MODAL CHARACTERISTICS OF PLANAR MULTI-STORY FRAME STRUCTURES

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ABSTRACT

In linear system, in-plane motions are decoupled from out-of-plane motions for planar frame structures. A theoretical method is proposed that permits the efficient calculations of modal characteristics of planar multi-story frame structures. There are $3 \times m$ beam components for a planar m-story frame structure. By analyzing the transverse and longitudinal motions of each component simultaneously and considering the compatibility requirements across each frame joint, the undetermined variables of the entire m-story frame structure system can be reduced to six, regardless of the number of stories, and that can be determined by the application of the boundary conditions. The main feature of this method is to decrease the dimensions of the matrix involved in the finite element methods and certain other analytical methods.

Keywords: Multi-story planar frame structures, Eigensolutions, Transfer matrix, Mode shapes.

1. INTRODUCTION

Frame structures are usually found in engineering designs, ranging from big scale building structures to micro-frame structures used in modern electronic equipments. These frame structures are often subjected to harmful vibrations. In linear system, in-plane motions are decoupled from out-of-plane motions for planar frame structures. The dynamic behaviors of frame structures can be predicted by using certain analytical and numerical methods such as the dynamic stiffness methods (DSM) and the finite element methods (FEM). The DSM employs the solutions of governing equations under harmonic nodal excitations as shape functions to formulate an analytical stiffness matrix. The method requires closed-form solutions of the governing equations which restrict the application areas [1]. The FEM has been very commonly used in recent years in this field. However, the FEM requires a large amount of computer memory and computation time, since many degrees of freedom are required for accurately solving dynamic problems for these structures [2, 3]. To solve this problem, various methods have been studied to overcome the disadvantages [2, 4, 5]. In most of the previous studies, the Euler-Bernoulli beam-theory models obtained by deriving the differential equations and the associated boundary conditions for a basic uniform Euler-Bernoulli beam are often used and discussed. Some researchers have also studied the different results between the Euler-Bernoulli beam-theory models and the Timoshenko beam theory [6]. Finally, it is possible to evaluate natural frequencies simply by finding roots of the *high order determinant* of the coefficient matrix of the linear system if accuracy of the eigensolutions is required. Lin and Ro [7] used a transfer matrix method to solve the dynamics of planar "serial-frame structures" and the order of the coefficient matrix to find the roots of the determinant can be reduced. However, its application is restricted to "serial-frame structures" in which only two beams are connected for every joint in the structure.

In this paper, a hybrid analytical/numerical method that permits the efficient computation of the eigensolutions for planar multi-story frame structures with various boundary conditions. For planar multi-story frame structures, the connecting elements for every joint are not restricted to two. The method is based on partitioning the frame structure to the sub-beam components and considering the transverse and longitudinal motions of each component and, by the compatibility requirements across each frame joint, the relationships of the six integration constants of the eigenfunctions between adjacent sub-beams can be determined. By using the method similar to the transfer matrix method [7, 11], as a consequence, the entire planar multi-story frame structures has only six unknown constants and which can be solved through the satisfaction of six boundary conditions. In this article, the eigenvalue problem is solved by using closed-form solutions.

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2. THEORETICAL MODEL

A typical m-story planar frame structure with $3 \times m$ elements is shown in Fig. 1. The angles between any two elements are right angles. This structure is partitioned into $3 \times m$ components with lengths L_1, L_2, …, L_{3m-1}, L_{3m}, and the positions of the joints are located by X_i, respectively, and the boundary points are X_{3m-2}, X_{3m}, enabling a substructure approach. In doing the vibration

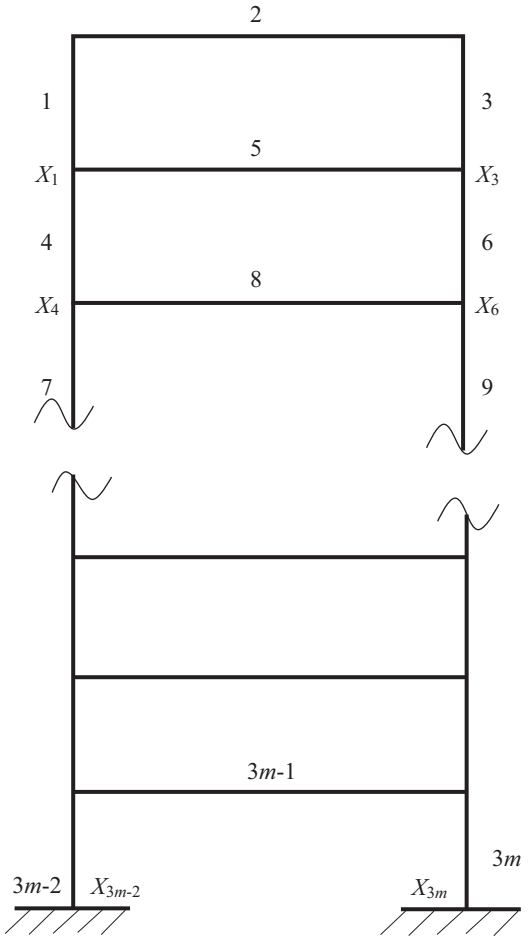


Fig. 1 A planar m-story frame structure with joints located at positions X_1, X_3, X_4, X_6… and boundaries X_{3m-2}, X_{3m} respectively; lengths of sub-beams are L_1, L_2, …, L_{3m-1}, L_{3m}.

analysis of this system for this study, each component member is analyzed by its transverse and longitudinal motions respectively. Let the X axis represent the longitudinal direction and the Y axis the transverse direction of each component member; then, the traditional vibration theories of an Euler-Bernoulli beam and the axial vibration of a rod are considered. The vibration amplitudes of the transverse and longitudinal displacements of the component i are denoted by $Y_i(X,T)$ and $U_i(X,T)$ and $i = 1, 2, \ldots, 3m$, as shown in Fig. 2. The entire system is now divided into $3m$ segments, wherein the total length of this frame system is L $(= L_1 + L_2 + \ldots + L_{3m})$. According to the literature [8-10], the equations of mo-

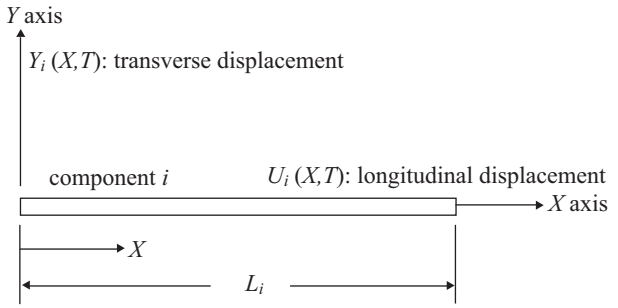


Fig. 2 Transverse and longitudinal motions of a component.

tion for each component, assumed with a uniform cross section, are

transverse motion:

$$EI\frac{\partial^4 Y_i(X,T)}{\partial X^4}+\rho A\frac{\partial^2 Y_i(X,T)}{\partial T^2}=0,\quad i=1,2,...,3m, \qquad (1)$$

longitudinal motion:

$$E\frac{\partial^2 U_i(X,T)}{\partial X^2}-\rho\frac{\partial^2 U_i(X,T)}{\partial T^2}=0,\quad i=1,2,...,3m, \qquad (2)$$

where E is Young's modulus of the material, I is the moment of inertia of the beam cross-section, ρ is the density of material, A is the cross-section area of the beam and T is time. The boundary conditions, the fixed-fixed supported case for this example, are

$$Y_{3m-2}(0,T)=Y'_{3m-2}(0,T)=U_{3m-2}(0,T)=0, \qquad (3a\text{-}c)$$

$$Y_{3m}(L_{3m},T)=Y'_{3m}(L_{3m},T)=U_{3m}(L_{3m},T)=0. \qquad (3d\text{-}f)$$

The top-story (or called the first story), as shown in Fig. 3a, of the structure is analyzed first. The transverse

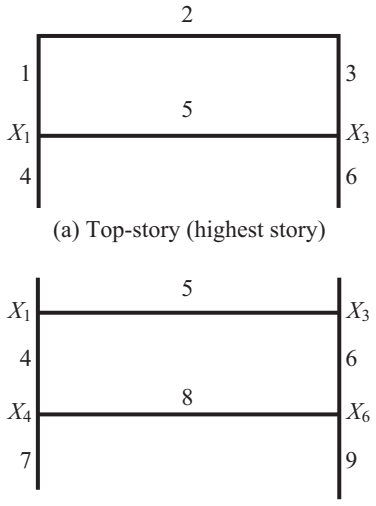


Fig. 3 (a) Top-story (first story) of the frame structure (b) Lower-story (second story, below the top-story) of the frame structure.

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and the longitudinal motions at the end of the segment constrain the motions of the adjacent segment(s). So the "compatibility conditions" enforce continuities in the displacement fields (in both transverse and longitudinal), slope, bending moment, shear force and axial force, respectively, across the joints X_1 and X_3 and can be expressed as:

for the joint X_1

$$Y_1(0,T) = Y_4(L_4,T), \quad \text{displacement continuity} \quad (4a)$$

$$U_1(0,T) = U_4(L_4,T), \quad \text{displacement continuity} \quad (4b)$$

$$Y_5(0,T) = U_4(L_4,T), \quad \text{displacement continuity} \quad (4c)$$

$$U_5(0,T) = -Y_4(L_4,T), \quad \text{displacement continuity} \quad (4d)$$

$$Y_1'(0,T) = Y_4'(L_4,T), \quad \text{slope continuity} \quad (4e)$$

$$Y_5'(0,T) = Y_4'(L_4,T), \quad \text{slope continuity} \quad (4f)$$

$$EIY_1''(0,T) = EIY_4''(L_4,T) - EIY_5''(0,T), \quad \text{moment continuity} \quad (4g)$$

$$EAU_1'(0,T) = EAU_4'(L_4,T) + EIY_5'''(0,T), \quad \text{axial force continuity, element 1} \quad (4h)$$

$$EAU_5'(0,T) = EIY_4'''(L_4,T) - EIY_1'''(0,T), \quad \text{axial force continuity, element 5} \quad (4i)$$

and for the joint X_3

$$Y_6(0,T) = Y_3(L_3,T), \quad \text{displacement continuity} \quad (5a)$$

$$U_6(0,T) = U_3(L_3,T), \quad \text{displacement continuity} \quad (5b)$$

$$Y_5(L_5,T) = -U_3(L_3,T), \quad \text{displacement continuity} \quad (5c)$$

$$U_5(L_5,T) = Y_3(L_3,T), \quad \text{displacement continuity} \quad (5d)$$

$$Y_6'(0,T) = Y_3'(L_3,T), \quad \text{slope continuity} \quad (5e)$$

$$Y_5'(0,T) = Y_3'(L_3,T), \quad \text{slope continuity} \quad (5f)$$

$$EIY_6''(0,T) = EIY_3''(L_3,T) + EIY_5''(L_5,T), \quad \text{moment continuity} \quad (5g)$$

$$EAU_6'(0,T) = EAU_3'(L_3,T) + EIY_5''(L_5,T), \quad \text{axial force continuity, element 6} \quad (5h)$$

$$EAU_5'(L_5,T) = EIY_3'''(L_3,T) - EIY_6'''(0,T). \quad \text{axial force continuity, element 5} \quad (5i)$$

All the assumptions in the above compatibility conditions are the same as the traditional analysis of the transverse vibrations of an Euler-Bernoulli beam and the axial vibrations of a rod. The frame angles are also assumed to be unchanged during the motions of the frame structure.

The following parameter transformations are introduced to the above equations:

$$y_i = \frac{Y_i}{L}, \quad x = \frac{X}{L}, \quad u_i = \frac{U_i}{L}, \quad t = \frac{T}{\sqrt{L}}, \quad l_i = \frac{L_i}{L}, \quad x_i = \frac{X_i}{L}. \quad (6a\text{-}f)$$

Thus, in each component of the first (top) story structure (refer to Fig. 3a), Eqs. (1) and (2) can then be expressed in a non-dimensional form as

$$\frac{EI}{L^3}\frac{\partial^4 y_i(x,t)}{\partial x^4} + \rho A \frac{\partial^2 y_i(x,t)}{\partial t^2} = 0, \quad i = 1,2,...,6, \quad (7)$$

$$\frac{E}{L}\frac{\partial^2 u_i(x,t)}{\partial x^2} - \rho \frac{\partial^2 u_i(x,t)}{\partial t^2} = 0, \quad i = 1,2,...,6. \quad (8)$$

The non-dimensional "compatibility conditions" across the joints X_1 and X_3 are (from Eqs. (4a-4i) and (5a-5i))

$$y_1(0,t) = y_4(l_4,t), \quad (9a)$$

$$u_1(0,t) = u_4(l_4,t), \quad (9b)$$

$$y_5(0,t) = u_4(l_4,t), \quad (9c)$$

$$u_5(0,t) = -y_4(l_4,t), \quad (9d)$$

$$y_1'(0,t) = y_4'(l_4,t), \quad (9e)$$

$$y_5'(0,t) = y_4'(l_4,t), \quad (9f)$$

$$y_1''(0,t) = y_4''(l_4,t) - y_5''(0,t), \quad (9g)$$

$$u_1'(0,t) = u_4'(l_4,t) + \frac{I}{AL^2} y_5'''(0,t), \quad (9h)$$

$$u_5'(0,t) = \frac{I}{AL^2} y_4'''(l_4,t) - \frac{I}{AL^2} y_1'''(0,t), \quad (9i)$$

and

$$y_6(0,t) = y_3(l_3,t), \quad (10a)$$

$$u_6(0,t) = u_3(l_3,t), \quad (10b)$$

$$y_5(l_5,t) = -u_3(l_3,t), \quad (10c)$$

$$u_5(l_5,t) = y_3(l_3,t), \quad (10d)$$

$$y_6'(0,t) = y_3'(l_3,t), \quad (10e)$$

$$y_5'(l_5,t) = y_3'(l_3,t), \quad (10f)$$

$$y_6''(0,t) = y_5''(l_5,t) + y_3''(l_3,t), \quad (10g)$$

$$u_6'(0,t) = u_3'(l_3,t) + \frac{I}{AL^2} y_5'''(l_5,t), \quad (10h)$$

$$u_5'(l_5,t) = \frac{I}{AL^2} y_3'''(l_3,t) - \frac{I}{AL^2} y_6'''(0,t). \quad (10i)$$

Similarly, the non-dimensional boundary conditions from Eqs. (3a) to (3f), for the example of fixed-fixed ends, can be written as

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$$y_{3m-2}(0,t) = y'_{3m-2}(0,t) = u_{3m-2}(0,t) = 0, , \quad \text{(11a-c)}$$

$$y_{3m}(l_{3m},t) = y'_{3m}(l_{3m},t) = u_{3m}(l_{3m},t) = 0. \quad \text{(11d-f)}$$

3. CALCULATION OF MODAL CHARACTERISTICS

The modal characteristics (eigensolutions) for cases of fixed-fixed ends boundary conditions are derived. The solutions of the other boundary conditions can also be obtained easily through a similar procedure. Using the separable solutions: $y_i(x,t) = w_i(x)e^{j\omega t}$ and $u_i(x,t) = v_i(x)e^{j\omega t}$ in Eqs. (7) and (8) leads to an associated eigenvalue problem,

$$w_i''''(x) - \lambda^4 w_i(x) = 0, \quad i = 1,2,...,6, \quad (12)$$

$$v_i''(x) + \gamma^2 v_i(x) = 0, \quad i = 1,2,...,6, \quad (13)$$

where

$$\lambda^4 = \frac{\rho A L^3 \omega^2}{EI} \quad \text{and} \quad \gamma^2 = \frac{\rho L \omega^2}{E}. \quad \text{(14a, b)}$$

From Eqs. (14a, b), the relationship between λ and γ can be expressed as

$$\gamma = \sqrt{\frac{I}{A}}\frac{1}{L}\lambda^2 = a\lambda^2, \quad (15)$$

where a is a constant ($a = \sqrt{\frac{I}{A}}\frac{1}{L}$). From Eqs. (9a-9i) and (10a-10i), the corresponding compatibility conditions across the joints X_1 and X_3 lead to

$$w_1(0) = w_4(l_4), \quad (16a)$$

$$v_1(0) = v_4(l_4), \quad (16b)$$

$$w_5(0) = v_4(l_4), \quad (16c)$$

$$v_5(0) = -w_4(l_4), \quad (16d)$$

$$w_1'(0) = w_4'(l_4), \quad (16e)$$

$$w_5'(0) = w_4'(l_4), \quad (16f)$$

$$w_1''(0) = w_4''(l_4) - w_5''(0), \quad (16g)$$

$$v_1'(0) = v_4'(l_4) + a^2 w_5'''(0), \quad (16h)$$

$$v_5'(0) = a^2 w_4'''(l_4) - a^2 w_1'''(0), \quad (16i)$$

and

$$w_6(0) = w_3(l_3), \quad (17a)$$

$$v_6(0) = v_3(l_3), \quad (17b)$$

$$w_5(l_5) = -v_3(l_3), \quad (17c)$$

$$v_5(l_5) = w_3(l_3), \quad (17d)$$

$$w_6'(0) = w_3'(l_3), \quad (17e)$$

$$w_5'(l_5) = w_3'(l_3), \quad (17f)$$

$$w_6''(0) = w_5''(l_5) + w_3''(l_3), \quad (17g)$$

$$v_6'(0) = v_3'(l_3) + a^2 w_5'''(l_5), \quad (17h)$$

$$v_5'(l_5) = a^2 w_3'''(l_3) - a^2 w_6'''(0). \quad (17i)$$

The general solutions of Eqs. (12) and (13), for each segment, are

$$w_i(x) = A_i \sin\lambda x + B_i \cos\lambda x + C_i \sinh\lambda x + D_i \cosh\lambda x, \quad i = 1,2,...,6, \quad (18)$$

$$v_i(x) = E_i \sin\gamma x + F_i \cos\gamma x = E_i \sin a\lambda^2 x + F_i \cos a\lambda^2 x, \quad i = 1,2,...,6, \quad (19)$$

where A_i, B_i, C_i, D_i, E_i and F_i are constants associated with the *i-th* component (i = 1, 2, …, 6). The constants in the fifth component (A_5, B_5, C_5, D_5, E_5 and F_5) are related to those in the first and the fourth components (A_1, B_1, C_1, D_1, E_1, F_1 and A_4, B_4, C_4, D_4, E_4, F_4) through the compatibility conditions in Eqs. (16c, 16d, 16f, 16g, 16h) and (16i), and which can be expressed as (Appendix Eqs. (A1)-(A6))

$$\underline{K}_5^1 \{R\}_5 = \underline{K}_4^1 \{R\}_4 + \underline{K}_1^1 \{R\}_1, \quad (20)$$

where

$$\underline{K}_5^1 \equiv \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \underline{K}_1^1 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a\lambda} & 0 \\ a\lambda & 0 & -a\lambda & 0 & 0 & 0 \end{bmatrix},$$

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$$\underline{K}_4^1=\begin{bmatrix} 0 & 0 & 0 & 0 & \sin a\lambda^2 l_4 & \cos a\lambda^2 l_4 \\ -\sin \lambda l_4 & -\cos \lambda l_4 & -\sinh \lambda l_4 & -\cosh \lambda l_4 & 0 & 0 \\ \cos \lambda l_4 & -\sin \lambda l_4 & \cosh \lambda l_4 & \sinh \lambda l_4 & 0 & 0 \\ -\sin \lambda l_4 & -\cos \lambda l_4 & \sinh \lambda l_4 & \cosh \lambda l_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{a\lambda}\cos a\lambda^2 l_4 & \frac{1}{a\lambda}\sin a\lambda^2 l_4 \\ -a\lambda \cos \lambda l_4 & a\lambda \sin \lambda l_4 & a\lambda \cosh \lambda l_4 & a\lambda \sinh \lambda l_4 & 0 & 0 \end{bmatrix},$$

and $\{R\}_i=[A_i\ B_i\ C_i\ D_i\ E_i\ F_i]^T$. The upper index i of the matrix $\underline{K}_j^i$ represents the related joint X_i and the lower index j represents the related coefficients $\{R\}_j$. From Eq. (20), the constants $\{R\}_5$ can be written as

$$\{R\}_5=(\underline{K}_5^1)^{-1}\underline{K}_4^1\{R\}_4+(\underline{K}_5^1)^{-1}\underline{K}_1^1\{R\}_1. \tag{21}$$

Similarly, the compatibility conditions in Eqs. (17c), (17d), (17f), (17g), (17h) and (17i) can be expressed as (Appendix Eqs. (A7)-(A12))

$$\underline{K}_5^3\{R\}_5=\underline{K}_3^3\{R\}_3+\underline{K}_6^3\{R\}_6, \tag{22}$$

where

$$\underline{K}_5^3\equiv\begin{bmatrix} \sin \lambda l_5 & \cos \lambda l_5 & \sinh \lambda l_5 & \cosh \lambda l_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin a\lambda^2 l_5 & \cos a\lambda^2 l_5 \\ \cos \lambda l_5 & -\sin \lambda l_5 & \cosh \lambda l_5 & \sinh \lambda l_5 & 0 & 0 \\ \sin \lambda l_5 & \cos \lambda l_5 & -\sinh \lambda l_5 & -\cosh \lambda l_5 & 0 & 0 \\ -\cos \lambda l_5 & \sin \lambda l_5 & \cosh \lambda l_5 & \sinh \lambda l_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos a\lambda^2 l_5 & -\sin a\lambda^2 l_5 \end{bmatrix},\quad \underline{K}_6^3\equiv\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a\lambda} & 0 \\ a\lambda & 0 & -a\lambda & 0 & 0 & 0 \end{bmatrix},$$

$$\underline{K}_3^3=\begin{bmatrix} 0 & 0 & 0 & 0 & -\sin a\lambda^2 l_3 & -\cos a\lambda^2 l_3 \\ \sin \lambda l_3 & \cos \lambda l_3 & \sinh \lambda l_3 & \cosh \lambda l_3 & 0 & 0 \\ \cos \lambda l_3 & -\sin \lambda l_3 & \cosh \lambda l_3 & \sinh \lambda l_3 & 0 & 0 \\ -\sin \lambda l_3 & -\cos \lambda l_3 & \sinh \lambda l_3 & \cosh \lambda l_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{a\lambda}\cos a\lambda^2 l_3 & \frac{1}{a\lambda}\sin a\lambda^2 l_3 \\ -a\lambda \cos \lambda l_3 & a\lambda \sin \lambda l_3 & a\lambda \cosh \lambda l_3 & a\lambda \sinh \lambda l_3 & 0 & 0 \end{bmatrix}.$$

From the reference Lin [7], the constants $\{R\}_3$ can be written as

$$\{R\}_3=\underline{T}_2\underline{T}_1\{R\}_1, \tag{23}$$

where

$$\underline{T}_1=\begin{bmatrix} \frac{1}{2}\cos \lambda l_1 & -\frac{1}{2}\sin \lambda l_1 & \frac{1}{2}\cosh \lambda l_1 & \frac{1}{2}\sinh \lambda l_1 & \frac{1}{2}\frac{1}{a\lambda}\cos a\lambda^2 l_1 & -\frac{1}{2}\frac{1}{a\lambda}\sin a\lambda^2 l_1 \\ \frac{1}{2}\sin \lambda l_1 & \frac{1}{2}\cos \lambda l_1 & -\frac{1}{2}\sinh \lambda l_1 & -\frac{1}{2}\cosh \lambda l_1 & \frac{1}{2}\sin a\lambda^2 l_1 & \frac{1}{2}\cos a\lambda^2 l_1 \\ \frac{1}{2}\cos \lambda l_1 & -\frac{1}{2}\sin \lambda l_1 & \frac{1}{2}\cosh \lambda l_1 & \frac{1}{2}\sinh \lambda l_1 & -\frac{1}{2}\frac{1}{a\lambda}\cos a\lambda^2 l_1 & \frac{1}{2}\frac{1}{a\lambda}\sin a\lambda^2 l_1 \\ -\frac{1}{2}\sin \lambda l_1 & -\frac{1}{2}\cos \lambda l_1 & \frac{1}{2}\sinh \lambda l & \frac{1}{2}\cosh \lambda l_1 & \frac{1}{2}\sin a\lambda^2 l_1 & \frac{1}{2}\cos a\lambda^2 l_1 \\ -a\lambda \cos \lambda l_1 & a\lambda \sin \lambda l_1 & a\lambda \cosh \lambda l_1 & a\lambda \sinh \lambda l_1 & 0 & 0 \\ -\sin \lambda l_1 & -\cos \lambda l_1 & -\sinh \lambda l_1 & -\cosh \lambda l_1 & 0 & 0 \end{bmatrix},$$

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$$\underline{T}_2=\begin{bmatrix}\frac{1}{2}\cos\lambda l_2 & -\frac{1}{2}\sin\lambda l_2 & \frac{1}{2}\cosh\lambda l_2 & \frac{1}{2}\sinh\lambda l_2 & \frac{1}{2}\frac{1}{a\lambda}\cos a\lambda^2 l_2 & -\frac{1}{2}\frac{1}{a\lambda}\sin a\lambda^2 l_2\\ \frac{1}{2}\sin\lambda l_2 & \frac{1}{2}\cos\lambda l_2 & -\frac{1}{2}\sinh\lambda l_2 & -\frac{1}{2}\cosh\lambda l_2 & \frac{1}{2}\sin a\lambda^2 l_2 & \frac{1}{2}\cos a\lambda^2 l_2\\ \frac{1}{2}\cos\lambda l_2 & -\frac{1}{2}\sin\lambda l_2 & \frac{1}{2}\cosh\lambda l_2 & \frac{1}{2}\sinh\lambda l_2 & -\frac{1}{2}\frac{1}{a\lambda}\cos a\lambda^2 l_2 & \frac{1}{2}\frac{1}{a\lambda}\sin a\lambda^2 l_2\\ -\frac{1}{2}\sin\lambda l_2 & -\frac{1}{2}\cos\lambda l_2 & \frac{1}{2}\sinh\lambda l_2 & \frac{1}{2}\cosh\lambda l_2 & \frac{1}{2}\sin a\lambda^2 l_2 & \frac{1}{2}\cos a\lambda^2 l_2\\ -a\lambda\cos\lambda l_2 & a\lambda\sin\lambda l_2 & a\lambda\cosh\lambda l_2 & a\lambda\sinh\lambda l_2 & 0 & 0\\ -\sin\lambda l_2 & -\cos\lambda l_2 & -\sinh\lambda l_2 & -\cosh\lambda l_2 & 0 & 0\end{bmatrix}.$$

Eq. (23) represents the relationships of the six constants in the component 1 and component 3. So Eq. (22) is written in the form

$$\{R\}_5=(\underline{K}_5^3)^{-1}\underline{K}_3^3\underline{T}_2\underline{T}_1\{R\}_1+(\underline{K}_5^3)^{-1}\underline{K}_6^3\{R\}_6. \tag{24}$$

Combining the Eqs. (21) and (24), the constants $\{R\}_1$ can be expressed by $\{R\}_4$ and $\{R\}_6$ as

$$((\underline{K}_5^3)^{-1}\underline{K}_3^3\underline{T}_2\underline{T}_1-(\underline{K}_5^1)^{-1}\underline{K}_1^1)\ \{R\}_1=(\underline{K}_5^1)^{-1}\underline{K}_4^1\ \{R\}_4-(\underline{K}_5^3)^{-1}\underline{K}_6^3\ \{R\}_6. \tag{25}$$

Let $\underline{M}_1\equiv(\underline{K}_5^3)^{-1}\underline{K}_3^3\underline{T}_2\underline{T}_1-(\underline{K}_5^1)^{-1}\underline{K}_1^1$, then the constants $\{R\}_1$ are

$$\{R\}_1=(\underline{M}_1)^{-1}(\underline{K}_5^1)^{-1}\underline{K}_4^1\{R\}_4-(\underline{M}_1)^{-1}(\underline{K}_5^3)^{-1}\underline{K}_6^3\{R\}_6. \tag{26}$$

Using the compatibility requirements Eqs. (16a, b, e) and (17a, b, e), the following equation can be obtained: (Appendix Eqs. (A13)-(A18))

$$\underline{K}_1^{1,3}\{R\}_1=\underline{K}_4^{1,3}\{R\}_4+\underline{K}_3^{1,3}\{R\}_3+\underline{K}_6^{1,3}\{R\}_6=\underline{K}_4^{1,3}\{R\}_4+\underline{K}_3^{1,3}\underline{T}_2\underline{T}_1\{R\}_1+\underline{K}_6^{1,3}\{R\}_6, \tag{27}$$

where

$$\underline{K}_1^{1,3}\equiv\begin{bmatrix}0&1&0&1&0&0\\0&0&0&0&0&1\\1&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\quad \underline{K}_4^{1,3}=\begin{bmatrix}\sin\lambda l_4&\cos\lambda l_4&\sinh\lambda l_4&\cosh\lambda l_4&0&0\\0&0&0&0&\sin a\lambda^2 l_4&\cos a\lambda^2 l_4\\\cos\lambda l_4&-\sin\lambda l_4&\cosh\lambda l_4&\sinh\lambda l_4&0&0\\0_1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

$$\underline{K}_3^{1,3}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\\sin\lambda l_3&\cos\lambda l_3&\sinh\lambda l_3&\cosh\lambda l_3&0&0\\0&0&0&0&\sin a\lambda^2 l_3&\cos a\lambda^2 l_3\\\cos\lambda l_3&-\sin\lambda l_3&\cosh\lambda l_3&\sinh\lambda l_3&0&0\end{bmatrix},\quad \underline{K}_6^{1,3}\equiv\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&-1&0&-1&0&0\\0&0&0&0&0&-1\\-1&0&-1&0&0&0\end{bmatrix}.$$

The constants $\{R\}_1$ in Eq. (27) are expressed as

$$\{R\}_1=(\underline{M}_2)^{-1}\underline{K}_4^{1,3}\{R\}_4+(\underline{M}_2)^{-1}\underline{K}_6^{1,3}\{R\}_6, \tag{28}$$

where $\underline{M}_2\equiv(\underline{K}_1^{1,3}-\underline{K}_3^{1,3}\underline{T}_2\underline{T}_1)$.

Combining the Eqs. (26) and (28), the relationships between $\{R\}_4$ and $\{R\}_6$ are determined as

$$\{R\}_6=\underline{K}_{F2}\{R\}_4, \tag{29}$$

where

$$\underline{K}_{F2}\equiv((\underline{M}_1)^{-1}(\underline{K}_5^3)^{-1}\underline{K}_6^3+(\underline{M}_2)^{-1}\underline{K}_6^{1,3})^{-1}\ ((\underline{M}_1)^{-1}(\underline{K}_5^1)^{-1}\underline{K}_4^1-(\underline{M}_2)^{-1}\underline{K}_4^{1,3}). \tag{30}$$

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Through Eq. (29), the six constants in the component 4 (A_4, B_4, C_4, D_4, E_4 and F_4) can be mapped into those of the component 6 (A_6, B_6, C_6, D_6, E_6 and F_6), thereby reducing the number of independent constants of the top-story frame to six.

Next, for the second story (lower of the top story), as shown in Fig. 3b, the structure of the second story is exactly the same as that of the first (top) story except that the numbers of the components are different. The relationships of the constants $\{R\}_4$ and $\{R\}_6$ in the second story have been obtained in Eq. (29) similar to Eq. (23) in the first-story. Using the similar procedure described above (the joints in this case are X_4 and X_6), the relationships between the constants $\{R\}_7$ and $\{R\}_9$ in the second story can then be obtaioned. Then, following the same method on the other stories (from top down) step by step, finally, the relationships between the constants $\{R\}_{3m-2}$ and $\{R\}_{3m}$ can be obtained as

$$\{R\}_{3m} = \underline{K}_{Fm} \{R\}_{3m-2}. \tag{31}$$

Thus, the six constants in the component $3m$-2 (A_{3m-2}, B_{3m-2}, C_{3m-2}, D_{3m-2}, E_{3m-2} and F_{3m-2}) can be mapped into those of the component $3m$ (A_{3m}, B_{3m}, C_{3m-2}, D_{3m}, E_{3m} and F_{3m}), thereby reducing the number of independent constants of the entire m-story frame structure to only six. These six remaining constants (A_{3m-2}, B_{3m-2}, C_{3m-2}, D_{3m-2}, E_{3m-2} and F_{3m-2}) can be determined through the satisfaction of the boundary conditions.

The boundary conditions, from Eqs. (11a)-(11f), are

$$w_{3m-2}(0) = w'_{3m-2}(0) = v_{3m-2}(0) = 0, \tag{32a-c}$$

$$w_{3m}(l_{3m}) = w'_{3m}(l_{3m}) = v_{3m}(l_{3m}) = 0. \tag{32d-f}$$

After substituting the right-side boundary conditions, Eqs. (32d-f), lead to (Appendix Eqs. (A19)-(A21))

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \underline{B}_{3\times 6} \{R\}_{3m}, \tag{33}$$

where

$$\underline{B}_{3\times 6} = \begin{bmatrix} \sin\lambda l_{3m} & \cos\lambda l_{3m} & \sinh\lambda l_{3m} & \cosh\lambda l_{3m} & 0 & 0 \\ \cos\lambda l_{3m} & -\sin\lambda l_{3m} & \cosh\lambda l_{3m} & \sinh\lambda l_{3m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin a\lambda^2 l_{3m} & \cos a\lambda^2 l_{3m} \end{bmatrix}.$$

Substitution of Eq. (31) into Eq. (33) and use of the boundary conditions at the left support, Eqs. (32a-c), leads to (Appendix Eqs. (A22)-(A24))

$$\begin{aligned} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} &= \underline{B}_{3\times 6} \{R\}_{3m} = \underline{B}_{3\times 6} \underline{K}_{Fm} \{R\}_{3m-2} = \underline{B}_{3\times 6} \underline{K}_{Fm} \begin{Bmatrix} A_{3m-2} \\ B_{3m-2} \\ C_{3m-2} \\ D_{3m-2} \\ E_{3m-2} \\ F_{3m-2} \end{Bmatrix} = \underline{R}_{3\times 6} \begin{Bmatrix} A_{3m-2} \\ B_{3m-2} \\ C_{3m-2} \\ D_{3m-2} \\ E_{3m-2} \\ F_{3m-2} \end{Bmatrix} \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{35} \end{bmatrix} \begin{Bmatrix} A_{3m-2} \\ B_{3m-2} \\ -A_{3m-2} \\ -B_{3m-2} \\ E_{3m-2} \\ 0 \end{Bmatrix} = \begin{bmatrix} r_{11}-r_{13} & r_{12}-r_{14} & r_{15} \\ r_{21}-r_{23} & r_{22}-r_{24} & r_{25} \\ r_{31}-r_{33} & r_{32}-r_{34} & r_{35} \end{bmatrix} \begin{Bmatrix} A_{3m-2} \\ B_{3m-2} \\ E_{3m-2} \end{Bmatrix}, \end{aligned} \tag{34}$$

where

$$\underline{R}_{3\times 6} = \underline{B}_{3\times 6} \underline{K}_{Fm} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} \end{bmatrix}.$$

Eq. (34) is a standard eigenvalue problem and the existence of non-trivial solutions requires

$$\det \begin{bmatrix} r_{11}(\lambda)-r_{13}(\lambda) & r_{12}(\lambda)-r_{14}(\lambda) & r_{15}(\lambda) \\ r_{21}(\lambda)-r_{23}(\lambda) & r_{22}(\lambda)-r_{24}(\lambda) & r_{25}(\lambda) \\ r_{31}(\lambda)-r_{33}(\lambda) & r_{32}(\lambda)-r_{34}(\lambda) & r_{35}(\lambda) \end{bmatrix} = 0. \tag{35}$$

This determinant provides the single (characteristic) equation for the solution of the eigenvalue λ_n. This equation is solved using the standard Newton-Raphson iterations or, for simplification, using the method shown in Fig. 4 [7] to obtain the eigenvalues. The coefficients of the eigenfunctions, $w_n(x)$ and $v_n(x)$, are obtained by back-substitution into Eqs. (34) and (29, 28, 24, 23) followed by Eqs. (18) and (19).

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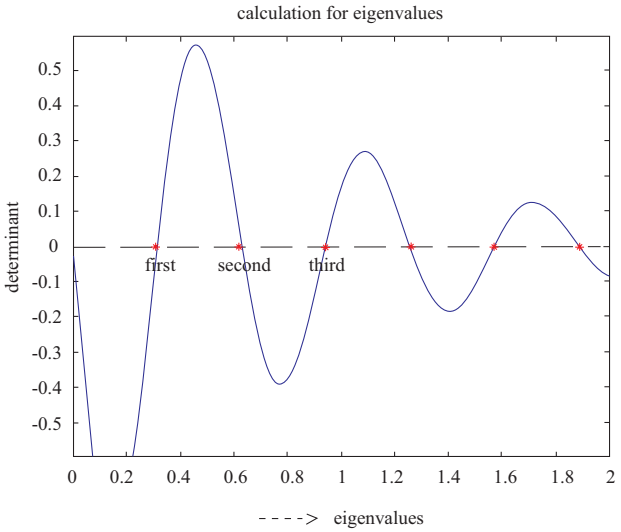


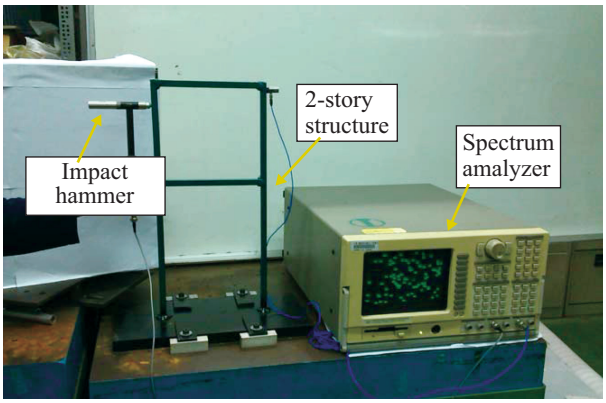
Fig. 4 Simple calculation of eigenvalues.

4. NUMERICAL RESULTS AND DISCUSSION

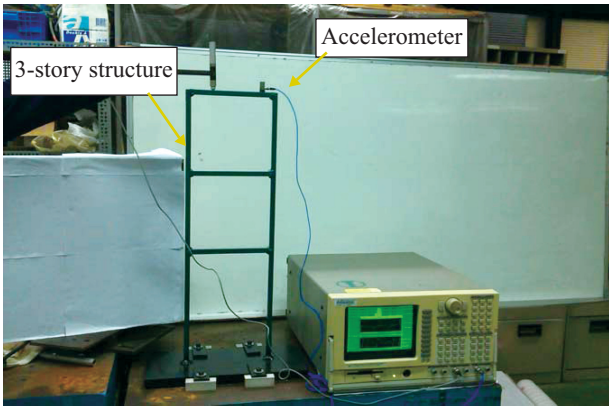
In order to present the method described in this article, some numerical examples are demonstrated. First is the case of a two-story planar frame structure, as shown in Fig. 5a. The cross-sections of all the components are square with width b = 12.7 *mm* and height h = 12.7 *mm*. Geometric dimensions are: $L_1 = L_2 = L_3 = L_5 = 0.2$ *m*, and $L_4 = L_6 = 0.3$ *m*. Material properties are: density ρ = 7,800 Kg/m^3, Young's modulus $E = 2.06 \times 10^{11}$ N/m^2. Table 1 shows the calculated lowest five eigenvalues and the corresponding natural frequencies of this two-story frame structure by using the proposed method in this article. The corresponding mode shapes are also calculated as shown in Fig. 6.

In order to validate the method presented in this article, the calculated numerical results are compared with the FEM method and the experimental results. Fig. 7a shows the measured frequency response function of the above two-story frame structure by experimental modal testing. The lowest five natural frequencies of this structure are measured as: Ω_1 = 72 Hz, Ω_2 = 296 Hz, Ω_3 = 592 Hz, Ω_4 = 768 Hz and Ω_5 = 828 Hz. The comparisons of the calculated natural frequencies from this article with the FEM method and the measured results are shown in Table 2.

Second is the example of a three-story frame structure as shown in Fig. 5b. All the cross-sections of the components are square with width b = 12.7 *mm* and height h = 12.7 *mm*. The geometric dimensions are: $L_1 = L_2 = L_3 = L_4 = L_5 = L_6 = L_8 = 0.2$ *m*, and $L_7 = L_9 = 0.3$ *m*. The material properties are: density ρ = 7,800 Kg/m^3, Young's modulus $E = 2.06 \times 10^{11}$ N/m^2. The measured frequency function of this structure is shown in Fig. 7b. Table 3 shows the comparisons of the calculated lowest five natural frequencies and the measured experimental natural frequencies of this structure and it is observed the errors are small and satisfactory. The



(a) Two-story frame structure



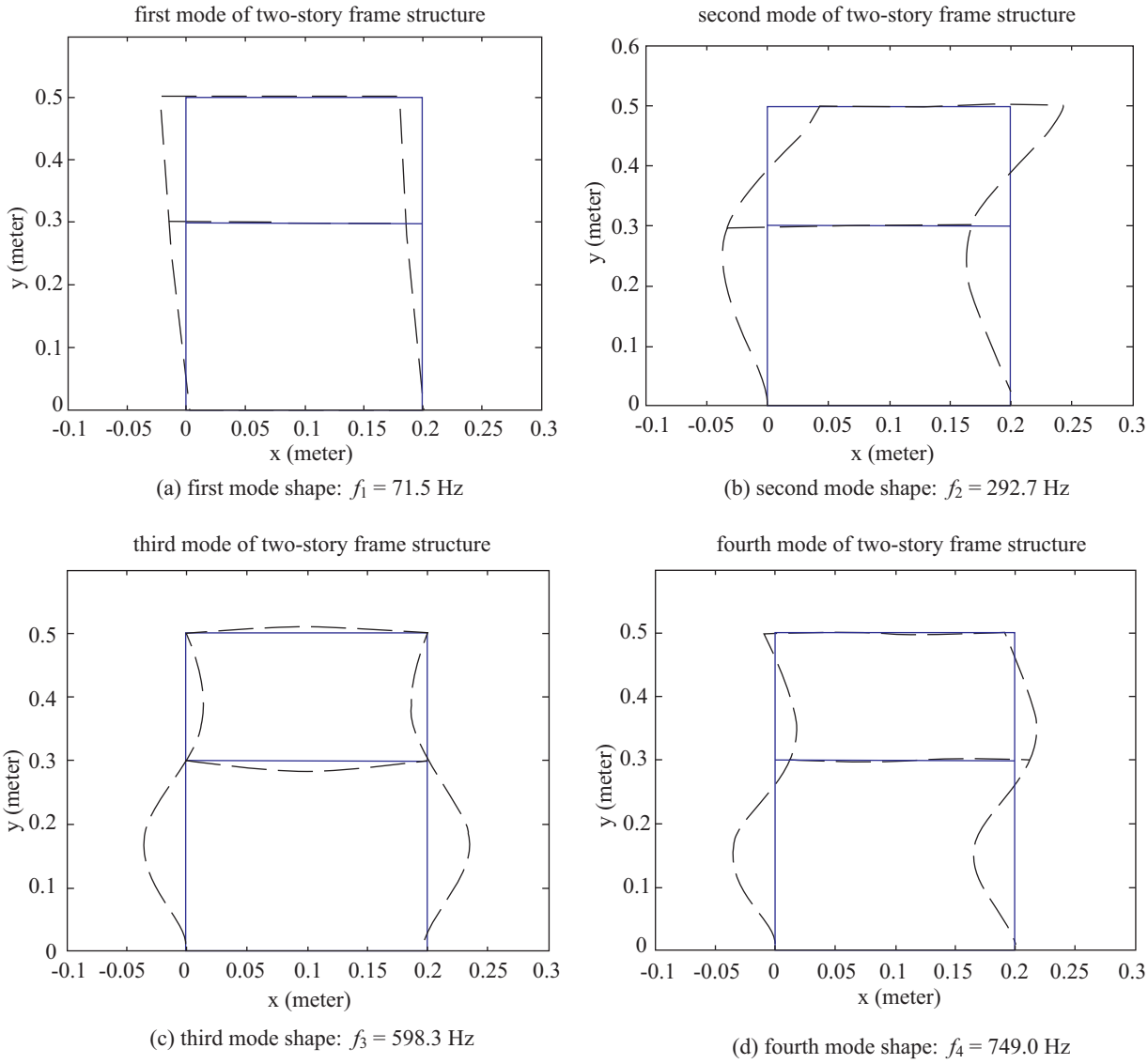
(b) Tree-story frame structure

Fig. 5 (a) Two-story frame structure (b)Three-story frame structure.

Table 1 Calculated eigenvalues of a two-story frame structure with dimensions and material properties: $L_1 = L_2 = L_3 = L_5 = 0.2$ *m*, $L_4 = L_6 = 0.3$ *m*, sectional height h = 12.7 *mm*, sectional width b = 12.7 *mm*, density ρ = 7,800 Kg/m^3 and Young's modulus $E = 2.06 \times 10^{11}$ N/m^2.

	λ_1	Λ_2	Λ_3	Λ_4	Λ_5
Calculated Eigenvalues	6.836	13.831	19.776	22.127	23.955
Corresponding Natural Frequency, Hz	71.5	292.7	598.3	749.0	877.9

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(a) first mode shape: f_1 = 71.5 Hz

(b) second mode shape: f_2 = 292.7 Hz

(c) third mode shape: f_3 = 598.3 Hz

(d) fourth mode shape: f_4 = 749.0 Hz

Fig. 6 Lowest four mode shapes of a two-story frame structure.

Table 2 Comparison of analytical and experimental results for a two-story frame structure with dimensions and material properties: $L_1 = L_2 = L_3 = L_5 = 0.2$ *m*, $L_4 = L_6 = 0.3$ *m*, sectional height $h = 12.7$ *mm*, sectional width $b = 12.7$ *mm*, density $\rho = 7{,}800$ *Kg*/*m*^{3} and Young's modulus $E = 2.06 \times 10^{11}$ *N*/*m*^{2}.

Calculated natural frequencies From this study	FEM Method (by ANSYS)	Measured natural frequencies (modal testing)
$\Omega_1 =$ 71.5 Hz	72.8 Hz	72 Hz
$\Omega_2 = 292.7$ Hz	303.3 Hz	296 Hz
$\Omega_3 = 598.3$ Hz	595.8 Hz	592 Hz
$\Omega_4 = 749.0$ Hz	757.8 Hz	768 Hz
$\Omega_5 = 877.9$ Hz	862.4 Hz	828 Hz

calculations of the corresponding mode shapes of this three-story frame structure are shown in Fig. 8.

Third is an example of a 10-story frame structure with 30 frame components. The cross-sections of all components are square with width $b = 12.7$ *mm* and height $h = 12.7$ *mm*. The geometric dimensions are: $L_1 = L_2 = L_3 = L_4 = L_5 = \ldots = L_{25} = L_{26} = L_{27} = L_{29} = 0.2$ *m* and boundary components $L_{28} = L_{30} = 0.3$ *m*. The material

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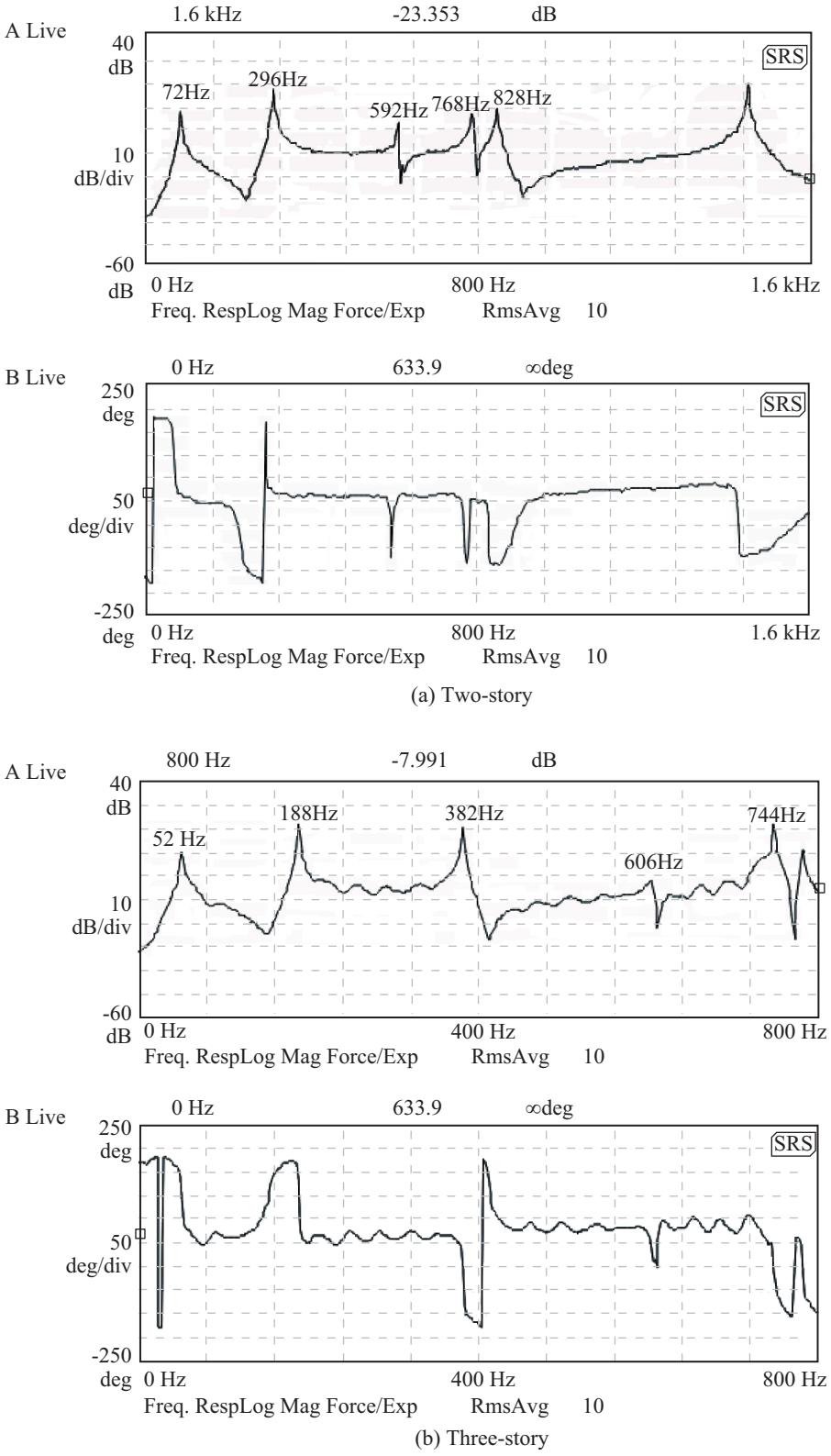


Fig. 7 Measured transfer functions (a) Two-story frame structure (b) Three-story frame structure.

properties are: density ρ = 7,800 Kg/m^3 and Young's modulus $E = 2.06 \times 10^{11}$ N/m^2. Using the method proposed in this article, the lowest five natural frequencies of this structure are easily calculated as: Ω_1 = 16.83 Hz, Ω_2 = 51.82 Hz, Ω_3 = 93.63 Hz, Ω_4 = 138.95 Hz and Ω_5 = 190.73 Hz.

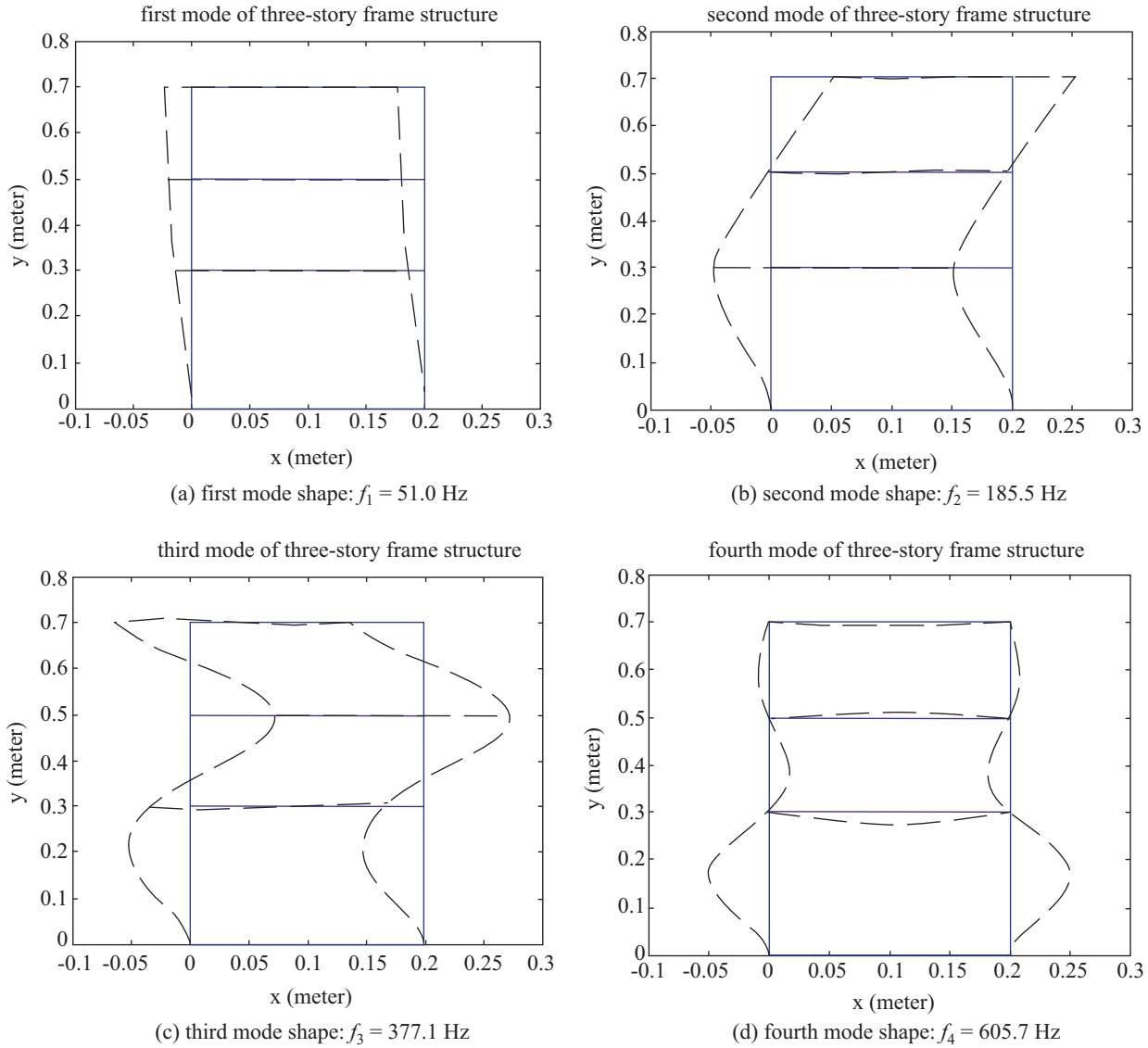
5. CONCLUSIONS

A hybrid analytical/numerical solution method that permits the efficient evaluation of eigensolutions for planar multi-story frame structures was developed. The method uses a numerical implementation to the analytic-

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Table 3 Comparison of analytical and experimental results for a three-story frame structure with dimensions and material properties: $L_1 = L_2 = L_3 = L_4 = L_5 = L_6 = L_8 = 0.2\ m$, $L_7 = L_9 = 0.3\ m$, sectional height $h = 12.7\ mm$, sectional width $b = 12.7\ mm$, density $\rho = 7{,}800\ Kg/m^3$ and Young's modulus $E = 2.06 \times 10^{11}\ N/m^2$.

Measured natural frequencies	Calculated natural frequencies from this study	Error(%)
$\Omega_1 =$ 52 Hz	51.0 Hz	– 1.9
$\Omega_2 = 188$ Hz	185.5 Hz	– 1.3
$\Omega_3 = 382$ Hz	377.1 Hz	1.3
$\Omega_4 = 606$ Hz	605.7 Hz	– 0.0
$\Omega_5 = 744$ Hz	748.1 Hz	0.6



(a) first mode shape: $f_1 = 51.0$ Hz

(b) second mode shape: $f_2 = 185.5$ Hz

(c) third mode shape: $f_3 = 377.1$ Hz

(d) fourth mode shape: $f_4 = 605.7$ Hz

Fig. 8 Lowest four mode shapes of a three-story frame structure.

cal equations of motion. Unlike all other methods, in which the dimensions of the matrix increase with the complexity of the structure, only six undetermined coefficients occur in the method proposed in this study, regardless of the number of stories. The main feature of this method is the possibility of reducing the dimension of the matrix involved in the finite element method and certain other analytical methods.

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APPENDIX

Derivation of matrix $\underline{K}_j^i$ and boundary conditions

Eqs. (16c, d, f, g, h, i)

$$w_5(0)=v_4(l_4),\quad B_5+D_5=E_4\sin a\lambda^2 l_4+F_4\cos a\lambda^2 l_4 \tag{A1}$$

$$v_5(0)=-w_4(l_4),\quad F_5=-(A_4\sin\lambda l_4+B_4\cos\lambda l_4+C_4\sinh\lambda l_4+D_4\cosh\lambda l_4) \tag{A2}$$

$$w_5'(0)=w_4'(l_4),\quad A_5+C_5=A_4\cos\lambda l_4-B_4\sin\lambda l_4+C_4\cosh h\lambda l_4+D_4\sinh\lambda l_4 \tag{A3}$$

$$w_1''(0)=w_4''(l_4)-w_5''(0),\quad -B_5+D_5=-A_4\sin\lambda l_4-B_4\cos\lambda l_4+C_4\sinh\lambda l_4+D_4\cosh\lambda l_4-(-B_1+D_1) \tag{A4}$$

$$v_1'(0)=v_4'(l_4)+a^2w_5'''(0),\quad a\lambda(-A_5+C_5)=-E_4\cos al_4+F_4\sin a\lambda^2 l_4+E_1 \tag{A5}$$

$$v_5'(0)=a^2w_4'''(l_4)-a^2w_1'''(0),\quad E_5=a\lambda[(-A_4\cos\lambda l_4+B_4\sin\lambda l_4+C_4\cosh\lambda l_4+D_4\sinh\lambda l_4)+(A_1-C_1)] \tag{A6}$$

Eqs. (A1) to (A6) can be expressed as:

$$\begin{bmatrix}0&1&0&1&0&0\\0&0&0&0&0&1\\1&0&1&0&0&0\\0&-1&0&1&0&0\\-1&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}\begin{Bmatrix}A_5\\B_5\\C_5\\D_5\\E_5\\F_5\end{Bmatrix}=\begin{bmatrix}0&0&0&0&\sin a\lambda^2l_4&\cos a\lambda^2l_4\\-\sin\lambda l_4&-\cos\lambda l_4&-\sinh\lambda l_4&-\cosh\lambda l_4&0&0\\\cos\lambda l_4&-\sin\lambda l_4&\cosh\lambda l_4&\sinh\lambda l_4&0&0\\-\sin\lambda l_4&-\cos\lambda l_4&\sinh\lambda l_4&\cosh\lambda l_4&0&0\\0&0&0&0&-\frac{1}{a\lambda}\cos a\lambda^2l_4&\frac{1}{a\lambda}\sin a\lambda^2l_4\\-a\lambda\cos\lambda l_4&a\lambda\sin\lambda l_4&a\lambda\cosh\lambda l_4&a\lambda\sinh\lambda l_4&0&0\end{bmatrix}\begin{Bmatrix}A_4\\B_4\\C_4\\D_4\\E_4\\F_4\end{Bmatrix}$$

$$+\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&-1&0&0\\0&0&0&0&\frac{1}{a\lambda}&0\\a\lambda&0&-a\lambda&0&0&0\end{bmatrix}\begin{Bmatrix}A_1\\B_1\\C_1\\D_1\\E_1\\F_1\end{Bmatrix}.$$

$$\rightarrow\quad \underline{K}_5^1\{R\}_5=\underline{K}_4^1\{R\}_4+\underline{K}_1^1\{R\}_1 \tag{20}$$

Eqs. (17c, d, f, g, h, i)

$$w_5(l_5)=-v_3(l_3),\quad A_5\sin\lambda l_5+B_5\cos\lambda l_5+C_5\sinh\lambda l_5+D_5\cosh\lambda l_5=-(E_3\sin a\lambda^2l_3+F_3\cos a\lambda^2l_3) \tag{A7}$$

$$v_5(l_5)=w_3(l_3),\quad A_3\sin\lambda l_3+B_3\cos\lambda l_3+C_3\sinh\lambda l_3+D_3\cosh\lambda l_3=E_5\sin a\lambda^2l_5+F_5\cos a\lambda^2l_5 \tag{A8}$$

$$w_5'(l_5)=w_3'(l_3),\quad A_5\cos\lambda l_5-B_5\sin\lambda l_5+C_5\cosh\lambda l_5+D_5\sinh\lambda l_5=A_3\cos\lambda l_3-B_3\sin\lambda l_3+C_3\cos\lambda l_3+D_3\sinh\lambda l_3 \tag{A9}$$

$$\begin{aligned}&w_6''(0)=w_5''(l_5)+w_3''(l_3),\\&\quad -B_6+D_6=-A_5\sin\lambda l_5-B_5\cos\lambda l_5+C_5\sinh\lambda l_5+D_5\cosh\lambda l_5\\&\quad +(-A_3\sin\lambda l_3-B_3\cos\lambda l_3+C_3\sin\lambda l_3+D_3\cosh\lambda l_3)\end{aligned} \tag{A10}$$

$$\begin{aligned}&v_6'(0)=v_3'(l_3)+a^2w_5'''(l_5),\\&\quad E_6=-E_3\cos a\lambda^2l_3+F_3\sin a\lambda^2l_3+a\lambda(-A_5\cos\lambda l_5+B_5\sin\lambda l_5+C_5\cosh\lambda l_5+D_5\sinh\lambda l_5)\end{aligned} \tag{A11}$$

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$$v_5'(l_5) = a^2 w_3'''(l_3) - a^2 w_6'''(0),$$
$$(E_5 \cos a\lambda^2 l_5 - F_5 \sin a\lambda^2 l_5) = a\lambda[(-A_3 \cos \lambda l_3 + B_3 \sin \lambda l_3 + C_3 \cos \lambda l_3 + D_3 \sinh \lambda l_3) - (-A_6 + C_6)] \tag{A12}$$

Eqs. (A7) to (A12) can be expressed as:

$$\begin{bmatrix} \sin \lambda l_5 & \cos \lambda l_5 & \sinh \lambda l_5 & \cosh \lambda l_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin a\lambda^2 l_5 & \cos a\lambda^2 l_5 \\ \cos \lambda l_5 & -\sin \lambda l_5 & \cosh \lambda l_5 & \sinh \lambda l_5 & 0 & 0 \\ \sin \lambda l_5 & \cos \lambda l_5 & -\sinh \lambda l_5 & -\cosh \lambda l_5 & 0 & 0 \\ -\cos \lambda l_5 & \sin \lambda l_5 & \cosh \lambda l_5 & \sinh \lambda l_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos a\lambda^2 l_5 & -\sin a\lambda^2 l_5 \end{bmatrix} \begin{Bmatrix} A_5 \\ B_5 \\ C_5 \\ D_5 \\ E_5 \\ F_5 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -\sin a\lambda^2 l_3 & -\cos a\lambda^2 l_3 \\ \sin \lambda l_3 & \cos \lambda l_3 & \sinh \lambda l_3 & \cosh \lambda l_3 & 0 & 0 \\ \cos \lambda l_3 & -\sin \lambda l_3 & \cosh \lambda l_3 & \sinh \lambda l_3 & 0 & 0 \\ -\sin \lambda l_3 & -\cos \lambda l_3 & \sinh \lambda l_3 & \cosh \lambda l_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{a\lambda} \cos a\lambda^2 l_3 & \frac{1}{a\lambda} \sin a\lambda^2 l_3 \\ -a\lambda \cos \lambda l_3 & a\lambda \sin \lambda l_3 & a\lambda \cosh \lambda l_3 & a\lambda \sinh \lambda l_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \\ E_3 \\ F_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a\lambda} & 0 \\ a\lambda & 0 & -a\lambda & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_6 \\ B_6 \\ C_6 \\ D_6 \\ E_6 \\ F_6 \end{Bmatrix}.$$

$$\rightarrow \quad \underline{K}_5^3 \{R\}_5 = \underline{K}_3^3 \{R\}_3 + \underline{K}_6^3 \{R\}_6 \tag{22}$$

Eqs. (16a, b, e) and (17a, b, e)

$$w_1(0) = w_4(l_4), \quad B_1 + D_1 = A_4 \sin \lambda l_4 + B_4 \cos \lambda l_4 + C_4 \sinh \lambda l_4 + D_4 \cosh \lambda l_4 \tag{A13}$$

$$v_1(0) = v_4(l_4), \quad F_1 = E_4 \sin a\lambda^2 l_4 + F_4 \cos a\lambda^2 l_4 \tag{A14}$$

$$w_1'(0) = w_4'(l_4), \quad A_1 + C_1 = A_4 \cos \lambda l_4 - B_4 \sin \lambda l_4 + C_4 \cosh \lambda l_4 + D_4 \sinh \lambda l_4 \tag{A15}$$

$$w_6(0) = w_3(l_3), \quad B_6 + D_6 = A_3 \sin \lambda l_3 + B_3 \cos \lambda l_3 + C_3 \sinh \lambda l_3 + D_3 \cosh \lambda l_3 \tag{A16}$$

$$v_6(0) = v_3(l_3), \quad F_6 = E_3 \sin a\lambda^2 l_3 + F_3 \cos a\lambda^2 l_3. \tag{A17}$$

$$w_6'(0) = w_3'(l_3), \quad A_6 + C_6 = A_3 \cos \lambda l_3 - B_3 \sin \lambda l_3 + C_3 \cosh \lambda l_3 + D_3 \sinh \lambda l_3 \tag{A18}$$

Eqs. (A13) to (A18) can be expressed as:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ E_1 \\ F_1 \end{Bmatrix} = \begin{bmatrix} \sin \lambda l_4 & \cos \lambda l_4 & \sinh \lambda l_4 & \cosh \lambda l_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin a\lambda^2 l_4 & \cos a\lambda^2 l_4 \\ \cos \lambda l_4 & -\sin \lambda l_4 & \cosh \lambda l_4 & \sinh \lambda l_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_4 \\ B_4 \\ C_4 \\ D_4 \\ E_4 \\ F_4 \end{Bmatrix}$$

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$$+\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\lambda l_3 & \cos\lambda l_3 & \sinh\lambda l_3 & \cosh\lambda l_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin a\lambda^2 l_3 & \cos a\lambda^2 l_3 \\ \cos\lambda l_3 & -\sin\lambda l_3 & \cosh\lambda l_3 & \sinh\lambda l_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \\ E_3 \\ F_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_6 \\ B_6 \\ C_6 \\ D_6 \\ E_6 \\ F_6 \end{Bmatrix}$$

$$\rightarrow \quad \underline{K}_1^{1,3}\{R\}_1 = \underline{K}_4^{1,3}\{R\}_4 + \underline{K}_3^{1,3}\{R\}_3 + \underline{K}_6^{1,3}\{R\}_6 \tag{27}$$

Satisfaction of the boundary conditions of Eqs. (18) and (19) at the right support, Eqs. (32d-f), requires

$$A_{3m}\sin\lambda l_{3m} + B_{3m}\cos\lambda l_{3m} + C_{3m}\sinh\lambda l_{3m} + D_{3m}\cosh\lambda l_{3m} = 0, \tag{A19}$$

$$A_{3m}\cos\lambda l_{3m} - B_{3m}\sin\lambda l_{3m} + C_{3m}\cosh\lambda l_{3m} + D_{3m}\sinh\lambda l_{3m} = 0, \tag{A20}$$

$$E_{3m}\sin a\lambda^2 l_{3m} + F_{3m}\cos a\lambda^2 l_{3m} = 0, \tag{A21}$$

which can be expressed as

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \sin\lambda l_{3m} & \cos\lambda l_{3m} & \sinh\lambda l_{3m} & \cosh\lambda l_{3m} & 0 & 0 \\ \cos\lambda l_{3m} & -\sin\lambda l_{3m} & \cosh\lambda l_{3m} & \sinh\lambda l_{3m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin a\lambda^2 l_{3m} & \cos a\lambda^2 l_{3m} \end{bmatrix} \{R\}_{3m} = \underline{B}_{3\times 6}\{R\}_{3m}, \tag{33}$$

Substitutions of the boundary conditions at the left support, Eqs. (18), (19) and (32a-c), lead to

$$B_{3m-2} + D_{3m-2} = 0, \tag{A22}$$

$$A_{3m-2} + C_{3m-2} = 0, \tag{A23}$$

$$F_{3m-2} = 0. \tag{A24}$$

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