($P^2 = P$) and Hermitian ($P^* = P$, where P^* is the adjoint of P) operators on H, it is well known that there is a natural one-to-one correspondence between $\pi(H)$ and $\mathscr{C}(H)$ which to each projection P associates the range of P, i.e. the closed linear subspace $M = \{Px\colon x \in H\}$, and, conversely, to each closed linear subspace M associates the projection P_M with range M defined by $P_M(x+y) = x$ where $x \in M$ and $y \in M'$. On $\pi(H)$ the relation defined by '$P \leqslant Q$ iff $P = P \circ Q$' is an order relation for which the above natural one-to-one correspondence is order preserving, i.e. $P_M \leqslant P_N$ iff $M \subseteq N$ holds. Moreover, it is easy to check that the mapping from $\pi(H)$ to $\pi(H)$, defined by the correspondence $P_M \to \mathbf{1} - P_M$, is an orthocomplementation and that the range of $\mathbf{1} - P_M$ is M'.

II. *Orthogonality.* DEFINITION 2·1. *Weak degenerate orthogonality. Let* $(\mathscr{P}, \leqslant)$ *be a poset. A binary relation* $\perp$ *on* $\mathscr{P}$ *is said to be a weak degenerate orthogonality iff the following conditions are satisfied*:

(1′) $x_0 \leqslant x$ *and* $x \perp y$ *imply* $x_0 \perp y$ (*Absorption property*),

(1″) $x \perp y$ *implies* $y \perp x$ (*Symmetry property*).

Let $\perp$ *be a relation of weak degenerate orthogonality on* $\mathscr{P}$ *then* $(\mathscr{P}, \leqslant, \perp)$ *is said to be a weak degenerate orthogonal poset (or wd*$\perp$*p) and the set*

$$\operatorname{Ker}\perp = \{x \in \mathscr{P}\colon x \perp x\}$$

is said to be the kernel of the orthogonality. We define $\{x\}^\perp = \{y \in \mathscr{P}\colon x \perp y\}$ *and denote with* $x^\perp$ *the general element of* $\mathscr{P}$ *which is orthogonal to* x. *If* $\mathscr{A} \subseteq \mathscr{P}$, $x \perp \mathscr{A}$ *means that* $x \perp a$, $\forall a \in \mathscr{A}$ *and the set of all these* x *is written* $\mathscr{A}^\perp$, *i.e.* $\mathscr{A}^\perp = \{x \in \mathscr{P}\colon x \perp \mathscr{A}\}$. *Finally*

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$\mathscr{A} \leqslant x$ *means that* $a \leqslant x$, $\forall a \in \mathscr{A}$. *Evidently condition* (1′) *and* (1″) *are equivalent to the following proposition*:

(1) $x_0 \leqslant x$ *and* $x \perp y$ *imply* $y \perp x_0$.

PROPOSITION 2·1. *If* $(\mathscr{P}, \leqslant, \perp)$ *is a* $wd\perp p$ *then*

(i) $x_0 \leqslant x$ *and* $x \in \mathrm{Ker}\perp$ *imply* $x_0 \in \mathrm{Ker}\perp$,

(ii) $x_0 \leqslant x$ *and* $x \perp x_0$ *imply* $x_0 \in \mathrm{Ker}\perp$,

(iii) $x \perp y$, $z \leqslant x$ *and* $z \leqslant y$ *imply* $z \in \mathrm{Ker}\perp$,

(iv) *if* $\mathscr{P}$ *has the least element* 0 *then* $0 \in \mathrm{Ker}\perp$,

(v) $y \leqslant x$ *implies* $\{x\}^\perp \subseteq \{y\}^\perp$.

Proof. (i) from $x \perp x$ and $x_0 \leqslant x$ we have $x \perp x_0$ and thus $x_0 \perp x_0$. Similarly (ii), (iii) and (iv) are obvious consequences of Definition 2·1. Finally, $y \leqslant x$ and $x \perp \{x\}^\perp$ lead to $y \perp \{x\}^\perp$; that is every element orthogonal to x is also orthogonal to y.

PROPOSITION 2·2. *In* $(\mathscr{P}, \leqslant, 0, \perp)$ *the following propositions are equivalent*:

(i) $\forall x \in \mathscr{P}$, *an element* $y_x \in \mathscr{P}$ *exists such that* $x \perp y_x$,

(ii) $0 \perp \mathscr{P}$.

Proof. Let (i) be true. $\forall x \in \mathscr{P}$, $y_x \in \mathscr{P}$ exists such that $x \perp y_x$ and $0 \leqslant y_x$ imply, by Definition 2·1, that $0 \perp x$, $\forall x \in \mathscr{P}$. On the contrary let (ii) be true. From $0 \perp x$, $\forall x \in \mathscr{P}$ we have that $\forall x \in \mathscr{P}$, $0 \in \mathscr{P}$ exists such that $x \perp 0$.

DEFINITION 2·2. *Weak orthogonality. Let* $(\mathscr{P}, \leqslant, 0, \perp)$ *be a poset with a weak degenerate orthogonality. Then* $\perp$ *is said to be a weak orthogonality iff* $\mathrm{Ker}\perp = \{0\}$. *In this case* $\mathscr{P}$ *is said to be a poset with a weak orthogonality (or* $w\perp p$*). Obviously in a* $w\perp p$, $x \perp y$ *implies that* $x \wedge y$ *exists and* $x \wedge y = 0$.

PROPOSITION 2·3. *Let* $(\mathscr{P}, \leqslant, 0)$ *be a poset, which satisfies one of the equivalent propositions in Proposition* 2·2, *then*

(i) *if* $\mathscr{P}$ *is a* $wd\perp p$ *we have* $\{0\}^\perp = \mathscr{P}$,

(ii) *if* $\mathscr{P}$ *is a* $w\perp p$ *we have* $\mathscr{P}^\perp = \{0\}$.

Proof. From (ii) Proposition 2·2 $\{0\}^\perp = \mathscr{P}$ follows at once. Moreover, from (ii) Proposition 2·2, we have that $\{0\} \subseteq \mathscr{P}^\perp$. But $\mathscr{P}^\perp \subseteq \mathrm{Ker}\perp$ and $\mathrm{Ker}\perp = \{0\}$ imply $\mathscr{P}^\perp = \{0\}$.

DEFINITION 2·3. *Strong orthogonality. Let* $(\mathscr{P}, \leqslant)$ *be a poset. A relation of weak, degenerate or not, orthogonality* $\perp$ *on* $\mathscr{P}$ *is said to be a strong, degenerate or not, orthogonality iff* $\perp$ *satisfies the condition*:

(3) $\qquad x_0 \perp \{x\}^\perp \quad$ *implies* $\quad x_0 \leqslant x$.

A poset with a strong non-degenerate orthogonality is more simply called poset with orthogonality.

PROPOSITION 2·4. *If* $(\mathscr{P}, \leqslant, \perp)$ *is a* $sd\perp p$ *then*:

(i) $\mathscr{P}^\perp \neq \varnothing$ *implies* 0 *exists in* $\mathscr{P}$ *and* $\mathscr{P}^\perp = \{0\}$,

(ii) *if* $y^\perp$ *exists such that* $\{x\}^\perp \leqslant y^\perp$ *then* $y \leqslant x$.

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Proof. (i) If $z \in \mathscr{P}$ exists such that $z \perp \mathscr{P}$ then from $\{x\}^\perp \subseteq \mathscr{P}$, $\forall x \in \mathscr{P}$ we have that $z \perp \{x\}^\perp$, $\forall x \in \mathscr{P}$ and thus (3) leads to $z \leqslant x$, $\forall x \in \mathscr{P}$ that is to $z = 0$.

(ii) If $x^\perp \leqslant y^\perp$, $\forall x^\perp$ by Definition 2·1 we have that $y \perp x^\perp$, $\forall x^\perp$ and from (3) $y \leqslant x$ follows.

From Proposition 2·3 and (i) Proposition 2·4 we have that if $(\mathscr{P}, \leqslant, 0, \perp)$ is $sd \perp p$ then $\mathscr{P}^\perp = \{0\}$.

DEFINITION 2·4. *Complete orthogonality. Let $(\mathscr{P}, \leqslant)$ be a poset. A weak (strong) degenerate (or not) orthogonality $\perp$ is said to be a complete weak (strong) degenerate (or not) orthogonality iff $\perp$ satisfies the condition*:

(4) $\max \{x\}^\perp$ *exists for every* $x \in \mathscr{P}$.

It is easy to prove that this condition is equivalent to the following propositions:

(4′) $x_\alpha \perp x$, $\forall \alpha \in A$ and $\vee_\alpha x_\alpha$ exists imply $\vee_\alpha x_\alpha \perp x$,

(4″) $\{x\}^\perp \neq \varnothing$, $\forall x \in \mathscr{P}$,

(4‴) $\sup \{x\}^\perp$, $\forall x \in \mathscr{P}$ exists.

A complete strong non-degenerate orthogonality is simply called complete orthogonality.

DEFINITION 2·5. *$\perp$-poset. A poset with a weak (strong) degenerate (or not) orthogonality is said to be weak (strong) degenerate (or not) $\perp$-poset iff the following condition is satisfied*:

$$x \perp y \quad \textit{implies that} \quad x \vee y \quad \textit{and} \quad x \wedge y \quad \textit{exist.}$$

We now give an example of a poset with an orthogonality relation which satisfies the weaker set of conditions but does not satisfy the stronger one, i.e. of an orthogonality which is strictly weak and degenerate. Before discussing our concrete example we must make some preliminary remarks. Let $(\mathscr{P}, \leqslant)$ be a poset, a subset $\mathscr{I}$ of $\mathscr{P}$ is called an order ideal, or simply an ideal, iff for every $x \in \mathscr{I}$ we have that $z \leqslant x$ implies $z \in \mathscr{I}$. It is easy to check that, once an ideal $\mathscr{I}$ of $\mathscr{P}$ has been chosen, the relation $\perp$ defined on by '$x \perp y$ iff $z \leqslant x$ and $z \leqslant y$ imply $z \in \mathscr{I}$' is an orthogonality relation for which $\mathrm{Ker} \perp = \mathscr{I}$. Let us consider the poset $\mathscr{P}$ whose diagram is shown in the figure. The set $\mathscr{I} = \{0, c\}$ is obviously an ideal of $\mathscr{P}$ and so we can associate to $\mathscr{P}$ an orthogonality relation according to the above considerations. This orthogonality is degenerate because its kernel is $\{0, c\}$, is not strong because $b \perp \{a\}^\perp$ and from this relation $b \leqslant a$ does not follow and is not complete because $\max \{a\}^\perp = \max \{0, e, c, f, g\}$ does not exist. Moreover, $d \perp e$ does not imply that $d \vee e$ exists and so $\mathscr{P}$ is not a $\perp$-poset. On the other hand, let $\mathscr{P}(X)$ be the power set of X ordered by the set inclusion. On $\mathscr{P}(X)$ the relation '$A \perp B$ iff $A \cap B = \varnothing$' is a complete orthogonality for which $\max \{A\}^\perp$ is A^c, the complement of A.

III. *Orthogonality and orthocomplementation.* This section is dedicated to the study of connexions between orthogonality and orthocomplementation. We demonstrate that the possibility of associating an orthocomplementation to a poset with orthogonality is necessarily linked to the requirement that orthogonality satisfies the completeness property.

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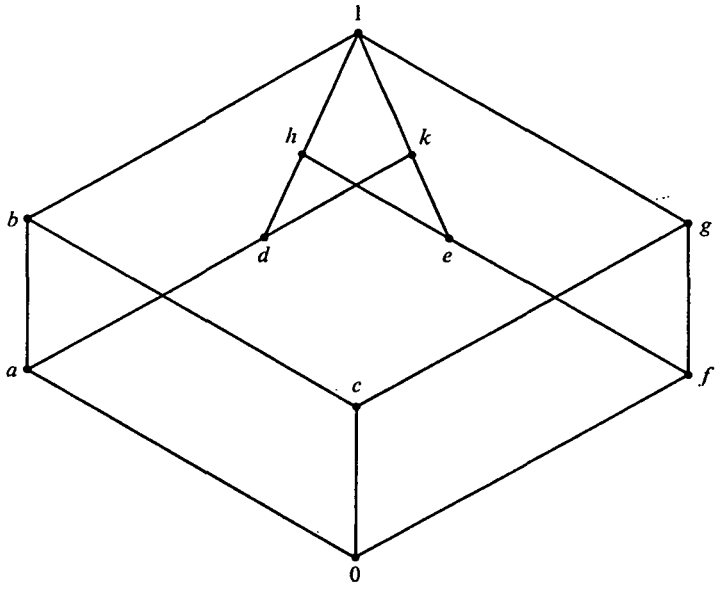


Fig. 1

THEOREM 3·1 (A). *Let* $(\mathscr{P}, \leqslant, ')$ *be a wdop. Then the following binary relation on* $\mathscr{P}$:

$$(a) \qquad x \perp y \quad \textit{iff} \quad x \leqslant y' \qquad \textit{is a weak degenerate and complete orthogonality relation for which the following properties are true}:$$

$$\max\{x\}^{\perp} = x', \tag{1}$$

$$\ker \perp = \mathscr{N}'. \tag{2}$$

(B) *Let* $(\mathscr{P}, \leqslant, \perp)$ *be a poset with weak degenerate and complete orthogonality. Then the mapping* $': \mathscr{P} \to \mathscr{P}$ *defined by the correspondence*

$$(b) \qquad x \to x' = \max\{x\}^{\perp} \qquad \textit{is a wdo, for which the following properties are true}:$$

$$x \leqslant y' \quad \text{iff} \quad x \perp y, \tag{1}$$

$$\mathscr{N}' = \ker \perp. \tag{2}$$

Proof. (A). We verify that the relation $\perp$ defined by (a) is an orthogonality relation.

(i) It follows from (1) Definition 1·1 that $x \leqslant (x')'$, $\forall x \in \mathscr{P}$. Hence we have that $x \perp x'$, $\forall x \in \mathscr{P}$.

(ii) If $x_0 \leqslant x$ and $x \leqslant y'$, from (1) and (2) Definition 1·1, $y \leqslant y'' \leqslant x_0'$ follows. Thus $x_0 \leqslant x$ and $x \perp y$ imply $x_0 \perp y$.

(iii) If $x_\alpha \leqslant x'$ and $\vee_\alpha x_\alpha$ exists then $\vee_\alpha x_\alpha \leqslant x'$. From (i) we have $x' \in \{x\}^{\perp}$, $\forall x \in \mathscr{P}$. But $x^{\perp} \perp x$ implies $x^{\perp} \leqslant x'$; hence $\max\{x\}^{\perp} = x'$, $\forall x \in \mathscr{P}$ exists.

(iv) $\operatorname{Ker} \perp = \{x \in \mathscr{P}: x \perp x\} = \{x \in \mathscr{P}: x \leqslant x'\} = \mathscr{N}'$.

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(B). The correspondence defined by (*b*) is evidently a mapping $\mathscr{P} \to \mathscr{P}$. This mapping is a wdo:

(i) From $x \perp x'$ we have $x \leqslant \max\{x'\}^{\perp} = x''$.

(ii) If $x \leqslant y$ Proposition 2·1 (v) leads to $\{y\}^{\perp} \subseteq \{x\}^{\perp}$; hence $y' \leqslant x'$ follows immediately.

(iii) $x \leqslant y'$ iff $x \perp y$. In fact $x \leqslant y' = \max\{y\}^{\perp}$ and (1′) Definition 2·1 implies $x \perp y$. Let $x \perp y$, then $x \in \{y\}^{\perp}$; hence $x \leqslant \max\{y\}^{\perp} = y'$.

(iv) (2) B follows immediately from (iii) B and (iv) A.

COROLLARY 1. (A) *Let* $(\mathscr{P}, \leqslant, 0, ')$ *be a wop. Then* (*a*) Theorem 3·1 *defines a weak complete orthogonality relation such that*

$$\{0\}^{\perp} = \mathscr{P} \quad \textit{and} \quad \mathscr{P}^{\perp} = \{0\}.$$

(B) *Let* $(\mathscr{P}, \leqslant, 0, \perp)$ *be a poset with weak complete orthogonality. Then the mapping* $': \mathscr{P} \to \mathscr{P}$ *defined by* (*b*) *is a wo such that the following properties are true*:

(1) $\mathbf{1}$ *exists in* $\mathscr{P}$ *and* $\mathbf{1} = 0'$;

(2) $x \wedge x'$ *exists and* $x \wedge x' = 0$, $\forall x \in \mathscr{P}$.

Proof. It is obvious.

COROLLARY 2. (A) *Let* $(\mathscr{P}, \leqslant, 0, ')$ *be an orthocomplemented poset. Then the binary relation defined by* (*a*) Theorem 3·1 *is a complete orthogonality such that* (A) *properties of Corollary* (1) *are true.*

(*b*) *Let* $(\mathscr{P}, \leqslant, 0, \perp)$ *be a poset with a complete orthogonality. Then the mapping* $': \mathscr{P} \to \mathscr{P}$ *defined by* (*b*) Theorem 3·1 *is an orthocomplementation such that* (B) *properties of Corollary* (1) *are true.*

Proof. (*a*) Let $'$ be an orthocomplementation. If $x_0 \perp \{x\}^{\perp}$ then $x^{\perp} \leqslant x_0'$ for $\forall x^{\perp}$; since $x' \in \{x\}^{\perp}$ we have as a particular case $x' \leqslant x_0'$ from which $x_0 \leqslant x$ follows.

(*b*) Let $\perp$ be a complete orthogonality. Since $x \leqslant x''$ it suffices to demonstrate that $x'' \leqslant x$ to conclude $'$ is an orthocomplementation. If z is such that $x' \perp z$, then $\max\{x\}^{\perp} \perp z$; this and $\forall x^{\perp} \leqslant \max\{x\}^{\perp}$ imply $z \perp \{x\}^{\perp}$. Therefore it follows from strong orthogonality that $z \in \{x'\}^{\perp}$ implies $z \leqslant x$; hence $x'' = \max\{x'\}^{\perp} \leqslant x$.

Remark. We can easily verify that if in Theorem 3·1 or in Corollaries (1) and (2), when speaking of $(\mathscr{P}, \leqslant, ')$ the term orthoposet is substituted for poset and when speaking of $(\mathscr{P}, \leqslant, \perp)$ the term $\perp$-poset is substituted for poset, then the propositions are still true owing to the perfect symmetry of definitions. Moreover, we can say that the set of orthoposets coincides with the set of complete $\perp$-posets.

IV. *Orthomodularity and* $\perp$*-modularity.* DEFINITION 4·1. *Distributive triple. Let* $(\mathscr{P}, \leqslant)$ *be a poset. If* $x_1, x_2, x_3 \in \mathscr{P}$ *we say that* (x_1, x_2, x_3) *is a distributive triple, written* $(x_1, x_2, x_3)\, D$, *provided*:

$$\exists\, (x_1 \vee x_2) \wedge x_3 = (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$$

($\exists$ *notation meaning that* $(x_1 \vee x_2)$, $(x_1 \wedge x_3)$, $(x_2 \wedge x_3)$, $(x_1 \vee x_2) \wedge x_3$, $(x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ *exist and satisfy the written relation*).

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DEFINITION 4·2. *Modular pair. Let* $(\mathscr{P}, \leqslant)$ *be a poset. If* $x_1, x_2 \in \mathscr{P}$ *we say that* (x_1, x_2) *is a modular pair, written* $(x_1, x_2)\, M$, *provided*:

$$x_0 \leqslant x_2 \quad \textit{implies} \quad (x_0, x_1, x_2)\, D.$$

Consequently we have that $(x_1, x_2)\, M$, *provided*:

$$\exists\, (x_0 \vee x_1) \wedge x_2 = x_0 \vee (x_1 \wedge x_2) \quad \textit{for every} \quad x_0 \in \mathscr{P} \quad \textit{such that} \quad x_0 \leqslant x_2.$$

DEFINITION 4·3. *Orthomodular poset. A poset* $(\mathscr{P}, \leqslant, ')$ *with (degenerate or not, weak or strong) orthocomplementation is said to be an orthomodular poset iff*:

$$x_1 \leqslant x_2' \quad \textit{implies} \quad (x_1, x_2)\, M.$$

Thus $\mathscr{P}$ *is an orthomodular poset iff*:

$$x_1 \leqslant x_2' \quad \textit{and} \quad x_0 \leqslant x_2 \quad \textit{imply} \quad \exists\, (x_0 \vee x_1) \wedge x_2 = x_0 \vee (x_1 \wedge x_2) \quad \textit{and} \quad x_1 \wedge x_2 \in \mathscr{N}'.$$

In the special case of non-degenerate orthocomplementation we have:

$$x_1 \leqslant x_2' \quad \textit{and} \quad x_0 \leqslant x_2 \quad \textit{imply} \quad \exists\, (x_0 \vee x_1) \wedge x_2 = x_0.$$

DEFINITION 4·4. *$\perp$-modular poset. A poset* $(\mathscr{P}, \leqslant, \perp)$ *with (degenerate or not, weak or strong) orthogonality is said to be an $\perp$-modular poset iff*:

$$x_1 \perp x_2 \quad \textit{implies} \quad (x_1, x_2)\, M.$$

An $\perp$-modular poset is then characterized by the following equivalent property:

$$x_0 \leqslant x \quad \textit{implies} \quad (x_0, x^{\perp}, x)\, D, \quad \forall\, x^{\perp} \in \{x\}^{\perp}.$$

Particularly if the orthogonality is not degenerate we have

$$x_0 \leqslant x \quad \textit{implies} \quad \exists\, (x_0 \vee x^{\perp}) \wedge x = x_0, \quad \forall\, x^{\perp} \in \{x\}^{\perp}.$$

The correlation between the notions of $\perp$-modularity and orthomodularity is expressed by the following theorem whose proof is obvious.

THEOREM 4·1. (*a*) *Let* $(\mathscr{P}, \leqslant, ')$ *be an orthomodular poset. Then* $(\mathscr{P}, \leqslant, \perp)$ *is a complete $\perp$-modular poset where the orthogonality is defined according to* (*a*) *Theorem* 3·1.

(*b*) *Let* $(\mathscr{P}, \leqslant, \perp)$ *be a complete $\perp$-modular poset. Then* $(\mathscr{P}, \leqslant, ')$ *is an orthomodular poset where the orthocomplementation is defined according to* (*b*) *Theorem* 3·1.

THEOREM 4·2. *If orthomodularity holds in a wdop then*

$$x_0 \leqslant x \quad \textit{implies} \quad (x_0, x', x)\, D. \quad \text{(OM)}$$

Proof. In fact setting $x' = x_1$ in the proposition '$x_1 \leqslant x'$ implies $(x_1, x)\, M$' we have $(x', x)\, M$, $\forall\, x \in \mathscr{P}$; that is $x_0 \leqslant x$ implies $(x_0, x', x)\, D$.

THEOREM 4·3. *In a weak orthoposet* $(\mathscr{P}, \leqslant, 0, ')$ *orthomodularity is equivalent to* (OM).

Proof. We must only prove that (OM) implies orthomodularity. From $x_0 \leqslant x$ and $x_1 \leqslant x'$ we have that $x_0 \leqslant x \leqslant x'' \leqslant x_1'$ and so $x_0 \vee x_1$ exists by (2) Definition 1·4 and x_0 is a lower bound of $\{x, x_0 \vee x_1\}$. If $z \leqslant x$ and $z \leqslant x_0 \vee x_1$, since $x_0 \vee x'$ exists, we have $z \leqslant x$ and $z \leqslant x_0 \vee x_1$. But the existence of $(x_0 \vee x') \wedge x$ implies that $z \leqslant (x_0 \vee x') \wedge x$,

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which leads to $z \leqslant x_0 \vee (x' \wedge x)$ by (OM). Hence it follows that $z \leqslant x_0$, since $'$ is non-degenerate. Therefore x_0 is the g.l.b. of $\{x, x_0 \vee x_1\}$ that is $x_0 = (x_0 \vee x_1) \wedge x$.

Note that in an orthoposet (OM) is equivalent to the following propositions:

$$x_0 \leqslant x \quad \text{implies} \quad x = (x \wedge x_0') \vee x_0; \quad \text{(OMi)}$$

$$x_0 \leqslant x \quad \text{and} \quad x \wedge x_0' = 0 \quad \text{imply} \quad x_0 = x. \quad \text{(OMii)}$$

Equivalence of (OMi) and (OMii) is easy to prove while equivalence of (OM) and (OMi) is an immediate consequence of Proposition 1·5. In fact, it follows from $(x_0 \vee x') \wedge x = x_0$ that $x_0' = (x_0' \wedge x'') \vee x'$; thus setting $x_0' = y$ and $x' = y_0$ in (OM) we have (OMi). Similarly we can prove that (OMi) implies (OM). Therefore in an orthoposet (OM), (OMi) and (OMii) are equivalent versions of orthomodularity.

V. *Morphisms.* DEFINITION 5·1. *Let* $(\mathscr{P}, \leqslant, \perp)$ *and* $(\mathscr{P}_1, \leqslant_1, \perp_1)$ *be two posets with weak degenerate orthogonality. A morphism from* $\mathscr{P}$ *to* $\mathscr{P}_1$ *is a mapping* $\phi: \mathscr{P} \to \mathscr{P}_1$ *such that the following properties hold*:

(1) $x \leqslant y$ *implies* $\phi(x) \leqslant_1 (y)$,

(2) $x \perp y$ *implies* $\phi(x) \perp_1 \phi(y)$.

Given a morphism ϕ *we define* $\operatorname{Ker}\phi \overset{\text{def}}{=} \phi^{-1}(\operatorname{Ker}\perp_1)$ *and* $\Im\phi \overset{\text{def}}{=} \phi(\mathscr{P})$. *Obviously* $\operatorname{Ker}\perp \subseteq \operatorname{Ker}\phi$. *The set of morphisms from* $\mathscr{P}$ *to* $\mathscr{P}_1$ *is denoted with* $\operatorname{Mor}(\mathscr{P}, \mathscr{P}_1)$. *A morphism is called endomorphism iff* $\mathscr{P} = \mathscr{P}_1$. *The set of endomorphism of* $\mathscr{P}$, *that is* $\operatorname{End}(\mathscr{P})$, *is a semigroup with regard to multiplicative operation*

$$(\alpha \circ \beta)(x) \overset{\text{def}}{=} \alpha(\beta(x)), \ \forall x \in \mathscr{P} \quad \text{if} \quad \alpha, \beta \in \operatorname{End}(\mathscr{P}).$$

DEFINITION 5·2. $\phi \in \operatorname{Mor}(\mathscr{P}, \mathscr{P}_1)$ *is said to be a homomorphism iff*:

(1) $x \vee y$ *exists in* $\mathscr{P}$ *implies that* $\phi(x) \vee_1 \phi(y)$ *exists in* $\mathscr{P}_1$ *and* $\phi(x) \vee_1 \phi(y) = \phi(x \vee y)$;

(2) $x \wedge y$ *exists in* $\mathscr{P}$ *implies that* $\phi(x) \wedge_1 \phi(y)$ *exists in* $\mathscr{P}_1$ *and* $\phi(x) \wedge_1 \phi(y) = \phi(x \wedge y)$.

The set of homomorphisms from $\mathscr{P}$ *to* $\mathscr{P}_1$ *is denoted with* $\operatorname{Hom}(\mathscr{P}, \mathscr{P}_1)$ *and the set of homomorphisms and endomorphisms of* $\mathscr{P}$ *with* $\operatorname{Hom}(\mathscr{P})$. *Let* $\mathscr{P}$ *be an* $\perp$*-poset, a morphism* $\phi: \mathscr{P} \to \mathscr{P}$ *is said to be an* $\perp$*-morphism iff* ϕ *satisfies the further condition*:

(3) $x \perp y$ *implies* $\phi(x \vee y) = \phi(x) \vee \phi(y)$ *and* $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$.

It is easy to prove that $\operatorname{Hom}(\mathscr{P})$ *is a subsemigroup of* $\operatorname{End}(\mathscr{P})$. *Moreover also the set of* $\perp$*-morphisms of* $\mathscr{P}$ *is a subsemigroup of* $\operatorname{End}(\mathscr{P})$.

DEFINITION 5·3. *A morphism* $\phi: \mathscr{P} \to \mathscr{P}_1$ *is called isomorphism if it is a bijection which satisfies*:

(1′) $\phi(x) \leqslant_1 \phi(y)$ *implies* $x \leqslant y$,

(2′) $\phi(x) \perp_1 \phi(y)$ *implies* $x \perp y$.

We denote with $\operatorname{Is}(\mathscr{P}, \mathscr{P}_1)$ *the set of isomorphisms from* $\mathscr{P}$ *onto* $\mathscr{P}_1$. *We can therefore consider* $\phi^{-1}: \mathscr{P}_1 \to \mathscr{P}$ *for every isomorphism* ϕ *from* $\mathscr{P}$ *onto* $\mathscr{P}_1$ *and* $\phi^{-1} \in \operatorname{Is}(\mathscr{P}, \mathscr{P}_1)$. *An isomorphism from* $\mathscr{P}$ *onto* $\mathscr{P}$ *is called automorphism. The set of automorphism of* $\mathscr{P}$ *is denoted with* $\operatorname{Aut}(\mathscr{P})$. *Obviously* $\operatorname{Aut}(\mathscr{P})$ *is a group.*

PROPOSITION 5·1. *If* $\phi \in \operatorname{Hom}(\mathscr{P}, \mathscr{P}_1) \cap \operatorname{Is}(\mathscr{P}, \mathscr{P}_1)$ *then*

$$\phi^{-1} \in \operatorname{Hom}(\mathscr{P}_1, \mathscr{P}) \cap \operatorname{Is}(\mathscr{P}_1, \mathscr{P}).$$

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Proof. In fact if $x_1, y_1 \in \mathscr{P}_1$ we can set $x_1 = \phi(x)$, $y_1 = \phi(y)$ by the bijectivity of ϕ. Therefore $\phi^{-1}(x_1 \vee_1 y_1) = \phi^{-1}(\phi(x) \vee_1 \phi(y))$. But (1) Definition 5·2 implies

$$\phi^{-1}(x_1 \vee_1 y_1) = \phi^{-1}(\phi(x \vee y)) = x \vee y = \phi^{-1}(x_1) \vee \phi^{-1}(y_1),$$

that is condition (1) Definition 5·2 is satisfied by ϕ^{-1}. Condition (2) Definition 5·2 is verified in a similar way.

It follows from this result that $\mathrm{Aut}(\mathscr{P}) \cap \mathrm{Hom}(\mathscr{P})$ is a subgroup of $\mathrm{Aut}(\mathscr{P})$.

PROPOSITION 5·2. *If* $\phi \in \mathrm{Is}(\mathscr{P}, \mathscr{P}_1)$ *then* $\mathrm{Ker}\,\phi = \mathrm{Ker}\perp$.

Proof. We have already seen that $\mathrm{Ker}\perp \subseteq \mathrm{Ker}\,\phi$. On the other hand $x \in \mathrm{Ker}\,\phi$ implies $\phi(x) \perp_1 \phi(x)$. From this and (2′) Definition 5·3 $x \perp x$ follows.

$\mathrm{Mor}(\mathscr{P}, \mathscr{P}_1)$ has the property of being a poset with respect to the following order relation:

$$\phi_1 \leqslant \phi_2 \quad \text{iff} \quad \phi_1(x) \leqslant_1 \phi_2(x), \quad \forall x \in \mathscr{P}. \tag{3}$$

Moreover $\mathrm{Mor}(\mathscr{P}, \mathscr{P}_1)$, provided with the binary relation:

$$\phi_1 \perp\!\!\!\perp \phi_2 \quad \text{iff} \quad \phi_1(x) \perp_1 \phi_2(x), \quad \forall x \in \mathscr{P} \tag{4}$$

is a poset with orthogonality, as it is easy to verify, for which

$$\mathrm{Ker}\perp\!\!\!\perp = \{\phi \in \mathrm{Mor}(\mathscr{P}, \mathscr{P}_1) : \phi(\mathscr{P}) \subseteq \mathrm{Ker}\perp_1\}.$$

The order relation just introduced satisfies the following:

THEOREM 5·1. *Let* $\{\phi_\alpha : \alpha \in A\}$ *be a family of elements of* $\mathrm{Mor}(\mathscr{P}, \mathscr{P}_1)$ *with index set* A. *Then we have that*:

$$\wedge_\alpha(\phi_\alpha(x)) \text{ exists for } \forall x \in \mathscr{P} \quad \text{implies} \quad \wedge_\alpha \phi_\alpha \text{ exists}$$

and

$$(\wedge_\alpha \phi_\alpha)(x) = \wedge_\alpha(\phi_\alpha(x)), \quad \forall x \in \mathscr{P}.$$

Proof. If $\wedge_\alpha(\phi_\alpha(x))$ exists for $\forall x \in \mathscr{P}$ the mapping $\Phi : \mathscr{P} \to \mathscr{P}_1$ defined by the relation $\Phi(x) = \wedge_\alpha(\phi_\alpha(x))$ belongs to $\mathrm{Mor}(\mathscr{P}, \mathscr{P}_1)$. In fact $x \leqslant y$ implies $\phi_\alpha(x) \leqslant \phi_\alpha(y)$, $\forall \alpha \in A$ by (1) Definition 5·1 and thus $\wedge_\alpha(\phi_\alpha(x)) \leqslant \wedge_\alpha(\phi_\alpha(y))$. Moreover, if $x \perp y$ then $\phi_\alpha(x) \perp_1 \phi_\alpha(y)$, $\forall \alpha \in A$ follows by (2) Definition 5·1. Hence we have $\wedge_\alpha(\phi_\alpha(x)) \perp_1 \phi_\alpha(y)$ and from this $\wedge_\alpha(\phi_\alpha(x)) \perp_1 \wedge_\alpha(\phi_\alpha(y))$ follows.

Since $\Phi(x) \leqslant \phi_\alpha(x)$, $\forall \alpha \in A$ and $\forall x \in \mathscr{P}$, from (3) $\Phi \leqslant \phi_\alpha$, $\forall \alpha \in A$ follows. That is Φ is a lower bound of the set $\{\phi_\alpha : \alpha \in A\}$.

Let $\phi \in \mathrm{Mor}(\mathscr{P}, \mathscr{P}_1)$ be any lower bound of $\{\phi_\alpha : \alpha \in A\}$. Then $\phi \leqslant \phi_\alpha$, $\forall \alpha \in A$ implies $\phi(x) \leqslant \phi_\alpha(x)$, $\forall \alpha \in A$ and $\forall x \in \mathscr{P}$. Hence we have $\phi(x) \leqslant \vee_\alpha(\phi_\alpha(x))$, $\forall x \in \mathscr{P}$, differently written $\phi(x) \leqslant \Phi(x)$, $\forall x \in \mathscr{P}$, and so $\phi \leqslant \Phi$ by (3).

Therefore Φ is the g.l.b. of $\{\phi_\alpha : \alpha \in A\}$.

VI. *Additive monoids and positive semi-rings.* It is known that a Boolean ring with unity (see (6), page 48) can become a Boolean algebra relatively to the binary operations $\wedge$ and $\vee$ defined by the relations $a \wedge b = a.b$ and $a \vee b = a + b - ab$. In this section, after the definition of additive monoid, we introduce the structure of a positive semi-ring and prove that the set of idempotent elements of a positive semi-ring ordered by the relation $\leqslant$ defined by '$x \leqslant y$ iff $x = x.y = y.x$' is an $\perp$-poset $\perp$-modular relatively to the orthogonality defined by '$x \perp y$ iff $x.y = y.x = 0$'.

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DEFINITION 6·1. *Additive monoid. An additive (i.e. Abelian) semigroup $\mathscr{B}$ is called an additive monoid iff the following conditions are satisfied*:

(1) *an element* 0 *exists in* $\mathscr{P}$ *such that* $x+0=x$, $\forall x\in\mathscr{B}$,

(2) $x+O_x=x$ *implies* $O_x=0$.

A regular monoid is an additive monoid for which the following condition is satisfied:

(3) $x+x_1=0$ *implies* $x=x_1=0$.

DEFINITION 6·2. *Positive semi-ring. A positive semi-ring is a set $\mathscr{A}$ in which two binary algebraic operations are given, addition and multiplication, such that*:

(1) $\mathscr{A}$ *is an additive monoid under the operation of addition,*

(2) $\mathscr{A}$ *is a semi-group with unity under the operation of multiplication,*

(3) *multiplication is right and left distributive with respect to addition.*

A positive semi-ring which is a regular monoid under addition is more simply called semi-ring.

An element $a\in\mathscr{A}$ is said to be idempotent iff $a\,.\,a=a^2=a$. The set of idempotent elements of $\mathscr{A}$ is denoted with $\mathscr{I}(\mathscr{A})$.

LEMMA 6·1. *In a positive semi-ring* $0\,.\,a=a\,.\,0=0$, $\forall a\in\mathscr{A}$.

Proof. From $a(b+0)=a\,.\,b$ we have that $ab+a\,.\,0=ab$ and so $a\,.\,0=0$. In a similar way we obtain $0\,.\,a=0$.

In a positive semi-ring 0 and 1 are idempotent elements. The binary relation $\leqslant$ defined on $\mathscr{I}(\mathscr{A})$ by the rule '$a\leqslant b$ iff $a=ab=ba$' is an order relation which we shall call the canonical order relation on $\mathscr{I}(\mathscr{A})$. This order relation is such that $0\leqslant a\leqslant 1$, $\forall a\in\mathscr{I}(\mathscr{A})$.

PROPOSITION 6·1. *In the poset $\mathscr{I}(\mathscr{A})$ the binary relation $\perp$ defined by*:

$$a\perp b \quad \text{iff} \quad ab=ba=0$$

is a weak non-degenerate orthogonality relation.

Proof. Obviously $\perp$ is a symmetric relation. Moreover, if $a_0\leqslant a$ and $a\perp b$ then $a_0b=(a_0a)b=a_0(ab)=0$ and $ba_0=b(aa_0)=(ba)a_0=0$ from which $a_0b=ba_0=0$ follows, concluding that $a_0\perp b$. The non-degeneration of $\perp$ is easy to prove.

PROPOSITION 6·2. *In the poset $\mathscr{I}(\mathscr{A})$ the following propositions are verified*:

(i) $ab=ba$ *implies* $ab\in\mathscr{I}(\mathscr{A})$ *and* $ab=a\wedge b$;

(ii) $a\perp b$ *implies* $a+b\in\mathscr{I}(\mathscr{A})$ *and* $a+b=a\vee b$.

Proof. (i) Let $ab=ba$. It follows from $(ab)^2=(ab)(ab)=a(ba)b=a^2b^2=ab$ that ab is an idempotent element of $\mathscr{A}$. Similarly we can prove that $ba\in\mathscr{I}(\mathscr{A})$. From $ab=a(ab)=a(ba)=(ab)a$ we have that $ab\leqslant a$. Similarly we obtain that $ab\leqslant b$. Thus ab is a lower bound of $\{a,b\}$. If c is such that $c\leqslant a$ and $c\leqslant b$, then $c(ab)=(ca)b=cb=c$ and $(ab)c=a(bc)=ac=c$ follow, and so $c\leqslant ab$. Therefore ab is the g.l.b. of $\{a,b\}$ in $\mathscr{I}(\mathscr{A})$, concluding that $a\wedge b=ab$.

(ii) Let $a\perp b$. From $(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2=a^2+b^2=a+b$ we have that $a+b\in\mathscr{I}(\mathscr{A})$. Moreover, $(a+b)a=a^2+ba=a$ and $a(a+b)=a^2+ab=a$

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imply $a \leqslant a+b$. Since by a similar procedure $b \leqslant a+b$ follows, then we conclude that $a+b$ is an upper bound of $\{a,b\}$. If c is such that $a \leqslant c$ and $b \leqslant c$, then

$$c(a+b) = ca+cb = a+b \quad \text{and} \quad (a+b)c = ac+bc = a+b$$

follow, and so $a+b \leqslant c$. Therefore $a+b$ is the l.u.b. of $\{a,b\}$ in $\mathscr{I}(\mathscr{A})$, i.e. $a \vee b = a+b$.

PROPOSITION 6·3. *If $\mathscr{A}$ is a semi-ring then in $\mathscr{I}(\mathscr{A})$ is verified the inverse proposition of* (ii) *Proposition* 6·2. *That is*

$$a+b \in \mathscr{I}(\mathscr{A}) \quad \textit{implies} \quad a \perp b \quad \textit{and} \quad a+b = a \vee b.$$

Proof. Since $a+b = (a+b)^2 = a+ab+ba+b$ then (2) Definition 6·1 implies $ab+ba = 0$ and from (3) Definition 6·1 $ab = ba = 0$ follows.

THEOREM 6·1. $(\mathscr{I}(\mathscr{A}), \leqslant, \perp)$ *is a weak non-degenerate $\perp$-modular $\perp$-poset.*

Proof. By Proposition 6·1 $(\mathscr{I}(\mathscr{A}), \leqslant, \perp)$ is a $w \perp p$. Moreover, by 6·2 we have that $a \perp b$ implies $a \vee b = a+b$ and $a \wedge b = 0$, so $\mathscr{I}(\mathscr{A})$ is a weak non-degenerate $\perp$-poset.

Let $a \perp b$ and $a_0 \leqslant b$. Then $a \wedge b$ exists and $a \wedge b = 0$. Since $a_0 \leqslant b$ implies $a_0 \perp a$, we have also that $a_0 \vee a$ exists and $a_0 \vee a = a_0 + a$. Hence

$$(a_0 \vee a)b = (a_0+a)b = a_0 b + ab = ba_0 + ba = b(a_0+a),$$

i.e. $(a_0 \vee a)b = b(a_0 \vee a)$. Using (i) Proposition 6·2 we have that $(a_0 \vee a) \wedge b$ exists and $(a_0 \vee a) \wedge b = (a_0+a)b = a_0$ and from Lemma 6·1, (ii) Proposition 6·2 and $a \wedge b = 0$ we have that $a_0 \vee (a \wedge b)$ exists and $a_0 \vee (a \wedge b) = 0$. Therefore we conclude that

$$\exists\, (a_0 \vee a) \wedge b = a_0 \vee (a \wedge b),$$

i.e. $a \perp b$ implies $(a,b)\,M$.

Let $\mathscr{B}$ and $\mathscr{B}_1$ be additive monoids. A mapping $\phi\colon \mathscr{B} \to \mathscr{B}_1$ is said to be a morphism from $\mathscr{B}$ to $\mathscr{B}_1$ iff $\phi(x+y) = \phi(x)+\phi(y)$, $\forall x$ and $\forall y \in \mathscr{B}$; particularly ϕ is called an endomorphism if $\mathscr{B} = \mathscr{B}_1$ and an additive functional if $\mathscr{B}_1 = \mathbb{R}^+$. By $\mathrm{Mor}(\mathscr{B}, \mathscr{B}_1)$, $\mathrm{End}(\mathscr{B})$ and $\mathscr{F}(\mathscr{B})$ we denote, respectively, the set of morphisms from $\mathscr{B}$ to $\mathscr{B}_1$, the set of endomorphisms of $\mathscr{B}$ and the set of additive functionals of $\mathscr{B}$. On $\mathrm{Mor}(\mathscr{B}, \mathscr{B}_1)$ it is possible to introduce an algebraic structure. Precisely if ϕ_1 and ϕ_2 belong to $\mathrm{Mor}(\mathscr{B}, \mathscr{B}_1)$, then the rule

$$(\phi_1+\phi_2)(x) \overset{\text{def}}{=} \phi_1(x)+\phi_2(x), \quad \forall x \in \mathscr{B}$$

defines into $\mathrm{Mor}(\mathscr{B}, \mathscr{B}_1)$ an additive operation which satisfies conditions (1) and (2) of Definition 6·1. Therefore $\mathrm{Mor}(\mathscr{B}, \mathscr{B}_1)$, and likewise $\mathrm{End}(\mathscr{B})$ and $\mathscr{F}(\mathscr{B})$, are additive monoids under this addition operation. $\mathrm{End}(\mathscr{B})$ can be provided with a product or composition $\phi_1 \circ \phi_2$, $\forall \phi_1$ and $\forall \phi_2 \in \mathrm{End}(\mathscr{B})$ defined by the rule

$$(\phi_1 \circ \phi_2)(x) \overset{\text{def}}{=} \phi_1(\phi_2(x)), \quad \forall x \in \mathscr{B}.$$

It is easy to prove that $\mathrm{End}(\mathscr{B})$ is a positive semi-ring under the addition and the multiplication operations just introduced.

If $\phi \in \mathrm{Mor}(\mathscr{B}, \mathscr{B}_1)$ then $\phi(0) = 0_1$. In fact from $\phi(x) = \phi(x+0) = \phi(x)+\phi(0)$ and from (2) Definition 6·1 $\phi(0) = 0_1$ follows. Obviously this property is also verified on $\mathrm{End}(\mathscr{B})$ and $\mathscr{F}(\mathscr{B})$.

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A (pre) order defined on $\mathscr{B}_1$ induces *a* (pre) order on $\mathrm{Mor}(\mathscr{B},\mathscr{B}_1)$ by the relation $\phi_1 \leqslant \phi_2$ iff $\phi_1(x) \leqslant \phi_2(x)$, $\forall x \in \mathscr{B}$. Particularly we can introduce on $\mathscr{F}(\mathscr{B})$ the order relation induced from the ordered positive real line.

VII. *Additive triple.* Let $\mathscr{B}$ be an additive monoid; following (4) the triple $(\mathscr{B}, \mathrm{End}(\mathscr{B}), \mathscr{F}(\mathscr{B}))$ is called an additive information system.

This triple can be viewed as the mathematical scheme which allows for a description of a typical experimental situation including also quantum mechanics. Schematically, and without getting into some delicate problems which arise from the physical interpretation of elements of an additive information system, we can imagine that rules for constructing a precise apparatus producing a certain physical system are associated to every element x of $\mathscr{B}$. Each element ϕ of $\mathrm{End}(\mathscr{B})$ can be considered as an 'interaction' which acts on a physical system x of $\mathscr{B}$ producing, as final result, a new physical system $\phi(x)$ of $\mathscr{B}$. Finally, we can imagine that to every element f of $\mathscr{F}(\mathscr{B})$ rules are associated for constructing an apparatus which allows us to perform a well-defined observation on every physical system x. The result of this observation is expressed by the positive real number $f(x)$. According to these interpretation rules a triple $(x,\phi,f) \in (\mathscr{B}, \mathrm{End}(\mathscr{B}), \mathscr{F}(\mathscr{B}))$ represents a precise experimental device formed by a preparation, an interaction on the produced physical system and an observation; that is a usual enough situation in the experimental sciences. Of course various algebraic operations introduced on the sets constituting an additive information system must be interpreted by the above interpretative rules. This is in part the subject of another work of ours(7). In a first approximation the addition defined on $\mathscr{B}$ describes the physical situation of a 'mixture' of two physical systems. On $\mathscr{B}$ the binary relation $\leqslant$ defined by

$$x \leqslant y \quad \text{iff} \quad f(x) \leqslant f(y), \quad \forall f \in \mathscr{F}(\mathscr{B}) \tag{1}$$

is a partial preorder, as it is easy to prove, which is a partial order if and only if the following condition is verified:

$$f(x) = f(y), \quad \forall f \in \mathscr{F}(\mathscr{B}) \quad \text{implies} \quad x = y. \tag{2}$$

As seen at the end of VI, this preorder on $\mathscr{B}$ induces a preorder on $\mathrm{End}(\mathscr{B})$. In the following we will refer to these preorders.

PROPOSITION 7·1. *In* $(\mathscr{B}, \mathrm{End}(\mathscr{B}), \mathscr{F}(\mathscr{B}))$ *the following propositions are verified*:

(i) $x, y \in \mathscr{B}$ *and* $x \leqslant y$ *imply* $x+z \leqslant y+z$, $\forall z \in \mathscr{B}$,
(ii) $0 \leqslant x$, $\forall x \in \mathscr{B}$,
(iii) $x, y \in \mathscr{B}$ *and* $x \leqslant y$ *imply* $x \leqslant y+z$, $\forall z \in \mathscr{B}$,
(iv) $x, y \in \mathscr{B}$ *and* $x \leqslant y$ *imply* $\phi(x) \leqslant \phi(y)$, $\forall \phi \in \mathrm{End}(\mathscr{B})$,
(v) $0 \leqslant \phi$, $\forall \phi \in \mathrm{End}(\mathscr{B})$.

Proof. (i) $f(x) \leqslant f(y)$, $\forall f \in \mathscr{F}(\mathscr{B})$ implies

$$f(x)+f(z) \leqslant f(y)+f(z), \quad \forall f \in \mathscr{F}(\mathscr{B}) \quad \text{and} \quad \forall z \in \mathscr{B},$$

i.e. $f(x+z) \leqslant f(y+z)$; hence $x+z \leqslant y+z$, $\forall z \in \mathscr{B}$ follows.

(ii) and (iii) can be proved analogously.

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(iv) Let $f(x) \leqslant f(y)$, $\forall f \in \mathscr{F}(\mathscr{B})$. If $\phi_0 \in \mathrm{End}(\mathscr{B})$ then $f \circ \phi_0 \in \mathscr{F}(\mathscr{B})$, $\forall f \in \mathscr{F}(\mathscr{B})$. Thus

$$f(\phi_0(x)) \leqslant f(\phi_0(y)), \quad \forall f \in \mathscr{F}(\mathscr{B})$$

is verified; hence $\phi_0(x) \leqslant \phi_0(y)$, $\forall \phi_0 \in \mathrm{End}(\mathscr{B})$.

(v) is an immediate consequence of (ii) and (iv).

It follows from (i) and (ii) Proposition 7·1 that $(\mathscr{B}, \leqslant, +)$ is an additive monoid in which partial preorder defined by (1) is compatible with addition. Moreover, every non-null element of $\mathscr{B}$ is positive. The fact is also interesting that every endomorphism of $\mathscr{B}$ is isotone (see (6), page 2). The preorder induced on the positive semi-ring $(\mathrm{End}(\mathscr{B}), +, \circ, \leqslant)$ is evidently compatible with addition, i.e. $\phi_1 \leqslant \phi_2$ implies $\phi_1 + \phi \leqslant \phi_2 + \phi$, $\forall \phi \in \mathrm{End}(\mathscr{B})$. But compatibility is also satisfied between preorder and composition, as shown in the following:

PROPOSITION 7·2. (i) $\phi_1, \phi_2 \in \mathrm{End}(\mathscr{B})$ *and* $\phi_1 \leqslant \phi_2$ *imply*

$$\phi \circ \phi_1 \leqslant \phi \circ \phi_2 \quad \textit{and} \quad \phi_1 \circ \phi \leqslant \phi_2 \circ \phi, \quad \forall \phi \in \mathrm{End}(\mathscr{B}).$$

(ii) *Let* $\phi_1, \phi_2 \in \mathrm{End}(\mathscr{B})$. *If* $\phi \in \mathrm{End}(\mathscr{B})$ *is surjective then* $\phi_1 \circ \phi \leqslant \phi_2 \circ \phi$ *implies* $\phi_1 \leqslant \phi_2$.

Proof. (i) Let $\phi_1(x) \leqslant \phi_2(x)$, $\forall x \in \mathscr{B}$. Setting $x = \phi(\bar{x})$, $\forall \bar{x} \in \mathscr{B}$ we have that $\phi_1(\phi(\bar{x})) \leqslant \phi_2(\phi(\bar{x}))$, $\forall \bar{x} \in \mathscr{B}$ and thus $\phi_1 \circ \phi \leqslant \phi_2 \circ \phi$ follows. On the other hand, if $\phi_1 \leqslant \phi_2$ then $f(\phi_1(x)) \leqslant f(\phi_2(x))$, $\forall x \in \mathscr{B}$ and $\forall f \in \mathscr{F}(\mathscr{B})$. If $\phi_0 \in \mathrm{End}(\mathscr{B})$, then $f \circ \phi_0 \in \mathscr{F}(\mathscr{B})$ for $\forall f \in \mathscr{F}(\mathscr{B})$ and so $(f \circ \phi_0 \circ \phi_1)(x) \leqslant (f \circ \phi_0 \circ \phi_2)(x)$, $\forall f \in \mathscr{F}(\mathscr{B})$ and $\forall x \in \mathscr{B}$. Hence $\phi_0 \circ \phi_1 \leqslant \phi_0 \circ \phi_2$ follows for $\forall \phi_0 \in \mathrm{End}(\mathscr{B})$.

(ii) By assumption we have that for $\forall x \in \mathscr{B}$, $\bar{x} \in \mathscr{B}$ exists such that $x = \phi(\bar{x})$. Then $\phi_1(\phi(\bar{x})) \leqslant \phi_2(\phi(\bar{x}))$, $\forall \bar{x} \in \mathscr{B}$ implies $\phi_1(x) \leqslant \phi_2(x)$, $\forall x \in \mathscr{B}$, i.e. $\phi_1 \leqslant \phi_2$.

Note that the results obtained in this section are still true if $\mathscr{B}$ is an additive quasigroup (i.e. a set with a commutative binary operation) with unity because associativity and conditions (2) and (3) Definition 6·1 have never been used. Notions of morphism, endomorphism and additive functional can be consequently given.

DEFINITION 7·1. *Additive triple. Let* $\tilde{\mathscr{B}}$ *be an additive quasigroup with unity then* $\mathscr{C}(\tilde{\mathscr{B}}) = (\tilde{\mathscr{B}}, \mathrm{End}(\tilde{\mathscr{B}}), \mathscr{F}(\tilde{\mathscr{B}}))$ *is called an additive triple iff* (1) *is an order relation on* $\tilde{\mathscr{B}}$.

PROPOSITION 7·3. *Let* $\mathscr{C}(\tilde{\mathscr{B}})$ *be an additive triple then* $\tilde{\mathscr{B}}$ *is a regular monoid. Thus every additive triple is an additive information system.*

Proof. Since

$$f[(x+y)+z] = f(x+y)+f(z) = f(x)+f(y+z) = f[x+(y+z)], \quad \forall f \in \mathscr{F}(\tilde{\mathscr{B}})$$

we have $(x+y)+z = x+(y+z)$, and so the addition defined on $\tilde{\mathscr{B}}$ is associative. On the other hand if $O_x \in \tilde{\mathscr{B}}$ is such that $x + O_x = x$ then $f(x)+f(O_x) = f(x)$, $\forall f \in \mathscr{F}(\tilde{\mathscr{B}})$ follows; hence $f(O_x) = 0$, $\forall f \in \mathscr{F}(\tilde{\mathscr{B}})$, so that $O_x = 0$. Finally, $x + x_1 = 0$ implies

$$f(x)+f(x_1) = 0, \quad \forall f \in \mathscr{F}(\mathscr{B})$$

which leads to

$$f(x) = f(x_1) = 0, \quad \forall f \in \mathscr{F}(\mathscr{B}),$$

that is $x = x_1 = 0$.

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Experimental considerations pointed out at the beginning of this section apply to additive triples. In this context condition (2) has the following interesting interpretation: two physical systems can be considered indistinguishable if all the observations performed on them give the same results.

Let $\mathscr{C}(\widetilde{\mathscr{B}})$ be an additive triple. We introduce on $\mathrm{End}(\widetilde{\mathscr{B}})$ the following binary relation:

$$\phi_1 \perp \phi_2 \quad \text{iff} \quad \phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 = 0. \tag{3}$$

Our goal is now to prove that this is a weak degenerate orthogonality relation with $\mathrm{Ker}\perp = \{\phi \in \mathrm{End}(\widetilde{\mathscr{B}}) : \phi^2 = 0\}$.

LEMMA. *Let $\mathscr{C}(\widetilde{\mathscr{B}})$ be an additive triple. Then the following properties are verified on* $\mathrm{End}(\widetilde{\mathscr{B}})$:

(i) $\quad \exists\, \phi_1 \vee \phi_2 \quad \textit{implies} \quad \phi_1 \vee \phi_2 \leqslant \phi_1 + \phi_2,$

(ii) $\quad \phi_1 \perp \phi_2 \quad \textit{and} \quad \phi_2 \perp \phi \quad \textit{imply} \quad (\phi_1 + \phi_2) \perp \phi.$

Proof. (iii) Proposition 7·1 implies $\phi_1(x) \leqslant \phi_1(x) + \phi_2(x)$, $\forall x \in \widetilde{\mathscr{B}}$, from which $\phi_1 \leqslant \phi_1 + \phi_2$ follows. Similarly we have that $\phi_2 \leqslant \phi_1 + \phi_2$, so that $\phi_1 \vee \phi_2 \leqslant \phi_1 + \phi_2$. (ii) Obviously $\phi \circ (\phi_1 + \phi_2) = (\phi_1 + \phi_2) \circ \phi = 0$.

THEOREM 7·1. $\mathrm{End}(\widetilde{\mathscr{B}})$ *with the binary relation defined by* (3) *is a wd*$\perp$*p. Moreover,* $(\mathrm{End}(\widetilde{\mathscr{B}}), \leqslant, \perp)$ *satisfies the properties*:

(i) $0 \perp \mathrm{End}(\widetilde{\mathscr{B}})$ (*see Proposition* 2·2),

(ii) *if* $\phi_1 \perp \phi$, $\phi_2 \perp \phi$ *and* $\phi_1 \vee \phi_2$ *exists then* $(\phi_1 \vee \phi_2) \perp \phi$.

Proof. The relation defined by (3) evidently satisfies the symmetry property and so we only need to prove the absorption property. Let $\phi \perp \phi_1$ and $\phi_0 \leqslant \phi$. Then

$$0 \leqslant \phi_0 \circ \phi_1 \leqslant \phi \circ \phi_1 = 0 \quad \text{and} \quad 0 \leqslant \phi_1 \circ \phi_0 \leqslant \phi_1 \circ \phi = 0$$

imply $\phi_0 \perp \phi_1$. On the other hand we have $\phi \circ 0 = 0 \circ \phi = 0$, $\forall \phi \in \mathrm{End}(\widetilde{\mathscr{B}})$ and thus (i) is true. Finally the properties of lemma imply $\phi_1 \vee \phi_2 \leqslant \phi_1 + \phi_2$ and $(\phi_1 + \phi_2) \perp \phi$, from which we conclude that $(\phi_1 \vee \phi_2) \perp \phi$.

It is interesting to remark that (i) Theorem 7·1 is condition (4″) Definition 2·4 while (ii) Theorem (7·1) is condition (4′) restricted to the case of a finite index set A.

On the positive semi-ring $\mathrm{End}(\widetilde{\mathscr{B}})$ we can apply the results obtained in section (VI). Precisely, if $\mathscr{I}(\widetilde{\mathscr{B}})$ denotes the idempotent elements of $\mathrm{End}(\widetilde{\mathscr{B}})$ and $\leqslant_1$ the canonical order relation, we have that $(\mathscr{I}(\widetilde{\mathscr{B}}), \leqslant_1, \perp)$ is a weak non-degenerate $\perp$-modular $\perp$-poset relatively to an orthogonality relation which is formally identical to (3). But the order relation which constitutes the 'support' of this orthogonality relation evidently differs from the order relation induced on $\mathscr{I}(\widetilde{\mathscr{B}})$ by $\mathrm{End}(\widetilde{\mathscr{B}})$. We now introduce a subset of $\mathscr{I}(\widetilde{\mathscr{B}})$ in which the order relation $\leqslant_1$ induced by $\mathscr{I}(\widetilde{\mathscr{B}})$ and the order relation $\leqslant$ induced by $\mathrm{End}(\widetilde{\mathscr{B}})$ are identical. For this purpose we say that an element $\phi \in \mathrm{End}(\widetilde{\mathscr{B}})$ is a '$\widetilde{\mathscr{B}}$-absorbent' iff $\phi \leqslant \mathbf{1}$. Obviously, if ϕ_1 and ϕ_2 are $\widetilde{\mathscr{B}}$-absorbents then also $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ are $\widetilde{\mathscr{B}}$-absorbents. Moreover, it is easy to prove that if ϕ is a $\widetilde{\mathscr{B}}$-absorbent and ϕ_1, ϕ_2 are elements of $\mathrm{End}(\widetilde{\mathscr{B}})$ such that $\phi_1 \perp \phi_2$ then $(\phi \circ \phi_1) \perp (\phi \circ \phi_2)$ and $(\phi_1 \circ \phi) \perp (\phi_2 \circ \phi)$. Finally, an element of $\mathrm{End}(\widetilde{\mathscr{B}})$ which is $\widetilde{\mathscr{B}}$-absorbent and idempotent is called '$\widetilde{\mathscr{B}}$-projection'. The set of $\widetilde{\mathscr{B}}$-projections is denoted with $\Pi(\widetilde{\mathscr{B}})$.

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THEOREM 7·2. *Let π_1 and π_2 be $\tilde{\mathscr{B}}$-projections. Then $\pi_1 \leqslant \pi_2$ is equivalent to $\pi_1 \leqslant_1 \pi_2$. Moreover $\pi_1 = \pi_1 \circ \pi_2$ iff $\pi_1 = \pi_2 \circ \pi_1$.*

Proof. Suppose $\pi_1 \leqslant \pi_2$. From $\pi_1 \leqslant \pi_2$ we have that $\pi_1^2 \leqslant \pi_1 \circ \pi_2$ by (i) Proposition 7·2 and this implies $\pi_1 \leqslant \pi_1 \circ \pi_2$. On the other hand $\pi_2 \leqslant \mathbf{1}$ leads to $\pi_1 \circ \pi_2 \leqslant \pi_1$ and so $\pi_1 = \pi_1 \circ \pi_2$. Moreover if $\pi_1 \leqslant \pi_2$ then $\pi_1^2 \leqslant \pi_2 \circ \pi_1$, from which $\pi_1 \leqslant \pi_2 \circ \pi_1$ follows. But $\pi_2 \leqslant \mathbf{1}$ implies $\pi_2 \circ \pi_1 \leqslant \pi_1$ and so $\pi_1 = \pi_2 \circ \pi_1$. On the contrary let

$$\pi_1 = \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$$

then from $\pi_1 \leqslant \mathbf{1}$ we have $\pi_2 \circ \pi_1 \leqslant \pi_2$ and thus $\pi_1 \leqslant \pi_2$.

THEOREM 7·3. *In $\Pi(\tilde{\mathscr{B}})$ the following statements are equivalent*:

(i) $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$,

(ii) $\pi_1 \circ \pi_2$ *and* $\pi_2 \circ \pi_1 \in \Pi(\tilde{\mathscr{B}})$,

(iii) $\pi_1 \wedge \pi_2$ *exists in* $\Pi(\tilde{\mathscr{B}})$

and in each of these cases we have that $\pi_1 \wedge \pi_2 = \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$.

Proof. (i) $\Rightarrow$ (ii) is analogous to the proof of (i) Proposition 6·2.

(ii) $\Rightarrow$ (i). Let $\pi_1 \circ \pi_2$ be an element of $\Pi(\tilde{\mathscr{B}})$. From $\pi_2 \leqslant \mathbf{1}$ we have $\pi_1 \circ \pi_2 \leqslant \pi_1$ by (i) Proposition 7·2. Moreover, $\pi_1 \leqslant \mathbf{1}$ leads to $\pi_1 \circ \pi_2 \leqslant \pi_2$. Thus $\pi_1 \circ \pi_2$ is a lower bound of $\{\pi_1, \pi_2\}$. If $\pi \in \Pi(\tilde{\mathscr{B}})$ is such that $\pi \leqslant \pi_1$ and $\pi \leqslant \pi_2$ then $\pi \circ \pi_2 \leqslant \pi_1 \circ \pi_2$ follows from $\pi \leqslant \pi_1$. But by the previous theorem $\pi \leqslant \pi_2$ implies $\pi = \pi \circ \pi_2$ and so $\pi \leqslant \pi_1 \circ \pi_2$. Therefore $\pi_1 \circ \pi_2$ is the g.l.b. of $\{\pi_1, \pi_2\}$ in $\Pi(\tilde{\mathscr{B}})$, concluding that $\pi_1 \wedge \pi_2$ exists in $\Pi(\tilde{\mathscr{B}})$ and $\pi_1 \wedge \pi_2 = \pi_1 \circ \pi_2$. Similarly if we repeat these passages substituting 1 with 2 and 2 with 1 we obtain that $\pi_2 \circ \pi_1$ is the g.l.b. of $\{\pi_1, \pi_2\}$ in $\Pi(\tilde{\mathscr{B}})$ if $\pi_2 \circ \pi_1 \in \Pi(\tilde{\mathscr{B}})$. Hence $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$ follows by the unicity of the g.l.b.

(i) $\Rightarrow$ (iii) is implicit in the proof of (i) Proposition 6·2.

(iii) $\Rightarrow$ (i). If $\pi_1 \wedge \pi_2$ exists in $\Pi(\tilde{\mathscr{B}})$ then $\pi_1 \wedge \pi_2 \leqslant \pi_1$ and $\pi_1 \wedge \pi_2 \leqslant \pi_2$ imply $\pi_1 \wedge \pi_2 \leqslant \pi_1 \circ \pi_2$ and $\pi_1 \wedge \pi_2 \leqslant \pi_2 \circ \pi_1$ by idempotence and (i) Proposition 7·2. But we have that $\pi_1 \circ \pi_2 \leqslant \pi_1$ and $\pi_1 \circ \pi_2 \leqslant \pi_2$, since π_1 and π_2 are also $\tilde{\mathscr{B}}$-absorbents, and so $\pi_1 \circ \pi_2 \leqslant \pi_1 \wedge \pi_2$. Similarly we can prove that $\pi_2 \circ \pi_1 \leqslant \pi_1 \wedge \pi_2$. Therefore we conclude that $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = \pi_1 \wedge \pi_2$.

Obviously $(\Pi(\tilde{\mathscr{B}}), \leqslant, \perp)$ is a w$\perp$p for which (i) and (ii) properties, Theorem 7·1, hold. But we cannot apply the results of Theorem 6·1 since $\pi_1 \perp \pi_2$ does not generally imply $\pi_1 + \pi_2 \in \Pi(\tilde{\mathscr{B}})$. However it is easy to prove that the binary relation $\underline{\perp}$ defined on $\Pi(\tilde{\mathscr{B}})$ by:

$$\text{`}\pi_1 \underline{\perp} \pi_2 \quad \text{iff} \quad \pi_1 \perp \pi_2 \quad \text{and} \quad \pi_2 + \pi_2 \in \Pi(\tilde{\mathscr{B}})\text{'} \tag{4}$$

is an orthogonality relation and $(\Pi(\tilde{\mathscr{B}}), \leqslant, \underline{\perp})$ is a weak $\perp$-modular $\perp$-poset.

This result is very interesting, if we take into consideration the so-called 'logical' approach to the foundation of Quantum Mechanics. Here (see (3) and (8)) the concept of 'event' or 'yes–no experiment', interpreted as an experiment which admits only one of two alternatives, is assumed as a primitive one. The aim of the logical approach is to find the minimal set of empirically well founded axioms from which the usual Hilbert space model of Quantum Mechanics may be deduced as a concrete realization.

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In view of this purpose, the events are usually assumed to be an orthomodular lattice, since the poset of projections of a Hilbert space (which are the events of this model) is an orthomodular poset. In recent works, the existence of the meet and join of any two events(4) or of the orthocomplement of any event(5) has been questioned from the point of view of their physical interpretation and in (4) it is shown that the existence in any case of these events makes the class of admissible quantum systems very narrow. In this context the additive information system not only represents a nice application of orthogonality concept but it is also a physically interesting example, in which the strong condition that the set of projections is an orthocomplemented lattice is not assumed; this gives rise to a structure in which the most physically reasonable properties of the Hilbert space model are preserved, at least in a weaker form, f.i. the fact that $(\Pi(\widetilde{\mathscr{H}}), \leqslant, \underline{\perp})$ is a $\perp$-modular $\perp$-poset.

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